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MORE ON FIBERWISE INVOLUTIONS OF FIBERED 3-MANIFOLDS

JEHPILL KIM[1)]

1. Introduction

This note is a study of those fiberwise involutions of a closed 3-manifold fibered over the circle S^1 with orbit spaces having the first Betti number greater than 1. What we first do is to detect such involutions by examining the behavior of the isomorphisms they induce on the first homology and the fundamental groups. We then find some conditions for an involution to be fiberwise relative to more than one fibering.

Let X be a closed 3-manifold and let h be a PL involution of X. If there is a fibering $g: X \longrightarrow S^1$ with connected fiber such that $g \circ h = g$, we shall say that h is *fiberwise* (or *fiberwise relative to* g whenever we want to be a little bit more specific).

As in [4], a *fibering of a group* G is defined to be an epimorphism $\varepsilon: G \longrightarrow \mathbf{Z}$ whose kernel is finitely generated. The obvious right action of Aut G on the set of fiberings of G induces a right action of Out G=Aut G/Inn G because the integers $\mathbf{Z}$ is an abelian group. Now the involution h determines an element of Out $\pi_1(X)$ as follows: Regard $h_\# : \pi_1(X, x_0) \rightarrow \pi_1(X, h(x_0))$, $x_0 \in X$, as an automorphism of $\pi_1(X, x_0)$ by identifying the two groups using a path between base points. Because this identification is unique up to inner automorphisms, we have the well defined outer automorphism class of $h_\#$, which will be denoted by $\hat{h}_\#$. We shall say that $\hat{h}_\#$ is *fiberwise* if $\pi_1(X, x_0)$ admits a fibering left unchanged by $\hat{h}_\#$.

We reserve the symbol $\mathrm{tr}\, h_*$ to denote the trace of the induced homomorphism $h_*: H_1(X) \longrightarrow H_1(X)$ restricted on the free part of the first homology group $H_1(X)$.

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Theorem 1. *Let h be a PL involution of a P^2-irreducible closed 3-manifold X such that* rank $H_1(X)$ + tr $h_*>2$. *Then h is fiberwise if and only if $\hat{h}_\#$ is fiberwise.*

This characterizes those fiberwise involutions h of X such that the orbit space Y of the $\mathbf{Z}_2$-action h generates on X has the first Betti number $\beta_1(Y)>1$ in view of Lemma 3 below. If h meets certain conditions of geometric nature, the conclusion of Theorem 1 remains true without the requirement imposed on the trace of h_*. See [3] and [4] for such conditions.

Considering the case where the orbit space Y is boundariless, we have

Theorem 2. *Let h be an involution of a closed 3-manifold X such that* (i) rank $H_1(X)+$tr $h_*>2$, *and* (ii) *the fixed point set of h has no 2-dimensional component. If there is a fibering $g: X \longrightarrow S^1$ with connected fiber $M=g^{-1}(1)$ making h fiberwise, then h is fiberwise relative to infinitely many fiberings $g_n: X \longrightarrow S^1$ distinguished by the homotopy classes (i.e., no two of the maps g_n are homotopic). If it is further assumed that the Euler characteristic of the fiber M is negative, then the fiberings g_n can be so chosen as to be distinguished also by the fibers (i.e., no two of the fibers $M_n=g_n(1)^{-1}$ are homeomorphic).*

Corollary. *Let h be an orientation preserving PL involution of an orientable P^2-irreducible closed 3-manifold X such that $\hat{h}_\#$ is fiberwise. If* rank $H_1(X)+$tr $h_*>2$, *then h is fiberwise relative to infinitely many fiberings distinguished by the homotopy classes. If it is further assumed that the fixed point set of h is not empty, h is fiberwise relative to infinitely many fiberings distinguished by the fibers.*

It is to be noted at this point that Byun [2] has given some conditions for h to be fiberwise in infinitely many ways. His result covers the case where h is free or $\pi_1(X, x_0)$ has nontrivial center, under the hypothesis rank $H_1(X)$ + tr $h_*>2$.

2. Some lemmas

Let $g: X \longrightarrow S^1$ be a fibering of a closed 3-manifold X with connected fiber $M=g^{-1}(1)$ and let $q: \tilde{X} \longrightarrow X$ be the covering space corresponding to ker $g_\#$. Since X cut along M is an $M\times I$, $\tilde{X}$ can be identified with $M\times\mathbf{R}$ so as to have $g(q(m,t))=e^{2\pi ti}$ for (m,t) in $M\times\mathbf{R}=\tilde{X}$. This identification then gives a flow φ on X defined by $\varphi(x,t)=q(m,u+t)$ where (m,u) is any point in $q^{-1}(x)\subset M\times\mathbf{R}=\tilde{X}$. Here, (m,u) is unique up to covering transformations and hence $\varphi: X\times\mathbf{R}\longrightarrow X$ is indeed a well defined steady flow on X. A flow of this type, determined on how X cut along M is identified to $M\times I$, is said to be *compatible* with the fibering $g: X\longrightarrow S^1$. *Stream lines* of the flow φ are curves of the form $\varphi(x\times\mathbf{R})$.

Now let h be an involution of X fiberwise relative to the fibering $g: X\longrightarrow S^1$ and let α denote h restricted to the fiber M. Then h lifts to an involution of the form $\tilde{h}=\alpha\times 1_{\mathbf{R}}$ for suitable product structure of the covering space $\tilde{X}=M\times\mathbf{R}$. This can be easily verified by using [5; Theorem A] if not quite trivial. Let $f: X\longrightarrow Y$ be the orbit map and define the fibering $g': Y\longrightarrow S^1$ by $g'(f(x))=g(x)$. The covering space $\tilde{Y}$ of Y corresponding to ker $g'_\#$ can be identified with $N\times\mathbf{R}$. Here, $N=f(M)$ is a compact connected 2-manifold possibly with boundary. The orbit space Y can then be regained as the image of the covering projection $q': N\times\mathbf{R}\longrightarrow Y$. In turn, the flow $\varphi: X\times\mathbf{R}\longrightarrow X$ induces a flow $\psi: Y\times\mathbf{R}\longrightarrow Y$ compatible with $g': Y\longrightarrow S^1$. To be precise, ψ is given by $\psi(f(x),t)=f(\varphi(x,t))$. Summarizing, we have

LEMMA 1. *Let h be an involution of a closed 3-manifold X. Suppose that there is a fibering $g: X\to S^1$ making h fiberwise, and let $g': Y\to S^1$ be the fibering of the orbit space Y induced by $g: X\longrightarrow S^1$. Then there is a flow $\varphi: X\times\mathbf{R}\longrightarrow X$ compatible with the fibering $g: X\longrightarrow S^1$ such that*

(i) *h sends stream lines of φ to stream lines of φ,*

(ii) *any stream line of φ meeting a 1-dimensional component of the fixed point set of h is a simple closed curve pointwise fixed under h, and*

(iii) *the correspondence $(f(x),t)\longrightarrow f(\varphi(x,t))$ gives a flow $\psi: Y\times\mathbf{R}\longrightarrow Y$ compatible with the fibering $g': Y\longrightarrow S^1$.*

Now let $T=S^1\times S^1$ and let $p: T\longrightarrow S$ be the projection to the second factor, i.e., $p(w,z)=z$. For $n\geq 1$, the torus T also admits the fiberings $p_n: T\longrightarrow S^1$ given by $p_n(\bar{z}^n, ze^{ti})=e^{nti}$. The fiber $p_n^{-1}(1)$ is the simple closed curve $W_n=\{(\bar{z}^n,z): z\in S^1\}$.

Let $g: X \longrightarrow S^1$ and $\varphi: X \times \mathbf{R} \longrightarrow X$ be as before. Whenever rank $H_1(X) > 1$, Neumann [6] has shown that X admits infinitely many fiberings by first finding a map $P: X \longrightarrow T$ inducing an epimorphism of fundamental groups such that $p \circ P$ agrees with $g: X \longrightarrow S^1$. What he has actually shown is that if n is large then there is a closed surface M_n in $P^{-1}(W_n)$ and a map $\tau: M_n \longrightarrow R$ subject to the condition: $\tau(x)$ is the smallest positive real number with $\varphi(x, \tau(x))$ in M_n. Each point of X is of the form $\varphi(x, t)$ for suitable x in M_n and $0 \leq t < \tau(x)$. The correspondence $\varphi(x, t) \longrightarrow e^{2\pi t i/\tau(x)}$ gives a fibering $g_n: X \longrightarrow S^1$ whose fiber is M_n. The homomorphism $(g_n)_*: H_1(X) \longrightarrow H_1(S^1)$ is the sum of $\eta = (p' \circ P)_*$ and ng_* where p' is the projection of $T = S^1 \times S^1$ to its first factor. Thus Neumann's result can be phresed as

Lemma 2. *Let X be a closed 3-manifold with* rank $H_1(X) > 1$, *let $g: X \longrightarrow S^1$ be a fibering of X with connected fiber $M = g^{-1}(1)$, and let $\varphi: X \times \mathbf{R} \longrightarrow X$ be a flow compatible with $g: X \longrightarrow S^1$. If n is sufficiently large, there is a fibering $g_n: X \longrightarrow S^1$ with connected fiber $M_n = g_n^{-1}(1)$ and a compatible flow $\varphi_n: X \times \mathbf{R} \longrightarrow X$ such that*

(i) *there is an epimorphism $\eta: H_1(X) \longrightarrow H_1(S^1)$ independent of n such that $\eta + ng_*$ agrees with $(g_n)_*: H_1(X) \longrightarrow H_1(S^1)$,*

(ii) $g_*(\ker (g_n)_*) = H_1(S^1)$ *and* $(g_n)_*(\ker g_*) = H_1(S^1)$,

(iii) *stream lines of φ_n are also stream lines of φ, and*

(iv) *if the Euler characteristic $\chi(M)$ is negative, then $\chi(M_n)$ becomes arbitrarily large negative as $n \longrightarrow \infty$.*

The first Betti number of the orbit space Y is determined by the automorphism h_* as follows.

Lemma 3. rank $H_1(Y) = \frac{1}{2}(\text{rank } H_1(X) + \text{tr } h_*)$.

Proof. We reproduce the proof given in [2] for the sake of completeness. First regard h_* as an automorphism of $H_1(X; \mathbf{Q})$. Then h_* can be represented by a diagonal matrix having diagonal entries 1 or -1 since the minimal polynomial of h_* divides $x^2 - 1$. Accordingly, the lemma is true for rational homology by virtue of [1; Theorem III. 2.4]. The desired result for integral homology then follows from the universal coefficient theorem.

3. Proof of Theorem 1

To prove the nontrivial part of the theorem, let $\varepsilon : \pi_1(X, x_0) \longrightarrow \mathbf{Z} = \pi_1(S^1, 1)$ be a fibering relative to which $\hat{h}_\#$ is fiberwise. By making use of [4; Theorem 2], we can find a fibering $g : X \longrightarrow S^1$ with connected fiber such that either (i) h is fiberwise relative to g, or (ii) $g(h(x)) = -g(x)$ for all x. In the first case, there remains nothing to be proved. If (ii) holds, consider the fibering $g' : Y \longrightarrow S^1$ given by $g'(f(x)) = f'(g(x))$ where $f' : S^1 \longrightarrow S^1$ is the double covering map sending z to z^2. To find a fibering of X making h fiberwise, observe that Y is boundariless since the orbit map $f : X \longrightarrow Y$ must be a double covering map in this case. Since rank $H_1(Y) > 1$ by Lemma 3, there is a fibering $g'_n : Y \longrightarrow S^1$ with connected fiber as given in Lemma 2. Let $\varepsilon_n : \pi_1(X, x_0) \longrightarrow \mathbf{Z} = \pi_1(S^1, 1)$ be the homomorphism induced by the map $g_n = g'_n \circ f$. Since the orbit map f gives a homeomorphism between the fibers $M = g^{-1}(1)$ and $N = g'^{-1}(1)$ in this case, ε_n must be epic by virtue of Lemma 2. Also by Lemma 2, $g'_\#$ sends $K'_n = \ker (g'_n)_\#$ onto im $g'_\#$. Accordingly, ker ε_n is finitely generated as it can be identified with the index 2 subgroup $f_\#(\ker \varepsilon_n) = K'_n \cap g'^{-1}_\#(\mathrm{im}\, f'_\#)$ of K'_n using the monomorphism $f_\#$. Thus ε_n is indeed a fibering of $\pi_1(X, x_0)$. That h is fiberwise follows now from [4; Theorem 1]. However, it might be better to observe that for each z in S^1 $g_n^{-1}(z)$ is a connected double covering space of $g'^{-1}_n(z)$ because of the fact that $f_\#(\ker \varepsilon_n)$ has index 2 in K'_n. If follows that $g_n : X \longrightarrow S^1$ is indeed a fibering of X making h fiberwise.

4. Proof of Theorem 2 and Corollary

If h is fiberwise relative to $g : X \longrightarrow S^1$, let $\varphi : X \times \mathbf{R} \longrightarrow X$ be a flow as in Lemma 1 so that $g(\varphi(x, t)) = g(x) e^{2\pi t i}$ and $h(\varphi(x, t)) = \varphi(h(x), t)$ for (x, t) in $X \times \mathbf{R}$. The induced flow $\phi : Y \times \mathbf{R} \longrightarrow Y$ given by $\phi(f(x), t = f(\varphi(x, t))$ is compatible with the fibering $g' : Y \longrightarrow S^1$ induced by $g : X \longrightarrow S^1$. The oribit space Y is a closed 3-manifold because of the hypothesis that the fixed point set F of h has no 2-dimensional component. Since rank $H_1(Y) > 1$ by Lemma 3, there are fiberings $g'_n : Y \longrightarrow S^1$ and compatible flows $\phi_n : Y \times \mathbf{R} \longrightarrow Y$ as in Lemma 2 for infinitely many integers n. Let $g_n = g'_n \circ f$. Each $g_n : X \rightarrow S^1$, whenever defined, is a fibering of X making h fiberwise because of the definition of the flow $\phi : Y\times\mathbf{R} \longrightarrow Y$ and the fact that the flows ϕ and ϕ_n have the same stream lines as stated in Lemma 2. It is also a consequence of Lemma 2 that any two of the maps g_n are homotopically distinct. Finally, assume that the Euler characteristic of the fiber $M=g^{-1}(1)$, $\chi(M)$, is negative and we want to show that infinitely many $M_n=g_n^{-1}(1)$ are topologically distinct. To this end, let us denote $f(M)=g'^{-1}(1)$ by N. If the fixed point set F of h is empty, N is double covered by M. Hence $\chi(M)=2\chi(N)$ and so $\chi(N)$ is negative. Letting $N_n=g'_n{}^{-1}(1)$, we conclude from Lemma 2 that $\chi(M_n)=2\chi(N_n)$ becomes arbitrarily large negative as n becomes large. If F is not empty, M_n is a branched double covering space of N_n having $M_n\cap F$ as the set of branch points. Here, $M_n\cap F$ is indeed a finite set by Lemmas 1 and 2 because F is 1-dimensional. Since $\chi(M_n)=2\chi(N_n)-\#(M_n\cap F)$, it suffices to show that $\#(M_n\cap F)$, the number of points in $M_n\cap F$, becomes arbitrarily large in order to discriminate the fibers M_n topologically. For this, let C be a component of F and let ζ denote the homology class of the simple closed curve C oriented in accordance with the orientation of the base space S^1. Using Lemma 2, we see that

$$\#(M_n\cap C)=(g_n)_*(\zeta)=(g'_n)_*(f_*(\zeta))$$

becomes large. Thus $\#(M_n\cap F)$ becomes large and we have completed the proof of Theorem 2.

We now pass to the Corollary. The given condition certainly implies that h is fiberwise relative to some fibering $g : X \longrightarrow S^1$ by Theorem 1. To see that h is fiberwise in many ways, observe that the fixed point set F of h is either empty or everywhere 1-dimensional because h preserves the orientation. Accordingly, we may use Theorem 2 to conclude that h is fiberwise relative to infinitely many nonhomotopic fiberings. It remains to prove that the Euler characteristic of the fiber $M=g^{-1}(1)$ is negative if F is not empty. Since h is orientation preserving, $N=f(M)$ is an orientable closed 2-manifold. N cannot be a 2-sphere because

$$\text{rank } H_1(Y)=\frac{1}{2}(\text{rank} H_1(X)+\text{tr} h_*)>1$$

Hence $\chi(N)\leq 0$, and so $\chi(M)=2\chi(N)-\#(M\cap F)<0$. This completes the proof.

Dedication. The characterization given in the pioneering work [3] of fiberwise involutions with fixed points is due to Professor Kyung Whan

Kwun. This paper is respectfully dedicated to him.

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Seoul National University
Seoul 151-742, Korea