J. Korean Math. Soc. 26(1989), No. 2, pp. 203-213

APPROXIMATE SOLUTIONS FOR THE CARLEMAN EQUATION

S. K. Chung

1. Introduction

We consider the Carleman equation which is a system of semilinear partial differential equaitons;

(1. 1) $$u_t+u_x=v^2-u^2$$
$$v_t-v_x=u^2-v^2 \qquad t>0, \quad -\infty<x<\infty$$

with initial data

(1. 2) $$u(0, x)=\phi(x)$$
$$v(0, x)=\psi(x)$$

The above system has been studied by Carleman to model the spatio-temporal behaviour of the velocity distribution function of a gas whose molecules move parallel to x-axis with constant speed. In [7], Kolodner showed that for nonnegative inital data $\phi(x)$ and $\psi(x)$ in $C^1(R)$, there exist nonnegative functions u and v in $C^1(R\times[0,\infty))$ for the problem (1. 1) and (1. 2) by using fixed point arguments and some properties of the Riccati equation Thereafter, this problem has been studied by several authors [2, 3, 5, 6, 8]

In this paper, we study the problem (1. 1) and (1. 2) by a constructive approximation scheme in the idea of [4]. In section 2, we introduce an approximaiton scheme and construct approximate solutions. In Section 3, we show that the constructed solutions are bounded. In Section 4, we prove that the sequence of approximate solutions satisfies the conditions of Arzela's theorem. Finally, we estimates the error between the exact solution and approximate solutions in Section 5.

2. An Approximation Scheme

Let $\phi(x)$ and $\psi(x)$ be continuous nonnegative functions from R to R with continuous first derivatives. Further, let $\phi(x)$, $\psi(x)$, $|\phi'(x)|$,

Received December 16, 1988.
Revised April 27, 1989.

and $|\psi'(x)|$ be bounded.

We are looking for nonnegative functions $u(t, x)$ and $v(t, x)$ in $C^1([0, \infty)\times R)$ satisfying (1.1) and (1.2).

For any fixed positive number T, let P_m be a partition of $[0, T]$ given by

$$P_m=\{t_0=0<t_1<t_2\cdots<t_m=T\}.$$

Let $\delta_i=t_{i+1}-t_i$. Then we define the functions $\phi_i(x)$ and $\psi_i(x)$ as follows;

$$(2.1)\qquad \begin{aligned}&\phi_0(x)=\phi(x),\\ &\psi_0(x)=\psi(x),\\ &\phi_{i+1}(x)=\phi_i(x-\delta_i)+\delta_i[\psi_i^2(x-\delta_i)-\phi_i^2(x-\delta_i)],\\ &\psi_{i+1}(x)=\psi_i(x+\delta_i)+\delta_i[\phi_i^2(x+\delta_i)-\psi_i^2(x+\delta_i)].\end{aligned}$$

We now consider the following approximating sequences for (1.1) and (1.2).

$$(2.2)\qquad \begin{aligned}U_m(t, x)&=\phi_i(x-(t-t_i))+(t-t_i)[\psi_i^2(x-(t-t_i))\\ &\qquad-\phi_i^2(x-(t-t_i)),\\ V_m(t, x)&=\psi_i(x+(t-t_i))+(t-t_i)[\phi_i^2(x+(t-t_i))\\ &\qquad-\psi_i^2(x+(t-t_i))]\end{aligned}$$

for $(t, x)\in[t_i, t_{i+1}]\times R$.

3. Estimates

Let P_m be a partition of $[0, T]$. We take this partition so that $\delta=\delta_i=t_{i+1}-t_i\leqq 1/(2M)$, where M is a constant such that

$$0\leqq\phi(x),\ \psi(x)\leqq M.$$

Then we have the following lemma.

LEMMA 1. *The functions U_m and V_m are nonnegative and bonuded by M.*

Proof. Suppose ϕ_i and ψ_i are nonnegative and bounded by M. Then inductively, we get

$$\begin{aligned}M-U_m(t, x)&=M-\phi_i(x-(t-t_i))+(t-t_i)[M^2-\psi_i^2(x-(t-t_i))]\\ &\qquad-(t-t_i)[M^2-\phi_i^2(x-(t-t_i))]\\ &\geqq[M-\phi_i(x-(t-t_i))][1-(t-t_i)(M+\phi_i(x-(t-t_i))]\\ &\geqq[M-\phi_i(x-(t-t_i))][1-2M(t-t_i)]\\ &\geqq 0\end{aligned}$$

for $(t, x)\in[t_i, t_{i+1}]\times R$. Hence $U_m(t, x)\leqq M$.

And we have the following inequality;

$$\begin{aligned} U_m(t, x) &\geqq \phi_i(x-(t-t_i)) - (t-t_i)\phi_i^2(x-(t-t_i)) \\ &= \phi_i(x-(t-t_i))[1-(t-t_i)\phi_i(x-(t-t_i))] \\ &\geqq 0, \end{aligned}$$

for $t \in [t_i, t_{i+1}]$. This shows that $U_m(t, x)$ is nonnegative.

Similarly we can show that $V_m(t, x)$ is nonnegative and bounded by M.

We now show that $U_m(t, x)$ and $V_m(t, x)$ increase as $\phi(x)$ and $\psi(x)$ increase.

LEMMA 2. *Let $\phi(x) \leqq \tilde{\phi}(x)$ and $\psi(x) \leqq \tilde{\psi}(x)$. Let $U_m(t, x)$, $\tilde{U}_m(t, x)$, $V_m(t, x)$ and $\tilde{V}_m(t, x)$ be the corresponding approximate solutions of (1.1) to $\phi(x)$, $\tilde{\phi}(x)$, $\psi(x)$, and $\tilde{\psi}(x)$, respectively. Then*

$$U_m(t, x) \leqq \tilde{U}_m(t, x) \quad \text{and} \quad V_m(t, x) \leqq \tilde{V}_m(t, x).$$

Proof. If we assume that $\phi_i(x) \leqq \tilde{\phi}_i(x)$ and $\psi_i(x) \leqq \tilde{\psi}_i(x)$, then from (2.2), we have the following inequalities.

$$\begin{aligned} \tilde{U}_m(t, x) - U_m(t, x) &\geqq [\tilde{\phi}_i(x-(t-t_i)) - \phi_i(x-(t-t_i))] \times \\ &\quad [1-(t-t_i)\{\tilde{\phi}_i(x-(t-t_i)) + \phi_i(x-(t-t_i))\}] \\ &\geqq [\tilde{\phi}_i(x-(t-t_i)) - \phi_i(x-(t-t_i))][1-2M(t-t_i)] \\ &\geqq 0 \end{aligned}$$

for $(t, x) \in [t_i, t_{i+1}] \times R$.

Similarly, we have $\tilde{V}_m(t, x) \geqq V_m(t, x)$.

If we assume that there is a nonnegative constant M_0 such that

$$|\phi'(x)|, \quad |\psi''(x)| \leqq M_0,$$

then we have the following lemma

LEMMA 3. *There exists a positive number M^* such that*

$$|\phi_i'(x)|, \quad |\psi_i'(x)| \leqq M^*$$

for $i=0, 1, \ldots, m$.

Proof. Let $M_i = \sup \{|\phi_i'(x)|, \ |\psi_i'(x)| : x \in R\}$ Then since

$$\begin{aligned} \phi_{i+1}'(x) = \phi_i'(x-\delta_i) + \delta_i[2\psi_i(x-\delta_i)\psi_i'(x-\delta_i) \\ -2\phi_i(x-\delta_i)\phi_i'(x-\delta_i)], \end{aligned}$$

we have

$$\begin{aligned} |\phi_{i+1}'(x)| &\leqq M_i + 4\delta_i M M_i \\ &= (1+4M\delta_i) M_i \\ &\leqq \prod_{j=0}^{i} (1+4M\delta) M_0 \end{aligned}$$

$$\leqq\left(1+4M\frac{T}{m}\right)M_0$$
$$\leqq \exp\,(4MT)\,M_0$$

If we take $M^*=\exp\,(4MT)\,M_0$, then M^* is the required one.

Similarly, we can show that $|\psi_{i+1}'(x)|\leqq M^*$. Thus we complete the proof.

LEMMA 4. *Given $r>0$, if there exists a modulus of continuity ω such that $|\phi(x)-\phi(y)|$, $|\psi(x)-\psi(y)|\leqq\omega(|x-y|)$ for $|x|, |y|\leqq r^*=r+T$, then*

$$|U_m(t,x)-U_m(t,y)|,\quad |V_m(t,x)-V_m(t,y)|\leqq K\omega(|x-y|)$$

for $0<t\leqq T$, $|x|, |y|\leqq r$, where K is a constant independent of m.

Proof. We consider the case when $t_0<t<t_1$ and $|x|, |y|\leqq r^*-\delta_0$. From (2.2), we get

$$\begin{aligned}|U_m(t,x)-U_m(t,y)|&\leqq|\phi_0(x-(t-t_0))-\phi_0(y-(t-t_0))|\\&\quad\times[1+(t-t_0)|\phi_0(x-(t-t_0))+\phi_0(y-(t-t_0))|]\\&\quad+(t-t_0)|\psi_0(x-(t-t_0))-\psi_0(y(t-t_0))|\\&\quad\times|\phi_0(x-(t-t_0))+\phi_0(y-(t-t_0))|\\&\leqq\omega(|x-y)|)(1+2\delta_0M)+2\delta_0M\omega(|x-y|)\\&=(1+4M\delta_0)\,\omega(|x-y|)\end{aligned}$$

for $|x-(t-t_0)|$, $|y-(t-t_0)|\leqq r^*-\delta+(t-t_0)\leqq r^*$

In general, if $t_i<t\leqq t_{i+1}$ and $|x|, |y|\leqq r^*-(\delta_0+\ldots+\delta_i)$, $i=0, 1, \ldots, m-1$, then

$$\begin{aligned}|U_m(t,x)-U_m(t,y)|&\leqq\prod_{j=0}^{i}(1+4M\delta_j)\,\omega(|x-y|)\\&\leqq\left(1+\frac{4M}{m}T\right)^m\omega(|x-y|)\\&\leqq\exp\,(4MT)\,\omega(|x-y|).\end{aligned}$$

Similarly, we can show that $|V_m(t,x)-V_m(t,y)|\leqq\exp\,(4MT)\times\omega(|x-y|)$.

Hence by taking $K=\exp(4MT)$ we complete the proof.

LEMMA 5. *For any $A>0$, there exists a modulus of continuity function Ω such that*

$$|\phi_i'(x)-\phi_i'(y)|,\quad |\psi_i'(x)-\psi_i'(y)|\leqq\Omega(|x-y|)$$

for $|x|$, $|y|\leqq A$.

Proof. Since $\phi(x)$ and $\psi(x)$ have bounded continuous first derivatives, there exists a modulus of continuity function, say Ω_0, such that

$$|\phi_0'(x)-\phi_0'(y)|,\quad |\psi_0'(x)-\psi_0'(y)|\leqq \Omega_0(|x-y|)$$

for $|x|, |y|\leqq A^*=A+T$. Inductively we assume that

$$|\phi_i'(x)-\phi_i'(y)|,\quad |\psi_i'(x)-\psi_i'(y)|\leqq \Omega_i(|x-y)).$$

From (2.1), by taking derivatives with respect to x, we get

$$\begin{aligned}\phi_{i+1}'(x)-\phi_{i+1}'(y)=&\phi_i'(x-\delta_i)-\phi_i'(y-\delta_i)+2\delta_i\phi_i(x-\delta_i)\times\\ &[\psi_i'(x-\delta_i)-\psi_i'(y-\delta_i)]\\ &+2\delta_i\psi_i'(y-\delta_i)[\psi_i(x-\delta_i)-\psi_i(y-\delta_i)]\\ &-2\delta_i\phi_i(x-\delta_i)[\phi_i'(x-\delta_i)-\phi_i'(y-\delta_i)]\\ &-2\delta_i\phi_i(y-\delta_i)[\phi_i(x-\delta_i)-\phi_i(y-\delta_i)].\end{aligned}$$

Hence, for $i=0, 1, \ldots, m-1$,

$$\begin{aligned}|\phi_{i+1}'(x)-\phi_{i+1}'(y)|\leqq&(1-2\delta_i\phi_i(x-\delta_i))|\phi_i'(x-\delta_i)-\phi_i'(y-\delta_i)|\\ &+2\delta_i|\phi_i(x-\delta_i)||\psi_i'(x-\delta_i)-\psi_i'(y-\delta_i)|\\ &+2\delta_i|\psi_i'(y-\delta_i)||\psi_i(x-\delta_i)-\psi_i(y-\delta_i)|\\ &+2\delta_i|\phi_i'(y-\delta_i)||\phi_i(x-\delta_i)-\phi_i(y-\delta_i)|\\ \leqq&(1-2\delta_i\phi_i(x-\delta_i))\Omega_i(|x-y)|+2\delta_iM\Omega_i(|x-y|)\\ &+2\delta_iM^*\exp(4MT)\omega(|x-y|)\\ &+2\delta_iM^*\exp(4MT)\omega(|x-y|)\\ \leqq&(1+2\delta_iM)\Omega_i(|x-y|)\\ &+4\delta_iM^*\exp(4MT)\omega(|x-y|)\\ \leqq&\prod_{j=0}^{i}(1+2\delta_iM)\Omega_0(|x-y|)\\ &+4M^*\exp(4MT)\omega(|x-y|)\\ &\times[\delta_i+\delta_{i-1}(1+2\delta_{i-1}M)+\ldots+\prod_{j=0}^{i}(1+2\delta_jM)\delta_0]\\ \leqq&\prod_{j=0}^{i}(1+2\delta_jM)\Omega_0(|x-y|)\\ &+4M^*\exp(4MT)\omega(|x-y)\\ &\times\left[\frac{T}{m}+\frac{T}{m}\left(1+2\frac{T}{m}M\right)+\ldots+\frac{T}{m}\prod_{j=0}^{i}\left(1+2\frac{T}{m}M\right)\right]\\ \leqq&\exp(2MT)\Omega_0(|x-y|)\\ &+2\frac{M^*}{M}\exp(6MT)\omega(|x-y|)\end{aligned}$$

The same result can be obtained for $|\psi_i'(x)-\psi_i'(y)|$.

Therefore, by taking $\Omega(|x-y|)=\exp(2MT)\Omega_0(|x-y|)$

$$+2\frac{M^*}{M}\exp(6MT)\omega(|x-y|)$$

we complete the proof.

LEMMA 6. *For any* $B>0$, *there exists a function* $\omega^*: (0, \infty)\to R$ *with* $\lim_{\xi\to 0}\omega^*(\xi)=0$ *such that*

$$|\phi_{i+h}'(x)-\phi_i'(x)|,\quad |\psi_{i+h}'(x)-\psi_i'(x)|\leqq\omega^*(\Delta_{i,i+h})$$

for $1\leqq i\leqq m$, $h\geqq 1$, *and* $|x|\leqq B$, *where* $\Delta_{i,i+h}=\delta_i+\delta_{i+1}+\ldots+\delta_{i+h-1}$.

Proof. Since $\phi_{i+1}'(x)=\phi'(x-\delta_i)+2\delta_i[\psi_i(x-\delta_i)\psi_i'(x-\delta_i)$
$-\phi_i(x-\delta_i)\phi_i'(x-\delta_i)]$,
we obtain the following inequalities.

$$\begin{aligned}|\phi_{i+h}'(x)-\phi_i'(x-\delta_i)| &\leqq 2\delta_i(|\psi_i(x-\delta_i)\psi_i'(x-\delta_i)| \\ &\quad+|\phi_i(x-\delta_i)\phi_i'(x-\delta_i)|) \\ &\leqq 4MM^*\delta_i\end{aligned}$$

Inductively we obtain

$$|\phi_{i+h}'(x)-\phi_i'(x-\Delta_{i,i+h})|\leqq 4MM^*\Delta_{i,i+h}$$

and

$$\begin{aligned}|\phi_{i+h}'(x)-\phi_i'(x)| &\leqq \phi_{i+h}'(x)-\phi_i'(x-\Delta_{i,i+h})| \\ &\quad+|\phi_i'(x-\Delta_{i,i+h})-\phi_i'(x)| \\ &\leqq 4MM^*\Delta_{i,i+h}+\Omega(\Delta_{i,i+h}).\end{aligned}$$

We can obtain the same inequality for $|\psi_{i+h}'(x)-\psi_i'(x)|$. Thus, by taking $\omega^*(\xi)=4MM^*\xi+\Omega(\xi)$, we complte the proof.

4. Convergence of Approximate Solutions

Let $\{U_m\}$ and $\{V_m\}$ be approximate solutions for (1.1) and (1.2) corresponding to a partition P_m. We will show that the approximate solutions $\{U_m\}$ and $\{V_m\}$ converge to solutions u and v, respectively. First we state the Arzela's theorem due to [1].

THEOREM (*Arzela*). *Let* $\{f_n\}$ *be a sequence of real-valued functions defined on a compact subset* K *of a separable metric space such that*

(*a*) *The sequence* $\{f_n\}$ *is uniformly bounded on* K

(*b*) *Given any* $\varepsilon>0$, *there exists* $N>0$, $\delta>0$ *such that whenever* $n>N$, $|x_1-x_2|<\delta$, $x_1, x_2\in K$, *then* $|f_n(x_1)-f_n(x_2)|<\varepsilon$.

Then there exists a subsequence $\{f_{n_k}\}$ *of* $\{f_n\}$ *which converges uniformly to a continuous function* f *defined on* K.

Applying the Arzela's theorem to $\{U_m\}$ and $\{V_m\}$, we can now find subsequences $\{U_m\}$ and $\{V_m\}$ which converge uniformly on bounded subset of $[0, T]\times R$.

LEMMA 7. *The sequences* $\{U_m(t, x)\}$ *and* $\{V_m(t, x)\}$ *satisfy the conditions of Arzela's theorem.*

Proof. The condition (a) is satisfied by Lemma 1. We now show that the condition (b) is satisfied. We only consider the sequence $\{U_m(t, x)\}$ since the other one is quite similar. Denote a partition P_m as $P_m=\{t_0{}^m=0<t_1{}^m<\ldots<t_m{}^m=T\}$.

For $t, t^*\in[t_i{}^m, t_{i+1}{}^m]$ *and* $x, x^*\in R$, we clearly have

$$|U_m(t, x)-U_m(t^*, x^*)|\leqq N(|t-t^*|+|x-x^*|)$$

from the Lipschitz nature of $\phi_i(x)$.

If $t, t^*\in[0, T]$ do not belong to the same strip, say $t_{i-1}{}^m<t\leqq t_i{}^m\leqq t_j{}^m<t^*\leqq t_{j+1}{}^m$, then we have the following inequalities;

$$\begin{aligned}|U_m(t, x)-U_m(t^*, x^*)| &\leqq U_m(t, x)-U_m(t_i{}^m, x)|\\ &\quad+\sum_{s=i}^{j-1}|U_m(t_{s+1}, x)-U_m(t_s, x)|\\ &\quad+|U_m(t_j{}^m, x)-U_m(t^*, x^*)|\\ &\leqq N|t-t_i{}^m|+N\sum_{s=i}^{j-1}|t_{s+1}{}^m-t_s{}^m|\\ &\quad+N|x-x^*|+N|t_j{}^m-t^*|\\ &\leqq N(|t-t^*|+|x-x^*|)\end{aligned}$$

Hence the sequence $U_m(t, x)$ satisfies the conditions of Arzela's theorem.

We now define functions $\dfrac{\partial U_m(t, x)}{\partial x}$ as follows;

$$\text{(4. 1)}\qquad \frac{\partial U_m(t, x)}{\partial x}=\begin{cases}\phi_i'(x-(t-t_i{}^m))+2(t-t_i{}^m)[\psi_i(x-(t-t_i{}^m))\times\\ \quad\psi_i'(x-(t-t_i{}^m))-\phi_i(x-(t-t_i{}^m))\phi_i'(x-\\ \quad(t-t_i{}^m))],\quad t_i{}^m<t\leqq t_{i+1}{}^m,\\ \phi_i'(x),\quad t=t_i{}^m.\end{cases}$$

The funcitons $\dfrac{\partial V_m(t, x)}{\partial x}$ can be defined similarly. Then we have the following lemma.

LEMMA 8. *The sequences* $\left\{\dfrac{\partial U_m(t, x)}{\partial x}\right\}$ *and* $\left\{\dfrac{\partial V_m(t, x)}{\partial x}\right\}$ *satisfy the conditions of Arzela's theorem.*

Proof. we only consider one, since the other one can be proved similarly. By Lemma 4, the condition (a) is obviously satisfied. We now show that the condition (b) is satisfied.

For $t, t^*\in[t_i{}^m, t_{i+1}{}^m]$, we have

$$\left|\frac{\partial U_m(t,x)}{\partial x}-\frac{\partial U_m(t^*,x^*)}{\partial x}\right|\leqq\left|\frac{\partial U_m(t,x)}{\partial x}-\frac{\partial U_m(t^*,t)}{\partial x}\right|$$
$$+\left|\frac{\partial U_m(t^*,x)}{\partial x}-\frac{\partial U_m(t^*,x^*)}{\partial x}\right|$$
$$\leqq\Omega(|t-t^*|+|x-x^*|).$$

For $t_{i-1}{}^m<t\leqq t_i{}^m\leqq t_j{}^m\leqq t^*\leqq t_{j+1}{}^m$, we have

$$\left|\frac{\partial U_m(t,x)}{\partial x}-\frac{\partial U_m(t^*,x^*)}{\partial x}\right|\leqq\left|\frac{|\partial U_m(t,x)}{\partial x}-\frac{\partial U_m(t^*,x)}{\partial x}\right|$$
$$+\left|\frac{\partial U_m(t_i{}^m,x)}{\partial x}-\frac{\partial U_m(t_j{}^m,x)}{\varrho x}\right|$$
$$+\left|\frac{\partial U_m(t_j{}^m,x)}{\partial x}-\frac{\partial U_m(t^*,x^*)}{\partial x}\right|$$
$$\leqq\Omega(|t-t_i{}^m|)+\omega^*(\Delta_{i,j})$$
$$+\Omega(|t_j{}^m-t^*|+|x-x^*|).$$

This complete the proof.

The above lemma says that there exists a subsequence of $\{U_m(t,x)\}$ such that $\left\{\frac{\partial U_m(t,x)}{\partial x}\right\}$ converges uniformly on $[-r,r]$ and the limit funciton $u(t,x)$ has a continuous partial derivative with respect to x.

Now we let $P_m(t,x)$ be defined as follows;

(4. 2) $$P_m(t,x)=-\phi_i'(x-(t-t_i{}^m))$$
$$+[\phi_i^2(x-(t-t_i{}^m))-\phi_i^2(x-(t-t_i{}^m))]$$
$$+(t-t_i{}^m)[-2\phi_i(x-(t-t_i{}^m))\phi_i'(x-(t-t_i{}^m))$$
$$+2\phi_i(x-(t-t_i{}^m))\phi_i'(x-(t-t_i{}^m))]$$

for $t\in[t_i{}^m,t_{i+1}{}^m]$.

Then by Lemma 1 and Lemma 3, $P_m(t,x)$ is uniformly bounded. And from (4. 1) we have

$$P_m(t,x)+\frac{\partial U_m(t,x)}{\partial x}=\phi_i^2(x+(t-t_i{}^m))-\phi_i^2(x-(t-t_i{}^m)).$$

Furthermore,

$$P_m(t,x)+\frac{\partial U_m(t,x)}{\partial x}-V_m{}^2(t,x)+U_m{}^2(t,x)$$
$$=\phi_i^2(x+(t-t_i{}^m))-\phi_i^2(x-(t-t_i{}^m))$$
$$-[\phi_i(x+(t-t_i{}^m))+(t-t_i{}^m)\{\phi_i^2(x+(t-t_i{}^m))-\phi_i^2(x+(t-t_i{}^m))\}]^2$$
$$+[\phi_i(x-(t-t_i{}^m))+(t-t_i{}^m)\{\phi_i^2(x-(t-t_i{}^m))-\phi_i^2(x-(t-t_i{}^m))\}]^2$$
$$\to 0.$$

Noting $|\phi_i(x)|$, $|\phi_i'(x)|$, $|\phi_i(x)|$, and $|\phi_i'(x)|$ are uniformly bounded and $|t-t|\leqq\frac{T}{m}$, this convergence is uniform. Thus $P_m(t, x)$ converges to a continuous function. It follows that the limit functions U and V (proceeding similarly) satisfy (1. 1) in $[0, T]\times[-r, r]$.

We are now ready to state the main theorem as follows.

Main Theorem. For nonnegative bounded continuous funcitons $\phi(x)$ and $\psi(x)$ with bounded continuous first derivatives, there exist nonnegative funcitons $u(t, x)$ and $v(t, x)$ satisfying (1. 1). Furthermore, $u(t, x)$ and $v(t, x)$ can be obtained as limits of sequences $U_m(t, x)$ and $V_m(t, x)$ constructed by (2. 1) and (2. 2), respectively

5. Error Estimates

Let $u(t, x)$ and $v(t, x)$ be the exact solutions to (1. 1) and (1. 2), and $U_m(t, x)$ and $V_m(t, x)$ be the approximate solutions corresponding to a partition P_m. Let $a_m=u-U_m$ and $b_m=v-V_m$. Then we have the following relations.

$$(5.1)\qquad \begin{aligned}\frac{\partial a_m}{\partial t}+\frac{\partial a_m}{\partial x}&=v^2-u^2-[\psi_i^2(x-(t-t_i^m))-\phi_i^2(x-(t-t_i^m))],\\ \frac{\partial b_m}{\partial t}-\frac{\partial b_m}{\partial x}&=u^2-v^2-[\phi_i^2(x-(t-t_i^m))-\psi_i^2(x+(t-t_i^m))].\end{aligned}$$

We let

$$(5.2)\qquad \begin{aligned}\phi_i(x-(t-t_i^m))&=U_m(t, x)+R_i(t, x),\\ \psi_i(x+(t-t_i^m))&=V_m(t, x)+S_i(t, x).\end{aligned}$$

Then

$$\begin{aligned}|R_i(t, x)|&=|t-t_i^m|\,|\psi_i^2(x-(t-t_i^m))-\phi_i^2(x-(t-t_i^m))|\\ &\leqq 2M^2|t-t_i^m|\\ &\leqq 2M^2\|P_m\|,\end{aligned}$$

where $\|P_m\|=\min\{|t_{i+1}^m-t_i^m| : i=0, 1, \dots, m-1\}$.

Similarly, we have

$$|S_i(t, x)|\leqq 2M^2\|P_m\|.$$

From (5. 1) and (5. 2), we have the following system of partial differential equaitons

$$(5.3)\qquad \begin{aligned}\frac{\partial a_m}{\delta t}+\frac{\partial a_m}{\partial x}&=\alpha_{11}a_m+\alpha_{12}b_m+\rho_i,\\ \frac{\partial b_m}{\partial t}-\frac{\partial b_m}{\partial x}&=\alpha_{21}a_m+\alpha_{22}b_m+\rho_i^*,\end{aligned}$$

with

$$a_m(0, x) = b_m(0, x) = 0, \tag{5.4}$$

where

$$\alpha_{11} = -(u + U_m),$$
$$\alpha_{12} = v + V_m,$$
$$\alpha_{21} = -\alpha_{11},$$
$$\alpha_{22} = -\alpha_{12},$$
$$\rho_i = 2R_i U_m + R_i^2 - 2S_i V_m - S_i^2,$$
$$\rho_i^* = -\rho.$$

We observe that $|\rho_i|, |\rho_i^*| \leqq 8M^2(1+M\|P_m\|)\|P_m\|$ and $|\alpha_{i,j}| \leqq 2M$, $i, j = 1, 2$. Let $a = 2M$ and $b = 8M^2(1+M\|P_m\|)\|P_m\|$. Further, if we let $E_i = \max\{|a_m(t_i, x)|, |b_m(t_i, x)|\}$, then $E_0 = 0$. Now by an application of Haar's lemma [9] we get

$$|a_m(t, x)|, \quad |b_m(t, x)| \leqq E_i \exp(2\delta_i) + \frac{b}{2a}\{\exp(2a\delta_i) - 1\}$$

for $t \in [t_i^m, t_{i+1}^m]$. That is,

$$E_{i+1} \leqq E_i \exp(2a\delta_i) + \frac{b}{2a}\{\exp(2a\delta_i) - 1\}.$$

This last inequality gives

$$E_{i+1} \leqq \frac{b}{2a}\{\exp(2aT) - 1\} \text{ for } i = 0, 1, \dots, m-1.$$

References

1. J. Conlan, *The Cauchy Problem and the Mixed Boundary Value Problem for a Non-linear Hyperbolic Partial Differential Equation in Two Independent Variables*, Arch. Rat. Mech. Anal., **3**(1959), 355–380.
2. W. E. Fitzgibbon, *The Fluid Dynamical Limit of the Carleman Equation with Reflecting Boundary*, Nonlinear Analysis, T. M. A., **6**(1982), 695–702.
3. R. Illner and M. C. Reed, *The Decay of Solutions of the Carleman Model*, Math Mech. in the Appl. Sci., **3**(1981), 121–127.
4. R. Kannan and R. Ortega, *Approximate Solutions for the Cauchy Problem for a Semilinear Hyperbolic System*, J. Math. Kyoto Univ., **25**(1985), 393–403.
5. H. G. Kaper and G. K. Leaf, *Initial Value Problems for the Carleman Equation*, Nonlinear Analysis, T. M. A., **4**(1980), 343–362.
6. H. G. Kaper, G. K. Leaf, and S. Reich, *Convergence of Semigroups with an Application to the Carleman Equation*, Math. Mech. in the Appl. Sci, **2**(1980), 303–308.

7. I Kolodner, *On the Carleman's model for the Boltzmann equation and its generalizations,* Annali Mat. pura Appl., Series 4 **63**(1963) 11-32.
8. R. Temam, *Sur la résolution exacte et approchée d'un problème hyperbolique non linèaire de T. Carleman,* Arch. Rat. Mech. Anal., **35**(1969), 351-362.
9. M. Yamaguti and S. N. Niizeki, *On the Cauchy Problem for a semilinear hyperbolic system,* J. Math Kyoto Univ., **20**(1980), 625-634.

Seoul National University
Seoul 151-742, Korea