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HYPOELLIPTICITY OF A COMPLEX VECTOR FIELDS SYSTEM

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0. Introduction

In this work, we are concerned with the hypoellipticity of a system $\boldsymbol{L}$ of m linearly independent complex vector fields $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_m$ defined in an open subset Ω of R^{m+1}. We assume that $\boldsymbol{L}$ is a locally integrable Frobenius Lie algebra over $C^\infty(\Omega)$ (cf. (1. 2) and (1. 3)).

Using the simple fact that hypoellipticity of $Q=\sum_{j=1}^{m} L_j{}^*L_j$ implies that of $\boldsymbol{L}$ (cf. Lemma 2. 2), we get sufficient conditions for $\boldsymbol{L}$ to be hypoelliptic.

When L_j are analytic vector fields, M. S. Baouendi and F. Treves ([1]) found a necessary and sufficient condition for $\boldsymbol{L}$ to be analytic hypoelliptic, which is necessary for $\boldsymbol{L}$ to be hypoelliptic. However, it is not known, except when $m=1$, whether analytic hypoellipticity of $\boldsymbol{L}$ implies its hypoellipticity. In Section 4, we shall give a positive result to this question in some special case.

1. Preliminaries

Let $\boldsymbol{L}$ be a locally integrable m-dimensional Frobenius Lie-algebra of smooth ($C^\infty-$) complex vector fields on an open subset Ω of R^{m+1}. In other words, $\boldsymbol{L}$ is a set of smooth complex vector fields in Ω satisfying:

(1. 1) every point of Ω has an open neighborhood in which L is spanned by m linearly independent elements $L_1, \ldots, L_m$ of $\boldsymbol{L}$;

(1. 2) $\boldsymbol{L}$ is closed under the commutation bracket $[A, B]=AB-BA$;

(1. 3) $\boldsymbol{L}$ is locally integrable, that is, every point in Ω has an open

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neighborhood in which there is a smooth function Z satisfying $\boldsymbol{L}Z=0$ and $dZ\neq 0$ (here, $\boldsymbol{L}Z=0$ means that $LZ=0$ for all L in $\boldsymbol{L}$).

Note that in the real case, (1.3) comes from (1.2) by Frobenius theorem, which need not be true in the complex case (cf. [4, 7, 8]).

By shrinking Ω about a point, which we may take as the origin in R^{m+1}, we may choose a local coordinates $(t, x)=(t_1, \dots, t_m, x)$ in Ω for which we have (cf. [1, 2])

$$L_j=\frac{\partial}{\partial t_j}+\lambda_j(t, x)\frac{\partial}{\partial x}, \quad 1\leq j\leq m \tag{1.4}$$

$$Z(t, x)=x+i\phi(t, x), \tag{1.5}$$

where $\phi(t, x)$ is a real-valued C^∞-function on Ω with $\phi(0, 0)=0$.

With L_j as in (1.4), (1.2) means, that

$$[L_j, L_k]=0, \quad 1\leq j, k\leq m. \tag{1.6}$$

From (1.4) and (1.5), $L_jZ=0, 1\leq j\leq m$, gives

$$\lambda_j(t, x)=-i\left(1+i\frac{\partial\phi}{\partial x}\right)^{-1}\frac{\partial\phi}{\partial t_j}, \quad 1\leq j\leq m. \tag{1.7}$$

Let Σ be the characteristic set of $\boldsymbol{L}$, which is the intersection of characteristic set of all elements of $\boldsymbol{L}$. Since $\{L_j\}_1^m$ generates $\boldsymbol{L}$, we have from (1.4) and (1.5)

$$\Sigma=\{(t, x, \tau, \xi)\in\Omega\times R^{m+1}\setminus\{0\}\mid\tau_j=0, \ \frac{\partial\phi}{\partial t_j}(t, x)=0, 1\leq j\leq m\}. \tag{1.8}$$

Let C be the base projection of Σ;

$$C=\{(t, x)\in\Omega\mid\frac{\partial\phi}{\partial t_j}(t, x)=0, 1\leq j\leq m\}. \tag{1.9}$$

To avoid triviality, we assume that C is neither ϕ nor Ω.

2. Statements of the main results

Let $\boldsymbol{L}$ be the same as in Section 1.

Definition 2.1. $\boldsymbol{L}$ is hypoelliptic in Ω if for any open set U in Ω, any distribution u in U,

(2.1) $Lu\in C^\infty(U)$ for all L in $\boldsymbol{L}$ implies that $u\in C^\infty(U)$.

We need the following rather trivial fact.

Lemma 2.1. *Let $L_1, \dots, L_m$ be any set of (local) generators of $\boldsymbol{L}$. Then $\boldsymbol{L}$ is hypoelliptic if and only if for any open set U in Ω, any distribution u in U*

(2. 2) $\qquad L_j u \in C^\infty(U),\ 1 \leq j \leq m,$ *implies* $u \in C^\infty(U)$.

Therefore, from now on, we restrict our attention to the particular set of generators $L_j, 1 \leq j \leq m,$ of $\boldsymbol{L}$ given by (1. 4) and a C^∞-f unction $Z(t, x)$ given by (1. 5).

Let us set $H_t\phi = \left(\frac{\partial^2 \phi}{\partial t_j \partial t_k}\right)$to be the Hessian of ϕ as a function of t and $tr\ H_t\phi$ to be its trace. Note that the set C giveni by (1. 9) is just the set of critical points of ϕ as a function of t.

THEOREM 2. 1. *$\boldsymbol{L}$ is hypoelliptic if $tr\ H_t\phi \equiv 0$ on C and if*
(2. 3) $\qquad H_t\phi \neq 0$ *on* C.

COROLLARY 2. 1. *$\boldsymbol{L}$ is hypoelliptic if $tr\ H_t\phi \equiv 0$ on C and if*
(2. 4) $\qquad H_t\phi$ *is non-singular at any point of* C.

In passing, note that the condition (2. 4) implies that C (and so Σ) is a smooth submanifold of Ω (resp. of $T^*\Omega$) since $d\left(\frac{\partial \phi}{\partial t_j}\right),\ 1 \leq j \leq m,$ are linearly independent on C. Moreover, then, Σ is not involutive with respect to the cannonical symplectic structure on $T^*\Omega$ (cf. Lemma 3. 2). If the coefficients $\lambda_j(t, x)$ of $L_j,\ 1 \leq j \leq m,$ are independent of the variable x, (equivalently, if the function $\phi(t, x) = Im\ Z$ is independent of x), then, the commutation relations (1. 6) reads as

(2. 5) $$\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial \lambda_k}{\partial t_j},\ 1 \leq j, k \leq m.$$

Therefore, by Poincare lemma, in some open neighborhood of $t=0$ in R^m, there is a smooth function $\lambda(t)$ satisfying

(2. 6) $$\frac{\partial \lambda}{\partial t_j} = \lambda_j(t),\ 1 \leq j \leq m.$$

On the other hand, from (1. 7), we have, modulo some additive constants,

(2. 7) $$\phi(t) = i\lambda(t).$$

We note that, in this case, the condition (2. 4) reads as: ϕ(t) is a Morse function, that is, any critical point of ϕ is non-degenerate.

THEOREM 2. 2. *Assume that $Z = x + i\phi(t)$ and that $\phi(t)$ is a Morse function. Then, $\boldsymbol{L}$ is hypoelliptic if the index of any critical point of ϕ, which is the number of negative eigenvalues of $H_t\phi$ at that point, is neither* 0 *nor* m.

Let us consider the Laplacian Q associated with L_j's;

(2.8) $$Q=Q(t, x, D_t, D_x)=\sum_{j=1}^{m} L_j^* L_j,$$

where L_j^* is the formal adjoint of L_j with respect to L^2-inner product.

LEMMA 2.2. *If Q is hypoelliptic, then so is $\boldsymbol{L}$.*

Proof. Let u be a distribution in an open subset U of Ω, satisfying $L_j u \in C^\infty(U)$, $1 \leq j \leq m$. Then, Qu and so u must be in $C^\infty(U)$ since Q is hypoelliptic. By Lemma 2.1, $\boldsymbol{L}$ is also hypoelliptic.

3. Proof of Theorems 2.1 and 2.2.

In this section we shall prove the following theorem, which implies Theorems 2.1 and 2.2, by Lemma 2.2.

THEOREM 3.1. *The operator Q (given by (2.8)) is hypoelliptic under the same hypotheses as those in Theorem 2.1 or in Theorem 2.2.*

Let $q=q(t, x, \tau, \xi)$ be the principal symbol of Q:

(3.1) $$q=\sum_{j=1}^{m} |\tau_j+\lambda_j \xi|^2.$$

Hence, q is everywhere non-negative and vanishes exactly of order two on char$Q$, which is equal to Σ. On the other hand, the subprincipal symbol of Q, which is defined intrinsically at least on Σ, is given by

(3.2) $$\sigma_{sub}(Q)=-\xi\left[1+\left(\frac{\partial\phi}{\partial x}\right)^2\right]^{-1} \sum_{j=1}^{m} \frac{\partial^2\phi}{\partial t_j^2} \quad on\ \Sigma,$$

where ξ is any non-zero real number.

At any point ρ in Σ, let $E=E(\rho)$ be the quadratic form which begins the Taylor series expansion of q at ρ, which can be intrinsically defined since q vanishes of order 2 on Σ. Also, let $F=F(\rho)$ be the fundamental matrix associated with E, defined by

(3.3) $$E(u, v)=\sigma(u, Fv), \ u, v \in T_\rho(T^*\Omega),$$

where $\sigma=\sum_{j=1}^{m} d\tau_j \wedge dt_j + d\xi \wedge dx$ is the standard symplectic 2-form on $T^*\Omega$. We need the following fact for the endomorphism F of $T_\rho(T^*\Omega)$.

LEMMA 3.1. ([3, 5]) *We have*

(3.4) *F is nilpotent if and only if $F^2=0$.*

(3.5) *When Σ is a submanifold of $T^*\Omega$, it is involutive if and only if F is nilpotent at every point of Σ.*

If we write E, in the block forms, as $E=\begin{pmatrix} E_1 & E_2 \\ E_2{}^t & E_3 \end{pmatrix}$, where E_j, $1\leq j\leq 3$, are $(m+1)\times(m+1)$ matrices (e.g. entries of E_1 are mixed second partial derivatives of q with respect to the base variable (t, x)), then $F=\begin{pmatrix} E_2{}^t & E_3 \\ -E_1 & -E_2 \end{pmatrix}$. We can then see that $F^2\neq 0$ if and only if $E_2{}^2-E_1E_3\neq 0$, that is, either $(H_t\phi)^2\neq 0$ or $\sum_{k=1}^{m} \frac{\partial^2\phi}{\partial t_j\partial t_k} \frac{\partial^2\phi}{\partial t_k\partial x}\neq 0$ for some $j=1, \ldots, m$, on C which is equivalent to $H_t\phi\neq 0$ on C, since $H_t\phi$ is symmetric. Hence by Lemma 3.1 we have

LEMMA 3.2. *F is not nilpotent at any point of Σ if and only if the condition* (2.3) *holds. Also, under the condition* (2.4), *the submanifold Σ is not involutive.*

Now, the first part of the theorem 3.1 follows immediately from the following fact, due to L. Hörmander [3], which we simplify a little for our case.

PROPOSITION 3.2. ([3]) *Let P be a partial differential operator with p as its principal symbol. Suppose that p is non-negative everywhere and vanishes exactly of order 2 on its characteristic set. Then, P is hypoelliptic if $\sigma_{sub}(P)$ vanishes on the characteristic set and if the fundamental matrix F corresponding to $\frac{1}{2}Hp$ (Hp is the Hessian matrix of p) is not nilpotent at any characteristic point.*

For the second part of Theorem 3.1, we need;

PROPOSITION 3.3. ([5]) *Let P be the same as in proposition 3.2. Assume that its characteristic set Σ is a smooth submanifold of $T^*\Omega$, which is not involutive. Then, P is hypoelliptic if*

(3.6) $$Re\ \sigma_{sub}(P)+tr^+F>0 \text{ on } \Sigma.$$

Here, tr^+F is the sum of positive eigenvalues of $-iF$.

Proof of the second part of Theorem 3.1.: When $\phi(t)$ is a Morse function, its critical points are isolated so that we may concentrate on some open set which contains only one critical point of ϕ, say,

t^o. In a possibly smaller open neighborhood of t^o. by Morse lemma, we may write $\phi(t)$ as:

(3.7) $$\phi(t)=\phi(t^o)-t_1^2-\cdots-t_k^2+t_{k+1}^2+\cdots+t_m^2,$$

so that index of ϕ at $t=0$ is k, $0\leq k\leq m$.
In this case, at any characteristic point $(0,\cdots,0,x,0,\cdots,0,\xi)$, $\xi\neq 0$, F takes the form

(3.8) $$F=\begin{pmatrix} 0 & E_3 \\ -E_1 & 0 \end{pmatrix},\quad E_3=\begin{pmatrix} & I_m & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \text{ and } E_1=-\xi^2E_3$$

where I_m is the identity $m\times m$ matrix.
Hence, the matrix $-iF$ has only one positive eigenvalue $|\xi|$ of multiplicity m, so that the left hand side of the equation (3.6) reads as: $-\xi(m-2k)+m|\xi|$, which is positive unless k is either 0 or m.

4. Analytic hypoellipticity

When $\boldsymbol{L}$ is composed of (real–) analytic vector fields, analytic hypoellipticity of $\boldsymbol{L}$ is completely characterized in [1], by using the so–called "local constancy principle". Note that in this case, we do not need the assumption (1.3) since it is a consequence of (1.2) by the complex version of Frobenius theorem. To be precise, they have proved the following.

PROPOSITION 4.1. *Let* $\boldsymbol{L}$ *be a Frobenius Lie algebra of analytic complex vector fields of dimension m in* Ω. *If* $\boldsymbol{L}$ *is* (C^∞–) *hypoelliptic in* Ω, *then it is analytic hypoelliptic in* Ω *and the latter is equivalent to the fact that the map Z, which is now analytic, is open from* Ω *to C.*

Combining Proposition 4.1 and Theorem 2.1, we can get

THEOREM 4.1. *Let* $\boldsymbol{L}$ *be as in proposition* 4.1. *Then* $\boldsymbol{L}$ *is both* (C^∞–) *hypoelliptic and analytic hypoelliptic if tr* $H_t\phi\equiv 0$ *on C and if the condition* (2.3) (*in particular,* (2.4)) *holds.*

It was also pointed out in [1] that if $\boldsymbol{L}$ is generated by a single analytic complex vector field (i.e., $m=1$), then, due to the result in [6], the converse of proposition 4.1 also holds, since any nowhere vanishing vector field is fo principal type. For $m>1$, it is not known whether analytic hypoellipticity of $\boldsymbol{L}$ implies its hypoellipticity. However, when the function $\phi(t, x)$ is independent of the variable x, we can get the following partial positive result by combining Proposition 4.1 and Theorem 2.2.

THEOREM 4.2. *Suppose that $Z=x+i\phi(t)$ and that $\phi(t)$ is an analytic Morse function. Then, the following four statements are all equivalent.*

(4.1) *L is hypoelliptic;*

(4.2) *L is analytic hypoelliptic;*

(4.3) *the map Z is an open map from Ω to C;*

(4.4) *the index of any critical point of ϕ is neither 0 nor m.*

Proof. It only remains to show that (4.3) implies (4.4), which follows immediately from Morse lemma.

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