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A BASIC PRINCIPLE OF TOPOLOGICAL VECTOR SPACE THEORY

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Recently, P. Antosik and C. Swartz established an effectual matrix method in analysis ([1], [2], [3], [4]). The kernel of the matrix method is the Mikusinski-Antosik-Swartz basic matrix theorem ([2], [3]). Many important basic results of functional analysis and measure theory, such as the uniform boundedness principle, the Orlicz-Pettis theorem, Schur lemma, Phillips lemma, and the Nykodym boundedness theorem can be conveniently obtained from the basic matrix theorem ([1], [2], [3], [4]).

In this paper, we will improve the Mikusinski-Antosik-Swartz theorem. Specifically, we will get our matrix theorem only from the definition of topological vector space, though C. Swartz and P. Antosik got their theorem by the uniform structure theory of topological groups. Thus, we will show that the basic matrix theorem is the most fundamental principle in topological vector space theory.

Throughout the remainder of the paper, G will denote an Abelian topological group, N will denote the set of all symmetric zero neighborhoods of G. From the definition of G, we come to an immediate conclusion: For every neighborhood U of zero of G there is a $V \in N$ such that $V+V \subseteq U$.

We will only use this initial property of G.

THEOREM 1. *Let $x_{ij} \in G$ for $i, j \in \boldsymbol{N}$ such that $\lim_j x_{ij}=0$ for each $i \in \boldsymbol{N}$. The followings are equivalent.*

(1) $\lim_i x_{ij}=x_j$ *exists uniformly for* $j \in \boldsymbol{N}$;

(2) $\lim_i x_{ij}=x_j$ *exists for each* $j \in \boldsymbol{N}$, *and for every* $U \in N$ *there is a sequence* $\{p_i\} \subseteq \boldsymbol{N}$ *such that if* $p_i \le q_i \in \boldsymbol{N}$ *and* $\{j_n\} \subseteq \boldsymbol{N}$ *are arbitrary,*

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then $\sum_{k=1}^{m}\left[x_{q_{n_{k_0}} j_{n_k}}-x_{q_{n_{k_0}+1} j_{n_k}}\right]\in U$ *for some subsequence* $\{j_{n_k}\}\subseteq\{j_n\}$, $\{x_{q_{n_k}}\}\subseteq\{x_{q_n}\}$ *and* $m, k_0\in \boldsymbol{N}, m\geq k_0$.

Proof. (1)→(2) Let $U_0\in N$. Take a sequece $\{U_n\}\subseteq N$ such that $U_{n+1}+U_{n+1}\subseteq U_n$, for $n=0, 1, 2, \cdots$.

The condition (1) says that for every $i\in \boldsymbol{N}$ there is a $p_i\in \boldsymbol{N}$ such that $x_{pj}-x_j\in U_i$ for all $j\in \boldsymbol{N}$ and $p_i\leq p\in \boldsymbol{N}$.

We may assume that, $p_1<p_2<p_3<\cdots$.

Let $p_i\leq q_i\in \boldsymbol{N}$ and $\{j_n\}\subseteq \boldsymbol{N}$ be arbitrary. Then,

$$\begin{aligned}\sum_{n=1}^{5}\left[x_{q_5 j_n}-x_{q_{5+1} j_n}\right]&=\sum_{n=1}^{5}\left[x_{q_5 j_n}-x_{j_n}+x_{j_n}-x_{q_6 j_n}\right]\\ &\in (U_5+U_6)+(U_5+U_6)+(U_5+U_6)+(U_5+U_6)\\ &\qquad +(U_5+U_6)\\ &\subseteq U_4+U_4+U_4+U_4+U_4\\ &\subseteq U_3+U_3+U_3\\ &\subseteq U_2+U_2\\ &\subseteq U_1\\ &\subseteq U_0.\end{aligned}$$

(2)→(1). Suppose not. Then there is a $U_0\in N$ such that for any $p\in \boldsymbol{N}$ there exists $q>p$ and $j\in \boldsymbol{N}$ such that

$$x_{qj}-x_j\notin U_0 \qquad (*)$$

Take a sequence $\{U_n\}\subseteq N$ such that $U_n+U_n\subseteq U_{n-1}$ for all $n\in \boldsymbol{N}$. Let $\{p_n\}$ be the integer sequence such that the condition (2) ensured its existence with respect to U_2. There is a $q_1>p_1$ and $j_1\in \boldsymbol{N}$ such that $x_{q_1 j_1}-x_{j_1}\notin U_0$, by (*). But $\lim_p x_{pj_1}=x_{j_1}$, there is an integer $k_1>\max(q_1, p_2)$ such that $x_{kj_1}-x_{j_1}\in U_2$ if $k\geq k_1$.

Since $x_{q_1 j_1}-x_{j_1}=x_{q_1 j_1}-x_{kj_1}+x_{kj_1}-x_{j_1}$ and

$x_{q_1 j_1}-x_{j_1}\notin U_0$, we have that $x_{q_1 j_1}-x_{kj_1}\notin U_1$ *if* $k\geq k_1$.

There is a $q_2>k_1$ and j_2 in $\boldsymbol{N}$ such that

$$x_{q_2 j_2}-x_{j_2}\notin U_0.$$

Since $q_2>k_1$, it follows that

$$x_{q_1 j_1}-x_{q_2 j_1}\notin U_1,\quad x_{q_2 j_1}-x_{j_1}\in U_2.$$

Observe that $\lim_p x_{pj_1}=x_{j_1}$ and $\lim_p x_{pj_2}=x_{j_2}$,

there is a $k_2>\max(q_2, p_3)$ such that $x_{kj_1}-x_{j_1}\in U_3$ and $x_{kj_2}-x_{j_2}\in U_3$ if $k\geq k_2$. So, observing $x_{q_2 j_2}-x_{j_2}=x_{q_2 j_2}-x_{kj_2}+x_{kj_2}-x_{j_2}$, we have that $x_{q_2 j_2}-x_{kj_2}\notin U_1$ if $k\geq k_2$.

In this way, we can get integer sequences $\{q_i\}$, $q_i \geq p_i$, and $\{j_i\}$ such that

$$x_{q_i j_i} - x_{q_{i+1} j_i} \notin U_1, \quad x_{q_i j_l} - x_{j_l} \notin U_i \text{ if } 1 \leq l < i \qquad (**).$$

Set $i_1 = 1$, $y_{11} = x_{q_{i_1} j_{i_1}} - x_{q_{i_1}+1 j_{i_1}}$ Then $y_{11} \notin U_1$.

Since $\lim_j x_{ij} = 0$ for each i, there is an $i_2 \geq 6 > i_1$ such that

$$\begin{aligned} x_{q_{i_1} j_{i_2}} - x_{q_{i_1}+1 j_{i_2}} &= x_{q_{i_1} j_{i_2}} - 0 + 0 - x_{q_{i_1}+1 j_{i_2}} \\ &\in U_6 + U_6 \\ &\subseteq U_5 \\ &= U_{2+3} \end{aligned}$$

and, from (**),

$$\begin{aligned} x_{q_{i_2} j_{i_1}} - x_{q_{i_2}+1 j_{i_1}} &= x_{q_{i_2} j_{i_1}} - x_{j_{i_1}} + x_{j_{i_1}} - x_{q_{i_2}+1 j_{i_1}} \\ &\in U_{i_2} + U_{i_2+1} \\ &\subseteq U_6 + U_6 \\ &\subseteq U_{2+3}. \end{aligned}$$

Thus, using the notations

$$\begin{aligned} y_{12} &= x_{q_{i_1} j_{i_2}} - x_{q_{i_1}+1 j_{i_2}}, \\ y_{21} &= x_{q_{i_2} j_{i_1}} - x_{q_{i_2}+1 j_{i_1}}, \text{ and} \\ y_{22} &= x_{q_{i_2} j_{i_2}} - x_{q_{i_2}+1 j_{i_2}}. \end{aligned}$$

We have that y_{12} and $y_{21} \in U_{2+3}$ but $y_{22} \notin U_1$.

In this way, we have a matrix

$$(y_{nk})_{n,k}, \quad y_{nk} = x_{q_{i_n} j_{i_k}} - x_{q_{i_n}+1 j_{i_k}},$$

such that y_{nk} and $y_{kn} \in U_{n+3}$ if $1 \leq k < n$; $y_{nn} \notin U_1$, $\forall n \in \boldsymbol{N}$.

From condition (2), observing $i_k \geq k$ and hence $q_{i_k} \geq q_k \geq p_k$, there are $\{j_{i_{k_l}}\} \subseteq \{j_{i_k}\}$ and integers m, l_0, $(m \geq l_0)$ such that

$$\sum_{l=1}^{m} y_{k_{l_0} k_l} = \sum_{l=1}^{m} (x_{q_{i_{k_{l_0}}} j_{i_{k_l}}} - x_{q_{i_{k_{l_0}}}+1 j_{i_{k_l}}}) \in U_2$$

since the sequence $\{p_i\}$ is taken with respect to U_2. But

$$\begin{aligned} y_{k_{l_0} k_{l_0}} &= \sum_{l=1}^{m} y_{k_{l_0} k_l} - \sum_{l=1}^{l_0-1} y_{k_{l_0} k_{l_0}} - \sum_{l=l_0+1}^{m} y_{k_{l_0} k_l} \\ &\in U_2 + (U_{k_{l_0}+3} + U_{k_{l_0}+3} + \cdots + U_{k_{l_0}+3}) \\ &\quad + (U_{k_{l_0+1}+3} + U_{k_{l_0+2}+3} + \cdots + U_{k_m+3}) \\ &\subseteq U_2 + U_3 + U_3 \\ &\subseteq U_2 + U_2 \\ &\subseteq U_1. \end{aligned}$$

This contradicts the fact $y_{k_{l_0} k_{l_0}} \notin U_1$.

Now we can get the Mikusinski-Antosik-Swartz basic matrix theorem ([2]), Theorem 1).

THEOREM 2. *Let $x_{ij}\in G$ for $i, j\in \boldsymbol{N}$ satisfy*

(A) $\lim_i x_{ij}=x_j$ *exists for each* j *and*

(B) *for each integer sequence* (m_j) *there is a subsequence* (n_j) *of* (m_j) *such that* $\left\{\sum_{j=1}^{\infty} x_{in_j}\right\}_{i=1}^{\infty}$ *is Cauchy.*

Then $\lim_i x_{ij}=x_j$ *uniformly in* $j\in \boldsymbol{N}$.

Proof. We claim that $\lim_j x_{ij}=0$, $\forall i\in \boldsymbol{N}$.

Otherwise, say that $\lim_j x_{i_0 j}\neq 0$ for some $i_0\in \boldsymbol{N}$, then there is $U\in N$ and $j_1<j_2<\cdots$ such that $x_{i_0 j}\notin U$, $\forall k\in \boldsymbol{N}$.

Thus, $\sum_{k=1}^{\infty} x_{i_0 j_k}$ has no such convergent subseries. This contradicts (B).

Now let $U\in N$. Take $U_1, U_2\in N$ such that $U_2+U_2\subseteq U_1$, $U_1+U_1\subseteq U$.

Set $\{p_i\}=\{1, 2, 3, \cdots\}$. Then for every $p_i\leq q_i\in \boldsymbol{N}$ and $\{j_n\}\subseteq \boldsymbol{N}$ there is a $\{j_{n_k}\}\subseteq\{j_n\}$ such that $\left\{\sum_{k=1}^{\infty} x_{q_i j_{n_k}}\right\}_{i=1}^{\infty}$ is a Cauchy, by (B). Thus, there is an $i_0\in \boldsymbol{N}$ such that

$$\sum_{k=1}^{\infty} x_{q_i j_{n_k}}-\sum_{k=1}^{\infty} x_{q_{i+1} j_{n_k}}\in U_2 \text{ for each } i\geq i_0.$$

Take a $k_0\in \boldsymbol{N}$ such that $n_{k_0}\geq i_0$. Then there is an $m\geq k_0$ such that

$$\sum_{k>m} x_{q_{n_{k_0}} j_{n_k}}\in U_2,\quad \sum_{k>m} x_{q_{n_{k_0}+1} j_{n_k}}\in U_2.$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{m} x_{q_{n_{k_0}} j_{n_k}}-\sum_{k=1}^{m} x_{q_{n_{k_0}+1} j_{n_k}}\\
&=\sum_{k=1}^{\infty} x_{q_{n_{k_0}} j_{n_k}}-\sum_{k>m} x_{q_{n_{k_0}} j_{n_k}}-\left[\sum_{k=1}^{\infty} x_{q_{n_{k_0}+1} j_{n_k}}-\sum_{k>m} x_{q_{n_{k_0}+1} j_{n_k}}\right]\\
&=\sum_{k=1}^{\infty} x_{q_{n_{k_0}} j_{n_k}}-\sum_{k=1}^{\infty} x_{q_{n_{k_0}+1} j_{n_k}}+\sum_{k>m} x_{q_{n_{k_0}+1} j_{n_k}}-\sum_{k>m} x_{q_{n_{k_0}} j_{n_k}}\\
&\in U_2+U_2+U_2\\
&\subseteq U_1+U_1\\
&\subseteq U.
\end{aligned}$$

Thus, $\{x_{ij}\}$ satisfies the condition (2) of Theorem 1. So the desired result is obtained from Theorem 1.

The conditions (A) and (B) of Theorem 2 are sufficient for the uniform convergence of columns but are not necessary.

EXAMPLE. Consider the matrix $((1/i)e_j)_{i,j}$ in (c_0, weak), where $e_j=(0, 0, \cdots, 0, \underset{j-\text{th}}{1}, 0, 0, \cdots)$, $j\in \boldsymbol{N}$.

Clearly, each row tends to zero. But this matrix fails to keep the condition (B) because for each i the series $\sum_{j=1}^{\infty}(1/i)e_j$ has no such convergent subseries. So we can not get the uniform convergence of the columns from Theorem 2. Clearly, we get it from THEOREM 1. Of course, we can get it from $((1/i)e_j)_{i,j}$ itself directly.

To show the forces of the matrix theorem, we will give a very general version of the uniform boundedness principle which is due to P. Antosik and C. Swartz ([5]). But we would like to give a more general version.

DEFINITION 3. ([3]) Let (E, τ) be a topological vector space. A sequence $\{x_i\}$ in E is a $\tau-\kappa$ convergent sequence if each subsequence $\{x_{i_k}\}$ of $\{x_i\}$ has a subsequence $\{x_{i_{k_n}}\}$ such that the series $\sum_{n=1}^{\infty}x_{i_{k_n}}$ is τ-convergent to an element x. A subset $B\subseteq E$ is $\tau-\kappa$ bounded if for each $\{x_j\}\subseteq B$ and $\{t_j\}\in c_0$ the sequence $\{t_jx_j\}$ is $\tau-\kappa$ convergent.

Let X be a linear space, Y be a topological vector space and Γ be a family of linear maps from X to Y. Let $\tau(\Gamma)$ be the weakest topology on X such that each member of Γ is continuous. Clearly,

$x_j\xrightarrow{\tau(\Gamma)}0$ if and only if $T(x_j)\longrightarrow 0$, $\forall T\in\Gamma$.

It is easy to check that $\tau-\kappa$ boundedness implies τ-boundedness but they are same in Frechet space case ([3]).

THEOREM 4. *Let X be a linear space, Y be a topological vector space and Γ be a family of linear maps from X to Y. if Γ is pointwise bounded, i.e., $\{T(x) : T\in\Gamma\}$ is bounded in Y for each $x\in X$, then Γ is uniformly bounded on $\tau(\Gamma)-\kappa$ bounded sets and $\tau(\Gamma)-\kappa$ convergent sequences.*

Proof. Let $B\subseteq X$ be a $\tau(\Gamma)-\kappa$ bounded set. We have to prove that $A=\{T(x) : T\in\Gamma, x\in B\}$ is a bounded set in Y. If this is not true, then there is a balanced neighborhood $U_0\in N(Y)$ and a sequence $\{t_i\}$

$\in c_0$ and a sequence $\{T_i(x_i)\}\subseteq A$ such that $t_iT_i(x_i)\notin U_0$ for all i. Since U_0 is balanced, if $|t_i|T_i(x_i)\in U_0$ then $t_iT_i(x_i)\in U_0$. Thus, we can assume that $t_i\geq 0$.

Consider the matrix $(\sqrt{t_i}T_i(\sqrt{t_j}x_j))_{i,j}$. Since B is $\tau(\Gamma)-\kappa$ bonuded and $\sqrt{t_j}\rightarrow 0$, the sequence $\{\sqrt{t_j}x_j\}$ is $\tau(\Gamma)-\kappa$ convergent. Hence, if $\{j_n\}\subseteq \boldsymbol{N}$ then there is a $\{j_{n_k}\}\subseteq\{j_n\}$ such that $\sum_{k=1}^{\infty}\sqrt{t_{j_{n_k}}}\,x_{j_{n_k}}$ is $\tau(\Gamma)$-convergent, i. e., there is an $x\in$X such that

$$\sum_{k=1}^{\infty}T(\sqrt{t_{j_{n_k}}}\,x_{j_{n_k}})=\lim_m\sum_{k=1}^{m}T(\sqrt{t_{j_{n_k}}}\,x_{j_{n_k}})$$
$$=T(x),\quad \forall T\in\Gamma.$$

The sequence $\left\{\sum_{k=1}^{\infty}\sqrt{t_i}T_i(\sqrt{t_{j_{n_k}}}\,x_{j_{n_k}})\right\}_{i=1}^{\infty}=\{\sqrt{t_i}T_i(x)\}_{i=1}^{\infty}$ is a Cauchy sequence in Y because $\{T_i(x)\}_{i=1}^{\infty}$ is bounded and $\sqrt{t_i}\longrightarrow 0$. In fact, $\sqrt{t_i}T_i(x)\longrightarrow 0$ and $\lim_i\sqrt{t_i}T_i(\sqrt{t_j}x_j)=0$ for each $j\in\boldsymbol{N}$. Now, by THEOREM 2, $\lim_i\sqrt{t_i}T_i(\sqrt{t_j}x_j)=0$ uniformly in $j\in\boldsymbol{N}$, there is an i_0 such that if $i\geq i_0$ then $\sqrt{t_i}T_i(\sqrt{t_j}x_j)\in U_0$ for each $j\in\boldsymbol{N}$.

Thus, $t_iT_i(x_i)=\sqrt{t_i}T_i(\sqrt{t_i}x_i)\in U_0$, $\forall i\geq i_0$.

This is a contradiction. The second result can be obtained by similar arguments.

COROLLARY 5. *Let* X *be a Frechet space,* Y *be a topological vector space and* Γ *be a family of continuous linear operators from* X *to* Y. *Then if* Γ *is pointwise bounded it is uniformly bounded on bounded sets. Furthermore,* Γ *is equicontinuous.*

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