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SOME CHARACTERIZATIONS OF THE BLOCH SPACE IN THE BALL

DO YOUNG KWAK* AND SUNGYUN LEE

1. Introduction

Let $B=B_n$ be the unit ball in C^n and let $H(B)$ be the space of all holomorphic functions in the ball. For $z, \zeta \in C^n$, we let $\langle z, \zeta \rangle = \sum_{j=1}^n z_j \bar{\zeta}_j$ be the inner product and let $\|z\| = \langle z, z \rangle^{1/2}$ be the Euclidean norm. Let dv be the usual Lebesgue measure on C^n and let $d\sigma$ be the normalized surface measure on the boundary ∂B of B. For $q>0$, dv_q is the probability measure on B defined by

$$dv_q(z) = \frac{\Gamma(n+q)}{\pi^n \Gamma(q)} (1-\|z\|^2)^{q-1} dv(z).$$

We let

$$A_q^p(B) = \{f \in H(B), \|f\|_{p,q}^p = \int_B |f(z)|^p dv_q(z) < \infty\}$$

be the weighted Bergman spaces.

DEFINITION For $q>0$, we define B^q to be the space of all functions f in $H(B)$ such that

$$\|f\|_{B^q} = \sup_{z \in B} (1-\|z\|^2)^q |(R+n+1)f(z)| < \infty,$$

where $Rf(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z)$ is the radial derivative introduced in [7]. The definition does not change if one replaces $R+n+1$ by $R+1$, but one can get sharper norm estimate in some other context, see [6]. When $q=1$, B^1 become the usual Bloch space and there are plenty of papers about it, most of them are one variable results, see [1, 2, 4, 5, 8] for example. In this paper, we give some generalizations of one variable results, with the weight $(1-\|z\|^2)^q$. Other generalizations will

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appear in [6]. The Bloch space with weight appears in [3] but we have not seen it elsewhere.

2. Equivalent norms on the Bloch spaces

Let $\phi_a(z)$ be the automorphism of B given by

$$\phi_a(z)=\frac{a-P_a z-sQ_a z}{1-\langle z, a\rangle}$$

where $P_a z=\frac{\langle z, a\rangle}{\|a\|^2}a$, $Q_a z=(I-P_a)z$ and $s=\sqrt{1-\|a\|^2}$. It is easy to check that $\phi_a(0)=a$, $\phi_a(a)=0$, $\phi_a^2=I$ and the real Jacobian of $\phi_a(z)$ *is* $[(1-\|a\|^2)/|1-\langle z, a\rangle|^2]^{n+1}$. We let

$$B(a, r)=\{z\in B: \|\phi_a(z)\|<r\}$$

be the pseudo-hyperbolic ball of radius r.

LEMMA 1.

(1) $1-\|\phi_a(z)\|^2=\frac{(1-\|a\|^2)(1-\|z\|^2)}{|1-\langle z, a\rangle|^2}$

(2) *For any multi-index* $\alpha=(\alpha_1\dots\alpha_n)\in \boldsymbol{Z}_+^n$, *we have*

$$\int_{\partial B}|\zeta^\alpha|^2 d\sigma(\zeta)=\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

(3) *For a fixed* $0<r<1$, *the quotient* $v_q(B(a, r))/(1-\|a\|^2)^{n+q}$ *approaches a positive constant as* $\|a\|\to 1$.

(4) $\int_B \|z\|^2 dv_q(z)=\frac{n}{n+q}$.

Proof. For (1) and (2), see [7]. For (3), we use the change of variable $w=\phi_a(z)$ to get

$$\begin{aligned}\int_{B(a,r)}dv_q(w)&=\int_{B(0,r)}\left(\frac{1-\|a\|^2}{|1-\langle z, a\rangle|^2}\right)^{n+1}(1-\|\phi_a(z)\|^2)^{q-1}dv_1(z)\\&=\int_{B(0,r)}\left(\frac{1-\|a\|^2}{|1-\langle z, a\rangle|^2}\right)^{n+q}dv_q(z)\\&=(1-\|a\|^2)^{n+q}\int_{\|z\|<r}|1-\langle z, a\rangle|^{-2n-2q}dv_q(z)\end{aligned}$$

Since $1-r\le|1-\langle z, a\rangle|\le 1+r$, the result follows. (4) follows directly from the polar coordinate change:

$$\begin{aligned}\int_B\|z\|^2 dv_q(z)&=\frac{2\Gamma(n+q)}{\Gamma(n)\Gamma(q)}\int_0^1\rho^{2n-1}(1-\rho^2)^{q-1}\int_{\partial B}\|\rho_z\|^2 d\sigma(z)\,d\rho\\&=\frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q)}\int_0^1 s^n(1-s)^{q-1}ds=\frac{n}{n+q}.\end{aligned}$$

LEMMA 2. *For* $1\leq q$, *we have* $\|f\circ\phi_z\|_{Bq}\leq C_q(1-\|z\|^2)^{1-q}\|f\|_{Bq}$, *where* C_q *depends only on* q.

Proof. We let $w=\phi_z(\zeta)$ and compute $R(f\circ\phi_z)$. With $s=\sqrt{1-\|z\|^2}$,

$$\begin{aligned}\frac{\partial(f\circ\phi_z)}{\partial\zeta_i}&=\sum_j\frac{\partial f}{\partial w_j}\cdot\frac{\partial w_j}{\partial\zeta_i}\\ &=\frac{1}{(1-\langle\zeta,z\rangle)^2}\Big(\sum_j\frac{\partial f}{\partial w_j}\bar z_j z_j\frac{s(1-s)}{\|z\|^2}\\ &\quad-s\bar z_i\sum_j\frac{\partial f}{\partial w_j}\zeta_j-s\frac{\partial f}{\partial w_i}(1-\langle\zeta,z\rangle)\Big)\\ &=\frac{s}{(1-\langle\zeta,z\rangle)^2}\Big(\bar z_j\sum\frac{\partial f}{\partial w_j}(z_j\cdot\frac{1}{1+s}-\zeta_j)-\frac{\partial f}{\partial w_i}(1-\langle\zeta,z\rangle)\Big).\end{aligned}$$

Hence

$$\begin{aligned}R(f\circ\phi_z)(\zeta)&=\sum_{i=1}^n\zeta_i\frac{\partial(f\circ\phi_z)}{\partial\zeta_i}\\ &=\frac{s}{(1-\langle\zeta,z\rangle)^2}\Big(\langle\zeta,z\rangle\frac{\langle\nabla f,\bar z\rangle}{1+s}-\langle\nabla f,\bar\zeta\rangle\Big).\end{aligned}$$

But since $\langle\nabla f,\overline{\phi_z(\zeta)}\rangle=\frac{1}{1-\langle\zeta,z\rangle}\Big(\Big(\frac{1+s-\langle\zeta,z\rangle}{1+s}\Big)\langle\nabla f.\bar z\rangle-s\langle\nabla f,\bar\zeta\rangle\Big)$, we solve for $\langle\nabla f,\bar\zeta\rangle$ and combine it with the above equation to obtain

$$R(f\circ\phi_z)(\zeta)=\frac{1}{1-\langle\zeta,z\rangle}\langle\nabla f,\bar\phi_z-\bar z\rangle.$$

Now (1) of Lemma 1 gives

$$\begin{aligned}(1-\|\zeta\|^2)^q|R(f\circ\phi_z)(\zeta)|&\leq 2(1-\|\phi_z(\zeta)\|)^q\|\nabla f(\phi_z(\zeta))\|\\ &\quad\times\frac{|1-\langle\zeta,z\rangle|^{2q-1}}{(1-\|z\|^2)^q}\\ &\leq C_q\cdot\|f\|_{Bq}\cdot(1-\|z\|^2)^{1-q}.\end{aligned}$$

THEOREM 1. *Let* $1\leq q<2$ *and* $0<r<1$ *be fixed. Then the following statements are equivalent*:

(A0) $f\in B^q$;

(A1) $\sup_{z\in B}(1-\|z\|^2)^q|Rf(z)|<\infty$;

(A2) $\sup_{z\in B}(1-\|z\|^2)^q\|\nabla f(z)\|<\infty$;

(A3) $\sup_{z\in B}(1-\|z\|^2)^q|\langle\nabla f,\bar z\rangle|<\infty$;

(A4) $\sup_{z\in B}Q_f{}^q(z)<\infty$, *where*

$$Q_f{}^q(z)=\sup_{\|\zeta\|=1}(1-\|z\|^2)^q\frac{|\langle\nabla f(z)\cdot\bar\zeta\rangle|}{\sqrt{(1-\|z\|^2)\|\zeta\|^2+|\langle z,\zeta\rangle|^2}};$$

(B0) $\sup_{z\in B}(1-\|z\|^2)^{2q-1}(\|\nabla f(z)\|^2-|\langle\nabla f(z),\bar{z}\rangle|^2)$
$=\sup_{z\in B}(1-\|z\|^2)^{2q-2}\|\nabla(f\circ\phi_z)(0)\|^2<\infty$;

(B1) $\sup_{z\in B}(1-\|z\|^2)^{q-1}\|f\circ\phi_z-f(z)\|_{2,q}<\infty$;

(C) $\sup_{z\in B}\frac{(1-\|z\|^2)^{2q-2}}{v_q(B(z,r))}\int_{B(z,r)}|f(\zeta)-f_{B(z,r)}|^2dv_q(\zeta)<\infty$, *where*
$f_{B(z,r)}=\frac{1}{v_q(B(z,r))}\int_{B(z,r)}f(\zeta)\,dv_q(\zeta)$;

(D) $\sup_{z\in B}\frac{(1-\|z\|^2)^{2q-2}}{v_q(B(z,r))}\int_{B(z,r)}|f(\zeta)-f(z)|^2dv_q(\zeta)$;

Proof. The equivalence of (A0) through (A4) follows as in [8]. To see that (B0) implies (A2), we observe that from the proof of Lemma 2

$$\left|\frac{\partial}{\partial\zeta_i}(f\circ\phi_z)(0)\right|=\left|\frac{s(1-s)}{\|z\|^2}\bar{z}_i\sum_j\frac{\partial f}{\partial w_j}z_j-s\frac{\partial f}{\partial w_i}\right|.$$

Hence

$$\begin{aligned}\|\nabla(f\circ\phi_z(0))\|^2&=\sum_i\Bigg(\frac{s^2(1-s)^2|z_i|^2}{\|z\|^4}\left|\sum_j\frac{\partial f}{\partial w_j}z_i\right|^2+s^2\left|\frac{\partial f}{\partial w_i}\right|^2\\&\qquad-\frac{2s^2(1-s)}{\|z\|^2}Re\left[\bar{z}_i\sum_j\frac{\partial f}{\partial w_j}z_j\cdot\overline{\frac{\partial f}{\partial w_i}}\right]\Bigg)\\&=\frac{s^2(1-s)^2}{\|z\|^2}|\langle\nabla f,\bar{z}\rangle|^2-s^2\|\nabla f\|^2\\&\qquad-\frac{2s^2(1-s)}{\|z\|^2}|\langle\nabla f,\bar{z}\rangle|^2\\&=(1-\|z\|^2)(\|\nabla f\|^2-|\langle\nabla f,\bar{z}\rangle|^2).\end{aligned}$$

We decompose z as $a\cdot\frac{\nabla f(z)}{\|\nabla f(z)\|}+b\xi$, where $\langle\nabla f(z),\xi\rangle=0$, $\|\xi\|=1$. Then

$$\begin{aligned}\|\nabla(f\circ\phi_z)(0)\|^2&=(1-\|z\|^2)(\|\nabla f\|^2-|\langle\nabla f,\bar{z}\rangle|^2)\\&=(1-\|z\|^2)(1-|a|^2)\|\nabla f\|^2\geq[(1-\|z\|^2)\|\nabla f\|]^2.\end{aligned}$$

Multiplying $(1-\|z\|^2)^{2q-2}$ on both sides, we see that (B0) implies (A2). To see the converse, we consider the identity;

$$\|\nabla f\|^2=\frac{|\langle\nabla f(z),\bar{z}\rangle|^2}{\|z\|^2}+|\langle\nabla f(z),\bar{\xi}\rangle|^2,$$

where $\langle z,\xi\rangle=0$ and $\|\xi\|=1$. Then the quantity in (B0) becomes

$$\left(\frac{(1-\|z\|^2)^q|\langle\nabla f(z),\bar{z}\rangle|}{\|z\|}\right)^2+(1-\|z\|^2)^{2q-1}|\langle\nabla f,\bar{\xi}\rangle|^2.$$

It is clear that we only have to consider the case $\|z\|>1/2$. The first term is bounded by (A2), while the second term is bounded by (A4). This proves the imlpication (A2) + (A4) ⇒ (B0). Now we assume the condition (B1). Then

$$\frac{(1-\|z\|^2)^{2q-2}}{v_q(B(z,r))}\int_{B(z,r)}|f(\zeta)-f(z)|^2dv_q(\zeta)$$
$$=\frac{(1-\|z\|^2)^{2q-2}}{v_q(B(z,r))}\int_{B(0,r)}|f\circ\phi_z(w)-f(z)|^2\left(\frac{1-\|z\|^2}{|1-\langle w,z\rangle|^2}\right)^{n+q}dv_q(w)$$
$$\leq C(1-\|z\|^2)^{2q-2}\|f\circ\phi_z-f(z)\|_{2,q}^2 \text{ by (3) of Lemma 1.}$$

Hence (B1) implies (D). To see that (B1) implies (B0), we consider, for $f\in A_q^2(B)$

$$f(z)=\int_B\frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+q}}dv_q(\zeta).$$

Hence $\frac{\partial f}{\partial z_j}(0)=(n+q)\int_B f(\zeta)\bar{\zeta}_j dv_q(\zeta)$ and a use of the Cauchy-Schwarz inequality gives

$$(1)\qquad \|\nabla f(0)\|^2\leq(n+q)^2\int_B|f(\zeta)|^2dv_q(\zeta)=(n+q)^2\|f\|_{2,q}^2$$

Applying this to $f\circ\phi_z-f(z)$, we obtain

$$(1-\|z\|^2)(\|\nabla f(z)\|^2-|\langle\nabla f(z),\bar{z}\rangle|^2)\leq(n+q)^2\|f\circ\phi_z-f(z)\|_{2,q}^2.$$

Thus (B1) implies (B0). To show the converse, we observe the following;

$$|f(z)-f(0)|\leq\int_0^1|\langle\nabla f(tz),\bar{z}\rangle|dt\leq\int_0^1\|\nabla f(tz)\|dt.$$

Replacing f with $f\circ\phi_z$, we see that for $1<q<2$

$$|f\circ\phi_z(\zeta)-f(z)|\leq\|f\circ\phi_z\|_{Bq}\cdot\int_0^1(1-t^2\|\zeta\|^2)^{-q}dt\leq C\|f\circ\phi_z\|_{Bq}$$
$$\times((1-\|\zeta\|^2)^{1-q}-1).$$

Hence

$$\int_B|f\circ\phi_z(\zeta)-f(z)|^2dv_q(\zeta)\leq C\|f\circ\phi_z\|_{Bq}^2$$
$$\times\int_B(1-\|\zeta\|^2)^{2-2q}dv_q\leq C\|f\circ\phi_z\|_{Bq}^2.$$

The case $q=1$ is dealt with similarly. If we multiply by $(1-\|z\|^2)^{2q-2}$ on both sides, we get the implication (B0) ⇒ (B1) by Lemma 2. The implication (D) ⇒ (C) is trivial by the triangle inequality. Finally, we complete the proof by showing that (C) implies (B0). Using ploar coordinate and (2) of the Lemma 1, we see that

$$\int_{\|\zeta\|<r}f(\zeta)\bar{\zeta}_i dv_q(\zeta)$$

$$=\frac{\partial f}{\partial z_i}(0)\int_{B(0,r)}|\zeta_i|^2 dv_q(\zeta)=\frac{2\Gamma(n+q)}{\Gamma(n)\Gamma(q)}\cdot\frac{\partial f}{\partial z_i}(0)$$
$$\int_0^r \rho^{2n-1}(1-\rho^2)^{q-1}\int_{\partial B}|\rho\zeta_i|^2 d\sigma(\zeta)$$
$$=\frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q)}\cdot\frac{\partial f}{\partial z_i}(0)\int_0^r s^n(1-s)^{q-1}ds=C(r,q)\frac{\partial f}{\partial z_i}(0),$$

where $C(r,q)\to\frac{n}{n+q}$ *as* $r\to 1^-$. Hence we have

$$\|\nabla f(0)\|^2\leq\frac{r^2}{C(r,q)^2}\int_{\|\zeta\|<r}|f(\zeta)|^2 dv_q(\zeta).$$

Now fix $z\in B$ and replace f by $f\circ\phi_z-f_{B(z,r)}$ to obtain

$$(1-\|z\|^2)(\|\nabla f\|^2-|\langle\nabla f(z),\bar{z}\rangle|^2)$$
$$\leq\frac{1}{C(r,q)^2}\int_{\|\zeta\|<r}|f\circ\phi_2(\zeta)-f_{B(z,r)}|^2 dv_q(\zeta)$$
$$=\frac{1}{C(r,q)^2}\int_{B(z,r)}|f(w)-f_{B(z,r)}|^2\left(\frac{1-\|z\|^2}{|1-\langle w,z\rangle|^2}\right)^{n+q}dv_q(w)$$
$$\leq\frac{C}{v_q(B(z,r))}\int_{B(z,r)}|f(w)-f_{B(z,r)}|^2 dv_q(w)$$

where the last inequality is due to Lemma 1. Now we multiply by $(1-\|z\|^2)^{2q-2}$ on both sides to see that (C) implies (B0).

The above proof works word by word when B^q is replaced by the "Little Bloch" space $B_0{}^q$ which is defined in an obvious way:

$$B_0{}^q=\{f\in H(B):(1-\|z\|^2)^q|(R+n+1)f(z)|\to 0 \text{ as } \|z\|\to 1^-\}.$$

THEOREM 2. *Let* $1\leq q<2$ *and* $0<r<1$ *be fixed. Then the following statements are equivalent*:

(A0) $f\in B_0{}^q$;

(A1) $\lim_{\|z\|\to 1^-}(1-\|z\|^2)^q|Rf(z)|=0$;

(A2) $\lim_{\|z\|\to 1^-}(1-\|z\|^2)^q\|\nabla f(z)\|=0$;

(A3) $\lim_{\|z\|\to 1^-}(1-\|z\|^2)^q|\langle\nabla f,\bar{z}\rangle|=0$;

(A4) $\lim_{\|z\|\to 1^-}Q_f{}^q(z)=0$;

(B0) $\lim_{\|z\|\to 1^-}(1-\|z\|^2)^{2q-1}(\|\nabla f(z)\|^2-|\langle\nabla f(z),\bar{z}\rangle|^2)=0$;

(B1) $\lim_{\|z\|\to 1^-}(1-\|z\|^2)^{q-1}\|f\circ\phi_z-f(z)\|_{2,q}=0$;

(C) $\lim_{\|z\|\to 1^-}\frac{(1-\|z\|^2)^{2q-2}}{v({}_qB(z,r))}\int_{B(z,r)}|f(\zeta)-f_{B(z,r)}|^2 dv_q(\zeta)=0$;

(D) $\lim_{\|z\|\to 1^-}\frac{(1-\|z\|^2)^{2q-2}}{v_q(B(z,r))}\int_{B(z,r)}|f(\zeta)-f(z)|^2 dv_q(\zeta)=0.$

REMARK. The above results for the case $n=1$ and $q=1$ are due to S. Axler [2]. We used basically the same method to generalize them to several variables. The restriction $1\leq q$ is necessary only to prove the implication (B0)$\Rightarrow$(B1), because we use Lemma 2. For other case, the poor works for $0<q<2$. We thank the referee for pointing out this.

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Korea Institute of Technology
Taejon 305-701, Korea