J. Korean Math. Soc. **26**(1989), No. 2, pp. 301-310

A CENTRAL LIMIT THEOREM IN A MEAN FIELD MODEL

Chi Hoon Choi, Woo Chul Kim and Jong Woo Jeon

1. Introduction

Let $\{X_i^{(n)} : i=1, \cdots, n\}$ $(n=1, 2, \cdots)$ be a triangular array of dependent and identically distributed random variables with the joint distribution given by

$$z_n^{-1}\exp\{-\beta H_n(x_1, \cdots, x_n)\}\prod_{i=1}^{n} dP(x_i) \tag{1.1}$$

where $\beta>0$ is a constant, $H_n(\cdot)$ is a function on R^n, P is a probability measure on R^1 and z_n is a normalization constant. The form of the joint distribution in (1.1) is often considered in statistical mechanics. There, $X_i^{(n)}$ is the magnetic spin at i-th site, β inverse temperature and H_n Hamiltonian which represents the energy of the system.

When H_n takes the particular form

$$H_n(x_1, \cdots, x_n) = -(\sum_{i=1}^{n} x_i)^2/2n \tag{1.2}$$

the model (1.1) is usually called the mean field model or the Curie Weiss model in the statistical mechanics literature. The Curie–Weiss model has been important physically because it explains qualitatively the thermodynamic behavior of some physical quantities in phase transitions and critical phenomena. See Stanley [11] for reference.

In recent years, a number of results on the asymptotic diatribution of the total magnetizm $S_n=\sum_{i=1}^{n} X_i^{(n)}$ for the Curie–Weiss model have been established. References along this line are Simon & Griffiths [9] and Dunlop & Newman [4]. The latest results in this direction were obtained by Ellis and Newman [5], [6]. They established central limit theorems for the Curise–Weiss model and related them to the criticality of phase transitions. In this paper, we consider a generalized

Received March 11, 1989.
Thie research was supported by Korea Science and Engineering Foundation 1983-1985.

model in which $H_n(\cdot)$ takes the following form

$$H_n(x_1, \cdots, x_n) = -n\psi_Y\{(x_1+\cdots+x_n)/n\}, \tag{1.3}$$

where $\psi_Y(\cdot)$ is the cumulant generating function of some random variable Y. Obviously the Curie–Weiss model (1.2) becomes the special case of our generalized model (1.3) when Y is standard normal. Then we establish central limit theorems for the asymptotic distribution of S_n for this generalized model. As for the proofs, we utilize the particular form of H_n in (1.3) to apply the conditioning technique of Sethuraman [8] which considerably simplifies proofs of our results.

Recently, Chaganty and Sethuraman [2] obtained a result which is more general than ours. Their main tool is the local limit theorem for arbitrary sequences of random variables [1], while, in this paper, we use Daniel's result [3], of the uniform saddle point approximation of the probility density function for the sample mean. The arbitrariness of their result allows them to have more general form of Hamiltonian function H_n. However, lack of uniformity in their local limit theorem makes the proof much more technical and complicated, hiding the real nature on what is going on in this problem.

In Section 2 we state our main results under rather simple hypothesis. The proof is given in Section 3. In Section 4, some simple examples are given.

2. Statement of the main result

In this section, we define a Generalized Curie–Weiss model and state a main result of asymptotic distribution for randon variables occuring in the model.

Let $Y, Y_1, Y_2, \cdots$ be a sequence of independent and identically distributed random variables with the finite moment generating function $\phi_Y(t)$ for real values t such that $|t|<c\leq\infty$. Denoted by $\psi_Y(t)$ the cumulant generating function $\log \phi_Y(t)$ and by $\gamma_Y(u)$ the large deviation rate $\sup_{t\in(-c,c)} \{ut-\psi_Y(t)\}$ for $u\in R^1$. Let $f_n(\cdot)$ be the probability density function of $(Y_1+\cdots+Y_n)/n$. Under the certain conditions, Daniel [3] proved the uniform local limit theorems for $(Y_1+\cdots+Y_n)/n$ as follows;

i) (absolutely continuous case)

$$f_n(x) = (n/2\pi)^{1/2}\sigma^{-1}(t) \exp[-n\gamma_Y(x)]\cdot[1+o(1)], n\to\infty \tag{2.1}$$

holds uniformly in $x\in D_Y = \{\psi_Y'(t) \ : t\in I=(-a, a),\ 0<a<c\}$, where

$\sigma^2(t)=\psi_Y''(t)$.

(ii) (lattice case)

$$nP_n(k)/h=(n/2\pi)^{1/2}\sigma^{-1}(t)\exp[-n\gamma_Y(x)]\cdot[1+o(1)],\quad n\to\infty \quad (2.2)$$

holds uniformly in $x=a+kh\in D_Y$, where $nP_n(k)=Pr\{Y_1+\cdots+Y_n=an+kh\}$, $\sigma^2(t)=\psi_Y''(t)$; h is the maximal span of the distribution, k is an integer and a is a fixed real number.

For a simple proof of (2.1) and (2.2) without uniformity can be found in So and Jeon [10].

REMARK. For $u\in D_Y$, we have $\gamma_Y(u)=[us-\psi_Y(x)]<\infty$, where $s\in I$ satisfies $\psi_Y'(s)=u$.

DEFINITION 2.1. Let L_Y be the class of probability measure P such that

$$\int\phi_Y(x)\,dP<\infty \quad (2.3)$$

REMARK. Let $\psi_p(u)$ be the cumulant generating function of a probability measure $P\in L_Y$. It is seen from Fubini's theorem that the condition (2.3) impies that $\psi_p(u)$ is finite for $u\in D_Y$.

For a probability measure $P\in L_Y$, we define

$$G_Y(u)=\gamma_Y(u)-\psi_p(u) \quad \text{for } u\in D_Y,$$

where $\psi_p(u)$ is the cumulant generating function of P. The function G_Y will play a basic role in determining the asymptotic behavior of total magnetism S_n.

DEFINITION 2.2. A local minimum m for $G_Y(\cdot)$ is said to be of *type* k if

$$G_Y(m+u)-G_Y(m)=c_{2k}u^{2k}/(2k)!+o(u^{2k}),\quad \text{as } n\to 0, \quad (2.4)$$

where $c_{2k}=G_Y^{(2k)}(m)>0$.

REMARK. Let P be the symmetric Bernoulli measure, i.e. $P\{1\}=P\{-1\}=\frac{1}{2}$. Then, (2.3) holds for any Y with $\phi_Y(x)<\infty$.
In particular, when Y is a standard normal random variable,

$$G_Y(u)=\frac{1}{2}u^2-\log(\cosh u)$$

since

$$\gamma_Y(u)=\sup_s\{us-\log\phi_Y(s)\}=\sup_s\{us-\frac{1}{2}s^2\}=\frac{1}{2}u^2$$

and

$$\phi_p(u)=\cosh(u).$$

In this case, it can be easily shown that G_Y has a local minimum $m=0$ of type 2 and in fact 0 is the unique global minimum.

We consider trangular array of dependent random variables $X_j^{(n)}$, $j=1, 2, \cdots, n$, with joint distribution

$$dQ_n(x)=z_n^{-1}\phi_Y^n(s_n/n)\prod_{j=1}^n dP(x_j), \tag{2.5}$$

where $P\in L_Y$. Let $S_n=X_1^{(n)}+\cdots+X_n^{(n)}$.

We now state the main result for the case of absolutely continuous Y. A simple modification of the proof will give the same conclusion for the discrete case.

THEOREM 2.1. *Let $X_1^{(n)}, \cdots, X_n^{(n)}$ have joint distribution Q_n given by* (2.5). *Assume that Y satisfies condition* (2.1) *or* (2.2) *and G_Y has the unique global minimum of type k at m and also assume that*

$$\inf_{s\in(a,b)} G_Y(s)<\min\{\underline{\lim}_{s\to a} G_Y(s), \underline{\lim}_{s\to b} G_Y(s)\}, \tag{2.6}$$

where $(a, b)=D_Y$. Then,

$$(S_n-nm')/m''n^{1-1/2k}\xrightarrow{d}F_{k,c_{2k}} \tag{2.7}$$

where $F_{k,c_{2k}}$ is defined by

$$dF_{k,c_{2k}}=\begin{cases}N(0, 1/m''+1/(c_2) & \text{if } k=1\\ \exp\{-c_{2k}u^{2k}/(2k)!\}/\int\exp\{-c_{2k}u^{2k}/(2k)!\}du & \text{if } k>1\end{cases}$$

and $m'=\phi_p'(m)$, $m''=\phi_p''(m)$.

REMARK. In the special case $(a, b)=R^1$, we can dispense with the condition (2.6) since in this case (2.3) implies (2.6). (See Appendix)

3. Proof of the main result

Note that when G_Y has a global minimum of m, (2.4) becomes

$$G_Y(z)=G_Y(m)+c_{2k}(z-m)^{2k}/(2k)!+o(|z-m|)^{2k}) \tag{3.1}$$

as $z\to m$.

Let $h_n(z)$ be any function on $n^{1/2k}D_Y-m=\{x\in R^1 : x=n^{1/2k}u-m, u\in D_Y\}$ satisfying

$$h_n(z)=\exp[-n\{G_Y(zn^{-1/2k}+m)\}-G_Y(m)\}]\sigma^{-1}(t_n)\ [1+o(1)],$$

where $\sigma^2(t)=\psi_Y''(t)$, $\psi_Y'(t_n)=zn^{-1/2k}+m$ and $\psi_Y=\log\phi_Y$. We first prove following lemmas.

LEMMA 3.1. *Assvme $P\in L_Y$. Then $G_Y(\cdot)$ is bounded below and*

$$\int\cdots\int\phi_Y^n(s_n/n)\prod_{i=1}^n dP(x_i)<\infty \text{ for every } n \tag{3.2}$$

and there exists a positive integer n_0 such that

$$\int\exp\{-nG_Y(s)\}/\sigma(t)ds<\infty \text{ for each } n\geqq n_0, \tag{3.3}$$

where $\sigma^2(t)=\psi_Y''(t)$, $\psi_Y'(t)=s$ and $\psi_Y=\log\phi_Y$.

Proof. For real s, we have the inequality

$$\int\exp(sx)dP(x)\leqq\int\exp\{\gamma_Y(s)+\psi_Y(x)\}dP(x)$$
$$=\exp[\gamma_Y(s)]\int\phi_Y(x)dP(x)$$

from the definition of $\gamma_Y(\cdot)$. Thus G_Y is bounded below. By the convexity of $\psi_Y(\cdot)$, we have

$$\int\cdots\int\phi_Y^n(s_n/n)\prod_{i=1}^n dP(x_i)=\int\cdots\int\exp n\psi_Y(s_n/n)\prod_{i=1}^n dP(x_i)$$
$$\leqq\int\cdots\int\exp\{n(\Sigma\psi_Y(x_i)/n)\}\prod_{i=1}^n dP(x_i)$$
$$=\left[\int\phi_Y(x)dP\right]^n.$$

This proves (3.2). We next prove (3.3). By the assumption on Y, we may choose positive integer n_0 such that for any $n\geqq n_0$

$$|f_n(s)/\exp(-n\gamma_Y(s))-1|<1/2 \text{ for every } s\in D_Y,$$

where $f_n(\cdot)$ is p.d.f. of S_n/n. From this, we have for any $n\geqq n_0$

$$\frac{1}{2}\int\exp\{-nG_Y(s)\}\sigma^{-1}(t)ds\leqq\int f_n(s)\phi_Y^n(s)ds=\int\phi_Y^n(s_n/n)\prod_{i=1}^n dP(x_i).$$

LEMMA 3.2. *Let $P\in L_Y$ and suppose G_Y has the unique global minimum of type k at m and also satisfies*

$$\inf_{s\in(a,b)}G_Y(s)<\min\{\underline{\lim}_{s\to a}G_Y(s),\ \underline{\lim}_{s\to b}G_Y(s)\}$$

Then, as $n\to\infty$

$$\int_{|z|\geqq n^\delta}h_n(z)dz\to 0 \text{ for } 0<\delta<1/2k.$$

Proof. Let $g_n=\min\{G_Y(m-n^{\delta-1/2k}),\ G_Y(m+n^{\delta-1/2k})\}>0$. Then

$$\int_{|z|\geq n^\delta} h_n(z)\,dz = n^{1/2k}\int_{|z|\geq n^{\delta-1/2k}} h_n(zn^{1/2k})\,dz$$
$$= n^{1/2k}\int_{|z|\geq n^{\delta-1/2k}} \exp[-n\{G(m+z)-G(m)\}]\cdot\sigma^{-1}\cdot[1+o(1)]dz$$
$$\leqq (3/2)n^{1/2k}e^{-(n-n0)g_n}\int_{D_Y}\exp[-n_0\{G(z)-G(m)\}]\cdot\sigma^{-1}\cdot dz$$
$$\leqq o(n^{1/2k})e^{-ng_n}, \text{ by } (3.3)$$
$$= o(n^{1/2k})\exp\{-(C_{2k}n^{2k\delta}+o(n^{2k\delta})\}\to 0, \text{ as } n\to\infty.$$

LEMMA 3.3. *If G_Y has a local minimum of type k at m, then as $n\to\infty$,*

$$h_n(z)\to\exp\{-c_{2k}z^{2k}/(2k)!\}\cdot\sigma^{-1}(t_0) \text{ for each } z \tag{3.4}$$

where $\phi_Y'(t_0)=m$ and

$$I_{(|z|\leq\theta_n)}(z)h_n(z)\leqq(3/2)\cdot\exp\{-c_{2k}z^{2k}/2(2k)!\}\sigma^{-1}(t_0) \tag{3.5}$$

for all n sufficiently large, where $\theta_n\to\infty$ and $\theta_n\cdot n^{-1/2k}\to 0$ as $n\to\infty$.

Proof. Since $zn^{-1/2k}\to 0$ as $n\to\infty$ uniformly for $|z|<\theta_n$

$$h_n(z)=\exp[-nG_Y(m+zn^{-1/2k})-nG(m)]\sigma^{-1}(t_n)[1+o(1)]$$
$$=\exp-n\{c_{2k}z^{2k}n^{-1}/(2k)!+o(n^{-1}z^{2k})\}\cdot\sigma^{-1}(t_n)[1+o(1)]$$
$$=\exp\{-c_{2k}z^{2k}/(2k)!\}\cdot\sigma^{-1}(t_0)[1+o(1)] \text{ as } n\to\infty$$

uniformly for $|z|<\theta_n$, where $\phi_Y'(t_0)=m$, $\phi_Y'(t_n)=m+z_n{}^{-1/2k}$. This establishes both (3.4) and (3.5) since $c_{2k}>0$.

LEMMA 3.4. *If $P\in L_Y$ and G_Y has a unique global minimum of type k at m and inf $G_Y(s)<\min\{\underline{\lim}_{s\to a} G_Y(s),\ \underline{\lim}_{s\to b} G_Y(s)\}$, then as $n\to\infty$,*

$$\int h_n(z)\,dz\to\int h(z)\,dz, \tag{3.6}$$

where $h(z)=\exp\{-c_{2k}z^{2k}/(2k)!\}\sigma^{-1}(t_0)$ and $\phi_Y'(t_0)=m$.

Proof. Let δ be such that $0<\delta<1/2k$. Then as $n\to\infty$,

$$\left|\int h_n(z)\,dz-\int h(z)\right|dz$$
$$\leqq\int_{|z|\geq n^\delta}h_n(z)\,dz+\int_{|z|\geq n^\delta}h(z)\,dz+\int_{|z|<n^\delta}|h_n(z)-h(z)|dz\to 0,$$

since the first term tends to 0 by Lemma 3.2, the second term by the integrability of $h(z)$ and the third term by the dominated covergence theorem and Lemma 3.3.

Now to prove theorem, we first express Q_n in (3.5) as follows:

$$dQ_n(x)=z_n{}^{-1}\phi_Y{}^n(s_n/n)\prod_{i=1}^n dP(x_i)$$
$$=z_n{}^{-1}\int e^{s_nz}f_n(z)\,dz\prod_{i=1}^n dP(x_i)$$

$$=z_n^{-1}\int\exp\{s_n(m+z_n^{-1/2k})\}\prod_{i=1}^{n}dP(x_j)f_n(m+z_n^{-1/2k})_n{}^{-1/2k}dz$$
$$=z_n^{-1}(2\pi)^{-1/2}n^{(k-1)/2k}\int\prod_{i=1}^{n}\exp[x_i(m+zn^{-1/2k})-\psi_p(m+zn^{-1/2})]$$
$$dP(x_j)\cdot\exp[n\psi_p(zn^{-1/2k}+m)-n\gamma_Y(m+zn^{-1/2k})]\sigma^{-1}(t_n)$$
$$[1+o(1)]dz$$
$$=z_n^{-1}(2\pi)^{-1}n^{(k-1)/2k}\int\prod_{i=1}^{n}\exp[x_j(m+z_n^{-1/2k})-\psi_p(m+zn^{-1/2k})]$$
$$dp(x_j)\cdot\exp[-nG_Y(m+z_n^{-1/2k})]\sigma^{-1}(t_n)\cdot[1+o(1)]dz$$
$$=K_n^{-1}\int\prod_{i=1}^{n}dM_{n,z}(x_j)\,h_n(z)\,dz,$$

where

$$dM_{n,z}(s)=\exp\{x(m+zn^{-1/2k})-\psi_p(m+zn^{-1/2k})\}dP(x) \tag{3.7}$$
$$h_n(z)=\exp[-n\{G_Y(m+zn^{-1/2k})-G_Y(m)\}]\cdot\sigma^{-1}(t_n)[1+o(1)], \tag{3.8}$$
$$\psi_Y'(t_n)=m+zn^{-1/2k},\ \sigma^2(t)=\psi_Y''(t)\ \text{and}$$
$$K_n=z_n(2\pi)^{1/2}n^{-(k-1)/2k}\cdot e^{-nG_Y(m)}$$

Since $\int\cdots\int dQ_n(x)=1$ and $\int dM_{n,z}(x_j)=1$ for each z and j, we have from (3.3) that

$$K_n=\int h_n(z)\,dz \tag{3.9}$$

Thus

$$h_n^*(z)=h_n(z)/\int h_n(z)\,dz \tag{3.10}$$

is a density function for each n.

Now, as $n\to\infty$

$$\log E_{M_{n,z}}[\exp n^{-(1-1/2k)}S_n]$$
$$=n[\psi_p(tn^{-(1-1/2k)}+m+zn^{-1/2k})-\psi_p(m+zn^{-1/2k})]$$
$$=n[\psi_p'(m+zn^{-1/2k})tn^{-(1-1/2k)}+\frac{1}{2}\psi_p''(m+zn^{-1/2k})t^2n^{-(2-1/k)}+o(n^{-1})]$$
$$=n^{1/2k}\psi_p'(m+zn^{1/2k})t+\frac{1}{2}\psi_p''(m+zn^{-1/2k})t^2n^{-(1-1/k)}+o(1)$$
$$=n^{1/2k}t[\psi_p'(m)+\psi_p''(m)zn^{-1/2k}+o(n^{-1/2k})]$$
$$+\left(\frac{1}{2}\right)n^{-(1-1/k)}t^2[\psi_p''(m)+o(n^{-1/2k})]+o(1)$$
$$=n^{1/2k}t\psi_p'(m)+\psi_p''(m)zt+\frac{1}{2}n^{-(1-1/k)}t^2\psi_p''(m)+o(1)$$

This shows that under $M_{n,z}$,

$$(S_n-nm')/m''n^{1-1/2k}\xrightarrow{d}\begin{cases}\delta(s-z) & \text{if } k>1,\\ N(z,1/m'') & \text{if } k=1\end{cases} \tag{3.11}$$

where $m'=\phi_p'(m)=\gamma_Y'(m)$, $m''=\phi_p''(m)$. By (3.4) of Lemma 3.3, and Lemma 3.4, we have, as $n\to\infty$

$$h_n^*(z)\to h^*(z) \text{ for every } z, \tag{3.12}$$

where $h^*(z)=\exp\{-c_{2k}z^{2k}/(2k)!\}/\int\exp\{-c_{2k}z^{2k}/(2k)!\}dz$.

By applying theorem 2.1 of Sethuraman [8], (3.11) and (3.12), proof of the Theorem 2.1 is completed.

3. Examples

EXAMPLE 3.1. When Y is standard normal,

$$dQ_n(x)=z_n^{-1}\phi_Y^n(s_n/n)\prod_{i=1}^n dP(x_i)$$
$$=z_n^{-1}\exp(s_n^2/2n)\prod_{i=1}^n dP(x_i).$$

Thus, our generalized model is reduced to the familiar Curie–Weiss model since it is well known that $\gamma_Y(t)=t^2/2$ and $\sigma^2(t)=1$ and uniformity condition (2.1) holds trivally in this case. Also, condition (2.6) holds by the Remark under Theorem 2.1 since $D_Y=R^1$ in this case.

EXAMPLE 3.2. Let Y be symmetric Bernoulli random variable. Thus, the joint distribution in the generalized model is given by

$$dQ_n(x)=Z_n^{-1}\phi_Y^n(s_n/n)\prod_{i=1}^n dP(x_i)$$
$$=Z_n^{-1}[\cosh(s_n/n)]^n\prod_{i=1}^n dP(x_i).$$

In this case uniformity condition (2.2) was shown to be satistied by Daniel [3]. It is also known that

$$\gamma_Y(t)=1/2\{(1+t)\log(1+t)+(1-t)\log(1-t)\} \text{ for } |t|<1$$

Let P be the standard normal distribution. Then

$$G_Y(t)=1/2\{(1+t)\log(1+t)+(1-t)\log(1-t)\}-t^2/2$$

Now, for each t, $|t|<1$

$$\begin{aligned}G_Y(t)&=(1/2)\{(1+t)(t-t^2/2+t^3/3-t^4/4+\cdots)+\\&\quad(1-t)(-t-t^2/2-t^3/3-t^4/4-\cdots)\}-t^2/2\\&=(-t^2/2-t^4/4-\cdots)+(t^2+t^4/3+t^6/5+\cdots)-t^2/2\\&=t^4/(3\cdot4)+t^6/(5\cdot6)+\cdots>0 \text{ for } t\neq0.\end{aligned}$$

It is seen from the above expression of $G_Y(t)$ that the condition (2.6) is satisfied and also condition (2. 3) is satisfied since

$$\int \phi_Y(x)\,dP=\int \phi_p(x)\,dF_Y.$$

Thus the standard normal distribution belongs to L_Y with $k=2$ where Y is symmetric Bernoulli. From Theorem 2. 1, we thus have

$$S_n/n^{3/4}\xrightarrow{d}\exp(-z^4/12) \text{ since } c^4/4!=1/12.$$

5. Appendix

In this section we give a short proof of the remark made in section 2. We prove specifically that $G(s)\to\infty$ as $|s|\to\infty$.

For any s and $L>0$, we have the estimate

$$\begin{aligned}\int \exp(s\cdot x)\,dP(x) &\leqq P[-L, L]\exp L\cdot|s|+\int_{|x|>L}\exp(s\cdot x)\,dP\\ &\leqq P[-L, L]\exp L\cdot|s|+\int_{|x|>L}\exp\{\gamma_Y(s)+\phi_Y(x)\}\,dP\\ &\leqq \exp L|s|+\exp\gamma_Y(s)\int_{|x|>L}\phi_Y(x)\,dP\end{aligned}$$

By choosing $L>0$ large enough, we only need to show that

$$\gamma_Y(s)-L|s|\to\text{as } |s|\to\infty.$$

Now we consider the case $s>0$ and the other case can be proved similarly. For any $L'>L>0$, we have

$$\gamma_Y(s)=\sup_x\{sx-\phi_Y(x)\}\geqq sL'-\phi_Y(L')$$

Therefore, we have

$$\gamma_Y(s)-Ls\geqq(L'-L)s-\phi_Y(L')\to\infty \text{ as } s\to\infty$$

as desired.

References

1. Chaganty, N. R. and Sethuraman, J. *Limit theorems in the area of large deviations for some dependent random variables*, Ann. Probab. **15**(1987), 628-645.
2. Chaganty, N. R. and Sethuraman, J. *Large deviation local limit theorems for arbitrary sequences of random variables*, Ann. Probab. **13**(1985), 97-114.
3. Daniel, H. E., *Saddle point approximations in statistics*, Ann. Math. Statist., **25**(1954), 631-650.
4. Dunlop, F. and Newman, C. M., *Multicomponent field theories and classical*

rotators, Commun. Math. Phy., Vol. **44**(1978), 223-235.

5. Ellis, R. S. and Newman, C. M., *Limit theorems for sums of dependent random variables occurring in statistical mechanics*, Z. f. Wahrscheinlichkeits-theorie verw. Gebiete **44**(1978), 117-139.
6. Ellis, R. S., Newman, C. M., *Limit theorems for sums of dependent random variables occuring in statistical mechanics*, Z. f. Wahrscheinlichkeits-1heorie verw. Gebiete **51**(1980), 153-169.
7. Richter, W., *Local limit theorems for large deviations*, Theor. Probability Appl. **2**(1957), 206-220.
8. Sethuraman, J., *Some limit theorems for joint distributions*, Sankhyā, Ser. A, Vol. **23**(1961), 379-385.
9. Simon, B. and Griffiths, R. B., *The*$(\phi^4)_2$ *field theory as a classical Ising model*, Commun. Math. Phy. Vol. **33**(1973), 145-164.
10. So, B. S. and Jeon, J. W., *A local limit theorem for large deviations*, J. Korean Statist. Soc., **11**-2(1982), 88-93.
11. Stanley, H. E., *Introduction to phase transition and critical phenomena*, Oxford. New York(1971).

Seoul National University
Seoul 151-742, Korea