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TOPOLOGICAL AND ANALYTIC CLASSIFICATION OF PLANE CURVE SINGULARITIES DEFINED BY $z^n+a(y)z+b(y)$ WITH MUITIPLICITY n AND ITS APPLICATION

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0. Introduction

Let $V=\{(y,z) : f(y,z)=0\}$ be an analytic subvariety of a polydisc in C^2 with $(0,0)\in V$ and the only singular point where f is holomorphic near $(0,0)$ and square-free. Assume that f begins with terms of degree n : $f=f_n(y,z)+$terms of degree$>n$ where f_n is a homogeneous polynomial of degree n. Then we call n the multiplicity of f. If $n=2$, then we see easily that f is topologically and at the same time analytically equivalent to z^2+y^k. If $n\geq 3$, then let f be of the form : $f=z^n+a(y)y^\alpha z+b(y)y^\beta$ where α, β are positive integers; $a(y)$, $b(y)$ are nowhere vanishing holomorphic functions near $y=0$; n is the multiplicity of f. Then we will explicitly classify the f topologically, and analytically in section 2 except that $f=z^n+y^{n-1}z+ky^n$ is homogeneous, square-free and $n\geq 4$. As a result, we can completely classify plane curve singularities with multiplicity three topologically and analytically. Next, particularly let $f_t=z^3+y^4z+ty^6$ for any complex number t where each f_t is square-free. Let $\mu_t=\dim O/\Delta(f_t)$, $\nu_t=\dim O/(f_t, \Delta(f_t))$ and $\omega_t=\dim O/M(f_t)$ as C-vector spaces where O is the ring of germs of holomorphic functions near the origin in C^2 ; $\Delta(f)$ is the ideal in O generated by $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$; $(f, \Delta(f))$ is the ideal in O generated by f, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$; $M(f)$ is the ideal O generated by f, $y\frac{\partial f}{\partial y}$, $z\frac{\partial f}{\partial y}$, $y\frac{\partial f}{\partial z}$, $z\frac{\partial f}{\partial z}$. Then $\mu_t=\nu_t=10$ and $\omega_t=12$ for all t. Then even if we know that the invariance of Milnor's number implies the invariance of the

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topological type of f_t, we will prove that f_t and f_s are analytically equivalent if and only if $s^2=t^2$. Thus we prove that even if μ_t, ν_t and ω_t are constant, respectively, it does not imply analytic equivalence.

1. Known preliminaries

DEFINITION 1.1. Let $V=\{(y,z) : f(y,z)=0\}$ and $W=\{(y,z) : g(y,z)=0\}$ be germs of analytic varieties of a polydisc in C^2 where f, g are holomorphic and square-free near the origin and the origin is the only singular point of V and W, both. V and W are said to be topologically equivalent if there exists a germ at the origin of homeomorphisms $\phi : (U_1, 0)\to(U_2, 0)$ such ihat $\phi(V)=W$ and $\phi(0)=0$ where U_1 and U_2 are open subsets containing the origin in C^2. In this case we call $f(y,z)$ and $g(y,z)$ topologically equivalent and denote this relation by $f\sim g$. Also, V and W are said to be analytically equivalent if there exists a germ at the origin of biholomorphisms $\phi : (U_1, 0)\to(U_2, 0)$ such that $\phi(V)=W$ and $\phi(0)=0$ where U_1 and U_2 are open subsets containing the origin in C^2. In this case we call $f(y,z)$ and $g(y,z)$ analytically equivalent and denote this relation by $f\approx g$.

Let O be the ring of germs of holomorphic functions near the origin in C^2. Let f be a Weierstrass polynomial in z of multiplicity n. Then f has the form: $f=z^n+\cdots+b_iy^{\alpha_i}z^{n-i}+\cdots+b_ny^{\alpha_n}$ where the $b_i=b_i(y)$ are units in O and the α_i are positive integers for $1\leq i\leq n$.

Then we have the following well-known theorem.

THEOREM 1.2. *Let $f=z^n+\cdots+b_iy^{\alpha_i}z^{n-i}+\cdots+b_ny^{\alpha_n}$ be an irreducible Weierstrass polynomial in O. Then $\frac{\alpha_n}{n}\leq\frac{\alpha_i}{i}$ for all i. Moreover, if $\alpha_n=nk$ for some positive integer k, then $\frac{\alpha_n}{n}=\frac{\alpha_i}{i}$ for $1\leq i\leq n-1$.*

Proof. See [1, Theorem 2.2].

COROLLARY 1.3. *Assume that the hypotheses of Theorem 1.2 are satisfied. If n and α_n are relatively prime, then f is irreducible in O if and only if $\frac{\alpha_n}{n}\leq\frac{\alpha_i}{i}$ for $1\leq i\leq n-1$. Moreover, if $\frac{\alpha_n}{n}<\frac{\alpha_i}{i}$ for all $i\neq n$, then f and $z^n+y^{\alpha_n}$ are topologically equivalent near the origin without assumption of the irreducility of f.*

THEOREM 1.4 (Zariski, Milnor). *Let $f_t(y,z)$ be Weierstrass polyno-*

mials in z of the form $f_t=z^n+a_1(y,t)z^{n-1}+\cdots+a_i(y,t)z^{n-i}+\cdots+a_n(y,t)$ *where the* $a_i(y,t)$ *are holomorphic functions near* $y=0$ *with* $a_i(0,t)=0$ *and the* $a_i(y,t)$ *are complex continuous functions near* $t=0$ *for* $i=1,\cdots,n$. *Let the z-discriminant of* f_t *be* $D_t=y^{N_t}u(y,t)$ *near* $(y,t)=(0,0)$ *where the* $u(y,t)$ *are nowhere vanishing continuous near* $(y,t)=(0,0)$. *If* N_t *is constant near* $t=0$, *then the* f_t *are topologically equivalent near* $t=0$.

Let O_n denote the ring of germs of holomorphic functions near the origin in C^n for $n\geq 2$. If $(V,0)$ is a germ at the origin of a hypersurface in C^n, let $I(V)$ be the ideal of functions in O_n vanishing on V, and let f be a generator of $I(V)$. Let

$$A(V)=O_n/\left(f,\ \frac{\partial f}{\partial z_1},\ \cdots,\ \frac{\partial f}{\partial z_n}\right)O_n$$

and

$$B(V)=O_n/\left(f,\ z_i\frac{\partial f}{\partial z_j}\right)O_n \quad 1\leq i,j\leq n.$$

THEOREM 1.5 (Mather–Yau). *Suppose that* $(V,0)$ *and* $(W,0)$ *are germs of hypersurfaces in* C^n, *and* $V-0$ *is nonsingular. Then the following conditions are equivalent.*

(i) $(V,0)$ *and* $(W,0)$ *are biholomorphically equivalent.*

(ii) $A(V)$ *is isomorphic to* $A(W)$ *as a* C*-algebra.*

(iii) $B(V)$ *is isomorphic to* $B(W)$ *as a* C*-algebra.*

2. Topological and analytic classification of plane curve singularities defined by z^n+az+b with its multiplicity n

Let $V=\{(y,z):f=z^n-nay^{\alpha}z+(n-1)by^{\beta}=0\}$ be an analytic subvariety of a polydisc in C^2 where $a=a(y)$ and $b=b(y)$ are nowhere vanishing holomorphic functions near $y=0$, n is the multiplicity of f and f is square-free. Note that the z-discriminant of f is $D(f)=c(a^ny^{n\alpha}+b^{n-1}y^{\beta(n-1)})$ for some constant c. By a nonsingular change of coordinate, clearly f is analytically equivalent to $h=z^n+y^{\alpha}z+Ay^{\beta}$ where A is nowhere vanishing holomorphic near $y=0$ and $D(h)$ is not identically zero. So it is enough to classify h topologically, and analytically except that h is homogeneous and $n\geq 4$.

THEOREM 2.1. *The above V can be completely classified up to topologically equivalence. For brevity let* $f=z^n+y^{\alpha}z+Ay^{\beta}$ *be a generator of V. Then we have the following cases.*

(1): *Let f be irreducible in O, that is,* $\dfrac{\alpha}{n-1}>\dfrac{\beta}{n}$ *and* $(n,\beta)=1$ *by*

Corollary 1.3. *Then* $f\sim z^n+y^\beta$. *Also, if* $g=z^n+y^\gamma z+By^\delta$ *where* $\frac{\gamma}{n-1}>\frac{\delta}{n}$, $(n,\delta)=1$ *and* $B(y)$ *is a unit in* O, *then* $f\sim g$ *if and only if* $\beta=\delta$.

Now consider those f *which are reducible in* O.

(2): *Let* $\frac{\alpha}{n-1}<\frac{\beta}{n}$ *and* $\frac{\alpha}{n-1}\neq\frac{m}{n}$ *for any positive integer* m. *Then* $f\sim z^n+y^\alpha z$. *Also, if* $g=z^n+y^\gamma z+By^\delta$ *where* $\frac{\gamma}{n-1}<\frac{\delta}{n}$ *and* $\frac{\gamma}{n-1}\neq\frac{m}{n}$ *for any positive integer* m, *then* $f\sim g$ *if and only if* $\gamma=\alpha$.

(3): *Let* $\frac{\alpha}{n-1}\geq\frac{\beta}{n}$ *and* $(n,\beta)>1$. *If* $D(f)=y^{\beta(n-1)}u(y)$ *where* $u(y)$ *is a unit in* O, *then* $f\sim z^n+y^\beta$. *Also, if* $g=z^n+y^\gamma z+By^\delta$ *where* $\frac{\gamma}{n-1}\geq\frac{\delta}{n}$ *and* $(n,\delta)>1$ *and* $D(g)=y^{\delta(n-1)}v(y)$ *a unit in* O, *then* $f\sim g$ *if and only if* $\beta=\delta$.

(4): *Let* $\frac{\alpha}{n-1}=\frac{\beta}{n}$. *But if* $D(f)=y^{n\alpha+p}u(y)$ *and* p *is a positive integer where* $u(y)$ *is a unit in* O, *then* $f\sim z^n-ny^\alpha z+(n-1)y^\beta+y^{\beta+p}$. *Also, if* $g=z^n-ny^\gamma z+(n-1)y^\delta+y^{\delta+q}$ *where* $\frac{\gamma}{n-1}=\frac{\delta}{n}$ *and* q *is a positive integer, then* $f\sim g$ *if and only if* $\beta=\delta$ *and* $p=q$.

Proof of Theorem 2.1. The proof just follows from Theorems 1.2 and 1.4. Note that we add the assumption that $\frac{\alpha}{n-1}\neq\frac{m}{n}$ for any positive integer m in case (2) because we want to separate the above four cases topologically.

Thus any f of the form $z^n+y^\alpha z+Ay^\beta$ where $D(f)\neq0$ and A is a unit in O belongs to exactly one of the above cases topologically.

COROLLARY 2.2. *If the multiplicity of* f *is three, then* f *can be completely classified topologically.*

Therefore, to classify the above V analytically it is enough to consider each case in Theorem 2.1, respectively.

(i) Let $f=z^n+y^\alpha z+Ay^\beta$ be irreducible in O as in case (1) of Theorem 2.1. That is, $\frac{\alpha}{n-1}>\frac{\beta}{n}$ and $(n,\beta)=1$. Then analytic classfication of case (i) follows from Theorems 2.4 and 2.5.

Let $M(f)$ be the ideal in O generated by f, $z\frac{\partial f}{\partial z}$, $y\frac{\partial f}{\partial y}$, $y\frac{\partial f}{\partial z}$, $z\frac{\partial f}{\partial y}$.

Then by Theorem 1.5 (Mather–Yau) we are going to just compute $M(f)$ and dim $O/M(f)$ over the complex field.

LEMMA 2.3. *Let $f=z^n+y^\alpha z+Ay^\beta$ be irreducible in O where A is a unit in O. Then $f\approx g=z^n+y^\alpha z+y^\beta$.*

Proof. Compute $M(f)$ and $M(g)$ directly. Then we get that $M(f)=M(g)$ as ideal in O.

Thus if $f=z^n+y^\alpha z+Ay^\beta$ is irreducible in O where A is a unit in O, then it is enough to consider $g=z^n+y^\alpha z+y^\beta$.

THEOREM 2.4. *Let $f=z^3+y^\alpha z+y^\beta$, $(3,\beta)=1$ and $\frac{\alpha}{2}>\frac{\beta}{3}$. Then f is analytically equivalent to exactly one of $z^3+y^{\alpha-k}z+y^\beta$, $z^3+y^{\alpha-k+1}z+y^\beta$, $\cdots$, $z^3+y^{\beta-1}z+y^\beta\approx z^3+y^\beta$ where k is a greatest positive integer such that $\frac{\alpha-k}{2}>\frac{\beta}{3}$ and $-k+1\leq\beta$.*

Proof. It is enough to compute the dimension of $O/M(f)$ as vector space over the complex field. Now we can see that if α is strictly decreasing with $\alpha<\beta$ under the assumption, then the dimension is also strictly decreasing. So, by Theorem 1.5 the proof is done.

THEOREM 2.5. *Let either $f=z^n+y^\alpha z+y^\beta$ or $f=z^n+y^\beta$ with its multiplicity $n\geq 4$, $(n,\beta)=1$, and $\frac{\alpha}{n-1}>\frac{\beta}{n}$ if the monomial $y^\alpha z$ exists. Then f is analytically equivalent to exactly one of $z^n+y^{\alpha-k}z+y^\beta$, $z^n+y^{\alpha-k+1}z+y^\beta$, $\cdots$, $z^n+y^{\beta-1}z+y^\beta$, z^n+y^β where k is a greatest positive integer such that $\frac{\alpha-k}{n-1}>\frac{\beta}{n}$ and $\alpha-k+1\leqq\beta$.*

Proof. The proof is as same as that of Theorem 2.4. Note that if the multiplicity of f is $n\geqq 4$ and f is irreducible in O, then $z^n+y^{\beta-1}z+y^\beta$ and z^n+y^β are not analytically equivalent.

Consider other cases in Theorem 2.1.

(ii) Let $f=z^n+y^\alpha z+Ay^\beta$ where $\frac{\alpha}{n-1}>\frac{\beta}{n}$ and $\frac{\alpha}{n-1}\neq\frac{m}{n}$ for any positive integer m as in case (2) of Theorem 2.1. Then $f\approx z^n+y^\alpha z+y^\beta$ and so f is analytically equivalent to exactly one of $z^n+y^\alpha z+y^{\beta-k}$, $z^n+y^\alpha z+y^{\beta-k+1}$, $\cdots$, $z^n+y^\alpha z+y^{2\alpha-1}$ where k is a greatest positive integer such that $\frac{\alpha}{n-1}<\frac{\beta-k}{n}$.

(iii) Let $\frac{\alpha}{n-1} \geqq \frac{\beta}{n}$ with $(n, \beta)=1$ and $D(f)=y^{(n-1)\beta}u(y)$ where $f=z^n+y^\alpha z+Ay^\beta$ and $u(y)$ is a unit in O, as in case (3) of Theorem 2.1. Then we are going to consider the following three subcases only except for the case that $n=\beta$.

(iii$_a$) $\frac{\alpha}{n-1} > \frac{\beta}{n} > 1$ and $\frac{\beta}{n}$ is a positive integer: Then $f \approx z^n+y^\alpha z+y^\beta$ and so f is analytically equivalent to exactly one of $z^n+y^{(\beta n+n-\beta)/n}z+y^\beta$, $z^n+y^{(\beta n+2n-\beta)/n}z+y^\beta, \cdots, z^n+y^{\beta-1}z+y^\beta$. Moreover, if $g=z^n+y^\beta$ and $h=z^n+y^{\beta(n-1)/n}z$, then any two of f, g and h are not analytically equivalent.

(iii$_b$) $\frac{\alpha}{n-1} = \frac{\beta}{n} > 1$: Then $\frac{\beta}{n}$ is a postive integer. In this case $f \approx z^n+y^\alpha z+cy^\beta$ for some number c. If $g=z^n+y^\alpha z+dy^\beta$ for some number d, then $f \approx g$ if and only if $c^{n-1}=d^{n-1}$.

(iii$_c$) $\frac{\alpha}{n-1} > \frac{\beta}{n} > 1$ and $\frac{\beta}{n}$ is not an integer: Then f is analytically equivalent to exactly one $z^n+y^{\alpha-k}+y^\beta$, $z^n+y^{\alpha-k+1}z+y^\beta, \cdots, z^n+y^{\beta-1}z+y^\beta, z^n+y^\beta$ where k is a greatest positive integer such that $\frac{\alpha-k}{n-1} > \frac{\beta}{n}$.

(iv) Let $\frac{\alpha}{n-1} = \frac{\beta}{n}$ and $D(f)=y^{n\alpha+p}u(y)$ where f may be assumed to be $z^n-ny^\alpha z+(n-1)y^\beta+A(y)y^{\beta+p}$, p is a positive integer and $u(y)$, $A(y)$ are units in O. Then $f \approx z^n-ny^\alpha z+(n-1)y^\beta+y^{\beta+p}$. If $g \approx z^n-ny^\gamma z+(n-1)y^\delta+y^{\delta+q}$ where $\frac{\gamma}{n-1}=\frac{\delta}{n}$ and q is a positive integer, then $f \approx g$ if and only if $\beta=\delta$ and $p=q$.

To prove those cases (ii), (iii) and (iv), we need just compute the dimension of $O/M(f)$ as vector space over the complex field, except for the subcase (iii$_c$) only. Therefore, we are going to prove the subcase (iii$_c$) only.

For convenience let f and g be of the forms $f=z^3+y^4z+cy^6$ and $g=z^3+y^4z+dy^6$, respectively, with $cd \neq 0$. Let ϕ be a germ of biholomorphism near the origin in C^2 such that $\phi(z, y)=(w, v)$ and $\phi(0,0)=(0,0)$. Write $w=\Sigma a_{\alpha\beta}z^\alpha y^\beta$ and $v=\Sigma b_{\alpha\beta}z^\alpha y^\beta$ with $\alpha+\beta \geqq 1$ where $a_{00}=b_{00}=0$ and $a_{10}b_{01}-a_{01}b_{10} \neq 0$. Then if $f \approx g$, we get

$$(\Sigma a_{\alpha\beta}z^\alpha y^\beta)^3+(\Sigma b_{\alpha\beta}z^\alpha y^\beta)^4 \cdot \Sigma a_{\alpha\beta}z^\alpha y^\beta+c(\Sigma b_{\alpha\beta}z^\alpha y^\beta)^6=u(z^3+y^4z+dy^6)$$

where $u=\Sigma u_{\alpha\beta}z^\alpha y^\beta$ is a unit in Q. Comparing those coefficients of

z^3, y^3, z^2y^2, y^4z, y^6 on both sides, respectively, then we get following in order:

z^3 : $a_{10}{}^3=u_{00}$ implies $a_{10}\neq 0$.

y^3 : $a_{01}{}^3=0$ implies $a_{01}=0$.

z^2y^2 : $\frac{3!}{2!}a_{10}{}^2a_{02}=0$ implies $a_{02}=0$.

y^4z : $\frac{3!}{2!}a_{10}a_{02}{}^2+b_{01}{}^4a_{10}=u_{00}$ implies $b_{01}{}^4a_{10}=u_{00}$.

y^6 : $a_{02}{}^3+b_{01}{}^4a_{02}+cb_{01}{}^6=du_{00}$ implies $cb_{01}{}^6=du_{00}$.

From the above equations together we get

$$cb_{01}{}^2=da_{10} \text{ and } b_{01}{}^4=a_{10}{}^2.$$

Therefore, we conclude necessarily that $c^2=d^2$. Conversely, by a non-singular change of coordinates $(z, y) \to (z, wy)$ with $w^4=1$, then $c^2=d^2$ implies that $z^3+y^4z+cy^6$ and $z^3+y^4z+dy^6$ are analytically equivalent.

Similarly, if $f=z^3+y^{2m}z+cy^{3m}$ where $m\geqq 2$ is an integer, then we can easily show that $f\approx z^3+y^{2m}z+dy^{3m}$ if and only if $c^2=d^2$ because blow-ups preserve analytically equivalence. Moreover, by the similar method we can prove that if $f=z^n+y^{(n-1)m}z+cy^{nm}$ where $m\geqq 2$ is an integer, then $f\approx z^n+y^{(n-1)m}z+dy^{nm}$ if and only if $c^{n-1}=d^{n-1}$.

Thus we classified completely the analytic subvatiety $V=\{(y, z) : f=z^n+y^{\alpha}z+by^{\beta}=0\}$ near the origin where n is the multiplicity of f, f is square-free and $b=b(y)$ is a unit in O except for the case that f is homogeneous with $n\geqq 3$.

COROLLARY 2.6. *If the multiplicity of f is three, then f can be completely classfied analytically.*

Proof. It is just enough to consider the case that $f_t=z^3+y^2z+ty^3$ where each f_t is square-free and homogeneous. Note that the ideals $M(f_t)$ in O are independent of t. By Theorem 1.5, the f_t are analytically equivalent.

3. Some application

Let $V_t=\{(y, z) : f_t(y, z)=0\}$ be the family of germs of analytic varieties where $(0, 0)\in V_t$ are the only singular point and the f_t are square-free for each t.

$$\text{Let } \mu_t=\dim O/\Delta(f_t)$$
$$\nu_t=\dim O/(f_t, \Delta(f_t))$$

and $w_t = \dim O/M(f_t)$ as $\boldsymbol{C}$-vector space where $\Delta(f) = (f_y, f_z)$, $(f, \Delta(f)) = (f, f_y, f_z)$ and $M(f) = (f, zf_z, yf_y, zf_y, yf_z)$ are ideals in O.

THEOREM 3.1. *Let V_t be defined as above. If μ_t, ν_t and w_t are constant, respectively, then V_t may not be analytically equivalent.*

Proof. For example, let $f_t = z^3 + y^4 z + t y^6$. Then $\mu_t = \nu_t = 10$ and $w_t = 12$ if f_t is square-free for each t. But we know that f_t and f_s are analytically distinct if and only if $s^2 \neq t^2$.

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