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# DENOMINATORS IN THE COEFFICIENTS OF POWER SERIES SATISFYING LINEAR DIFFERENTIAL EQUATIONS

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## 1. Introduction

A formal power series with algebraic coefficients is said to be Eisensteinian or to satisfy the Eisenstein condition if there exists a nonzero rational integer $Q$ such that $Q^n a_n$ is an algebraic integer for all $n \geqq 1$. A classical theorem of Eisenstein states that if a power series with rational coefficients represents an algebraic function, then it is Eisensteinian (see e.g. Pólya and Szegö [6, p. 135]). In a recent work [3], we have generalized this theorem of Eisenstein as follows: let $f(z)$ be a power series with algebraic coefficients. If $f(z)$ satisfies an algebraic equation whose leading coefficient does not vanish identically and if each coefficient is an Eisensteinian power series, then $f(z)$ itself is Eisensteinian. Since algebraic functions satisfy linear differential equations with polynomial coefficients (see e.g. Mahler [4, p. 42]), it is natural to ask for similar arithmetic property for the coefficients of power series satisfying linear differential equations. Let us consider a few examples.

EXAMPLE 1. The exponential series $e^{z/2}=\sum_{n=0}^{\infty} z^n/n!2^n$ satisfies $2y'-y=0$. In this case, there is a nonzero rational integer $Q(=2)$ such that $n!Q^n$. coefficient of $z^n$ is an algebraic integer for $n \geqq 1$.

EXAMPLE 2. The series $y(z)=\sum_{n=0}^{\infty} z^n/(n!)^m$, $m$ being an integer$>1$, satisfies $\left(z\frac{d}{dz}\right)^m y-zy=0$. In this case, there is no nonzero rational integer $Q$ such that $n!Q^n$. coefficient of $z^n$ is integral for $n \geqq 1$. However, there is a polynomial $Q(z)=z^{m-1}$ such that $n!Q(1)Q(2)...Q(n)$.

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coefficient of $z^n$ is integral for $n \geqq 1$. Note that 0 is a singularity of the differential equation.

Example 3. The hypergeometric series

$$f(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\gamma(\gamma+1)\dots(\gamma+n-1)} z^n,$$

where $\alpha, \beta, \gamma$ are positive rationals, satisfies

$$z(1-z)y'' + (\gamma - (\alpha+\beta+1)z)y' - \alpha\beta\gamma = 0.$$

Here again 0 is a singularity of the differential equation and there is no nonzero rational integer $Q$ such that $n!Q^n$. coefficient of $z^n$ is integral for $n \geqq 1$. Yet, it is easy to see that there is a nontrivial polynomial $Q(z)$ with rational integral coefficients such that $n!Q(1)Q(2)\dots Q(n)$. coefficient of $z^n$ is integral for $n \geqq 1$.

These three examples are typical, as we shall see later, and lead us to the following definitions.

Definitions. Let $\sum_{n=1}^{\infty} a_n z^n$ be a formal power series with algebraic coefficients.

1) It is said to be *F–Eisensteinian* (or factorially Eisensteinian) if there exists a nonzero rational integer $Q$ such that $n!Q^n a_n$ is an algebraic integer for all $n \geqq 1$.
2) It is said to be *FA–Eisensteinian* (or factorially almost Eisensteinian) if there exist a nonnegative (rational) integer $I$ and a nontrivial polynomial $Q(z)$ (possibly depending on $I$), with rational integral coefficients and with $Q(j) \neq 0$ for each positive integer $j$, such that $(I+n)!Q(1)Q(2)\dots Q(n)a_{I+n}$ is an algebraic integer for all $n \geqq 1$.

In the next section, we prove that power series satisfying linear differential equations, with 0 not being one of the singularities and with F–Eisensteinian coefficients, are F–Eisensteinian. This corresponds to the first example considered. In section 3, we prove that power series satisfying linear differential equations, with 0 being one of the singularities and with F–Eisensteinian coefficients, are FA–Eisensteinian. This is illustrated in the second and third examples above. The proofs of both theorems are elementary. The original ideas of the proofs come from the works of Popken [7, Kap. I] and Hurwitz [1], respectively. The class of linear differential equations considered here contains most of the equations in classical analysis and mathematical physics.

## 2. Case where 0 is not a singularity

THEOREM 1. *Let* $y(z)=\sum_{n=0}^{\infty} a_n z^n$ *be a power series with algebraic coefficients. If* $y$ *satisfies a linear differential equation of the form*

$$\sum_{i=0}^{m} P_i(z) y^{(i)}(z)=0, \qquad (1)$$

*where* $m \geqq 1$, *all* $P_i$ $(i=0,1,\dots,m)$ *are F–Eisensteinian power series, and if* $P_m(0)\neq 0$, *then* $y$ *is F–Eisensteinian.*

*Proof.* Writing $D=d/dz$, for each nonnegative integer $j$, we get

$$D^j(P_r(z)y^{(r)}(z))=\sum_{k=0}^{j}\binom{j}{k} P_r^{(k)}(z)y^{(r+j-k)}(z),$$

for $r=0,1,\dots,m$. Taking the $j$th derivatives of the expressions in (1), and using this last identity, we have for $j\geqq 0$

$$P_m(z)y^{(m+j)}(z)=-\sum_{k=1}^{m+j}\Big(\sum_{s=0}^{k}\binom{j}{s}P_{m-k+s}{}^{(s)}(z)\Big)y^{(m+j-k)}(z). \qquad (2)$$

Let

$$P_i(z)=\sum_{n=0}^{\infty} p_{i,n}z^n \quad (i=0,1,\dots,m), \qquad (3)$$

where, by hypothesis, $p:=p_{m,0}\neq 0$. Putting $z=0$ in (2) yields

$$(m+j)!pa_{m+j}=-\sum_{k=1}^{m+j}\Big(\sum_{s=0}^{k}\binom{j}{s}s!p_{m-k+s,s}\Big)(m+j-k)!a_{m+j-k} \quad (j\geqq 0). \ (4)$$

Upon multiplying by a suitable rational integer in (4), we may assume that $p$ is a nonzero algebraic integer. Since each $P_i$ $(i=0,1,\dots,m)$ is F–Eisensteinian and there are only finitely many of them, then there is a nonzero rational integer $B$ such that $n!B^n p_{i,n}$ and $Bp_{i,0}$ are algebraic integers for $i=0,1,\dots,m$ and for all $n\geqq 1$. Let $A$ be a nonzero rational integer such that $Aa_0, Aa_1, \dots, Aa_{m-1}$ are algebraic integers. Setting $j=0$ in (4) and multiplying both sides of the equation by $AB^m$, we see that $m!pAB^m a_m$ is an algebraic integer. Similarly, setting $j=1$ in (4), we obtain $(m+1)!p^2AB^{m+1}a_{m+1}$ is an algebraic integer. By induction, it then follows that $(m+j)!p^{j+1}AB^{m+j}a_{m+j}$ is an algebraic integer. Taking $Q=AB\,\mathrm{Norm}(p)$, then $Q$ is a nonzero rational integer and $n!Q^n a_n$ is an algebraic integer for all $n\geqq 1$. This completes the proof of the theorem.

REMARKS. It follows immediately from Theorem 1 that for $n$ sufficiently large, if the coefficients $a_n$ are rational, then the greatest prime factor in the denominator of each $a_n$ is $\leqq n$.

## 3. Case where 0 is a singularity

THEOREM 2. *Let* $y(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with algebraic coefficients. If* $y$ *satisfies a linear differential equation of the form*

$$\sum_{i=0}^{m}P_i(z)y^{(i)}(z)=0, \tag{5}$$

*where* $m\geqq 1$, *all* $P_i$ $(i=0,1,\dots,m)$ *are F–Eisensteinian power series and* $P_m(z)\not\equiv 0$, *then* $y$ *is FA–Eisensteinian.*

*Proof.* The identity (2) in the proof of Theorem 1 remains valid, so evaluating $z=0$ in (2), we get for nonnegative integer $\lambda$,

$$\sum_{k=0}^{m+\lambda}F_k(\lambda)y^{(m+\lambda-k)}(0)=0, \tag{6}$$

where $F_k(\lambda):=\sum_{s=0}^{k}\binom{\lambda}{s}P_{m-k+s}{}^{(s)}(0)$. Let $P_i(z)$, $i=0,1,\dots,m$ be of the same shape as in (3). Since $P_m(z)\neq 0$, let $M$ be the least nonnegative integer for which

$$P_m{}^{(M)}(0)=M!\,p_{m,M}\neq 0.$$

Observe that $F_k(\lambda)$ is a polynomial in $\lambda$ of degree at most $k$ with algebraic coefficients. We claim that for each nonnegative integer $\lambda\geqq\max(0,M-m):=\lambda'$, say, not all $F_k(\lambda)$ are identically zero. This follows immediately from the fact that the leading coefficient in $F_M(\lambda)$ (i.e. the coefficient of $\lambda^M$) is $P_m{}^{(M)}(0)\neq 0$. Let then $k_0$, $0\leqq k_0\leqq M$ be the least nonnegative integer (independent of $\lambda$) such that $F(\lambda):=F_{k_0}(\lambda)\not\equiv 0$ as a polynomial in $\lambda$. Put

$\lambda_0=\max(\lambda',$ largest rational integral root of $F(\lambda))+1$,

so that $F(\lambda)\neq 0$ for all $\lambda\geqq\lambda_0$. Thus for all $\lambda\geqq\lambda_0$, (6) yields

$$(m-k_0+\lambda)!F(\lambda)a_{m-k_0+\lambda}=-\sum_{k=k_0+1}^{m+\lambda}(m+\lambda-k)!F_k(\lambda)a_{m+\lambda-k}. \tag{7}$$

Upon multiplying by a suitable rational integer in (7), we may assume that all coefficients of $F(\lambda)$, considered as a polynomial in $\lambda$, are algebraic integers and not all of them are 0. Setting $M_0=m-k_0+\lambda_0\geqq 1$, then (7) can be rewritten as

$$(M_0+n)!F(\lambda_0+n)a_{M_0+n}=-\sum_{k=1}^{M_0+n}F_{k_0+k}(\lambda_0+n)(M_0+n-k)!a_{M_0+n-k} \tag{8}$$

$$(n\geqq 1).$$

Since each $P_i$, $i=0,1,\dots,m$ is F–Eisensteinian and since there are only finitely many of them, then, by definition, there exists a nonzero rational integer $B$ such that $n!B^n p_{i,n}$ is an algebraic integer for $i=0, 1, \dots, m$ and all $n \geqq 1$. For brevity, with $n$ being positive integer, set

$$f(n) := F(\lambda_0+n) = F_{k_0}(\lambda_0+n) = \sum_{s=0}^{k_0} \binom{\lambda_0+n}{s} s!\, p_{m-k_0+s,s} \neq 0,$$
$$f_k(n) := F_{k_0+k}(\lambda_0+n) = \sum_{s=0}^{k_0+k} \binom{\lambda_0+n}{s} s!\, p_{m-k_0-k+s,s}.$$

Now consider two separate cases.

Case 1. $k_0 \geqq \lambda_0$.

Let $t_0$ be a nonnegative integer for which $k_0 = \lambda_0 + t_0$. Thus

$$f_k(t_0+j) = \sum_{s=0}^{\lambda_0+t_0+k} \binom{\lambda_0+t_0+j}{s} s!\, p_{m-\lambda_0-t_0-k+s,s} \quad (j \geqq 1).$$

Let $A = A(M_0, t_0)$ be a nonzero rational integer such that $Aa_0, Aa_1, \dots, Aa_{M_0+t_0}$ are algebraic integers. Putting $n = t_0+1$ in (8), we obtain
$(M_0+t_0+1)!f(t_0+1)a_{M_0+t_0+1} = -\sum_{k=1}^{M_0+t_0+1} f_k(t_0+1)(M_0+t_0+1-k)!a_{M_0+t_0+1-k}$. Here, we see that $(M_0+t_0+1)!f(t_0+1)AB^{\lambda_0+t_0+1}a_{M_0+t_0+1}$ is an algebraic integer. Putting $n=t_0+2$ in (8), we obtain $(M_0+t_0+2)!f(t_0+2)a_{M_0+t_0+2} = -\sum_{k=1}^{M_0+t_0+2} f_k(t_0+2)(M_0+t_0+2-k)!a_{M_0+t_0+2-k}$. Here, $(M_0+t_0+2)!f(t_0+2)f(t_0+1)AB^{2(\lambda_0+t_0+1)}a_{M_0+t_0+2}$ is an algebraic integer. It follows by induction that

$$(M_0+t_0+n)!f(t_0+n)f(t_0+n-1)\dots f(t_0+1)AB^{n(\lambda_0+t_0+1)}a_{M_0+t_0+n}$$

is an algebraic integer for each $n \geqq 1$. Let $f_0(z)$ be the product of $f(z)$ and all its conjugate polynomials (i. e. all polynomials having the coefficients running through all possible conjugates in the smallest extension field containing them). Thus $f_0(z)$ is a nontrivial polynomial with rational integral coefficients. Set

$$I = M_0+t_0 \text{ and } Q(z) = AB^{\lambda_0+t_0+1} f_0(t_0+z).$$

Then $Q(z)$ is a nontrivial polynomial with rational integral coefficients, $Q(j) \neq 0$ for each integer $j \geqq 1$, and $(I+n)!Q(1)Q(2)\dots Q(n)a_{I+n}$ is an algebraic integer for all $n \geqq 1$.

Case 2. $k_0 < \lambda_0$.

Let $t_1$ be a positive integer for which $\lambda_0 = k_0 + t_1$. Thus

$$f_k(n) = \sum_{s=0}^{k_0+k} \binom{k_0+t_1+n}{s} s!\, p_{m-k_0-k+s,s} \quad (n \geqq 1).$$

Let $A = A(M_0)$ be a nonzero rational integer such that $Aa_0, Aa_1, \dots, Aa_{M_0}$ are algebraic integers. Putting $n=1$ in (8), we have

$$(M_0+1)!f(1)a_{M_0+1}=-\sum_{k=1}^{M_0+1}f_k(1)(M_0+1-k)!a_{M_0+1-k}.$$

Here we see that $(M_0+1)!f(1)AB^{k_0+t_1+1}a_{M_0+1}$ is an algebraic integer. Similarly, putting $n=2$ in (8), we get

$$(M_0+2)!f(2)a_{M_0+2}=-\sum_{k=1}^{M_0+2}f_k(2)(M_0+2-k)!a_{M_0+2-k}.$$

Here we see that $(M_0+2)!f(2)f(1)AB^{2(k_0+t_1+1)}a_{M_0+2}$ is an algebraic integer. It follows by induction that

$(M_0+n)!f(n)f(n-1)\ldots f(1)AB^{n(k_0+t_1+1)}a_{M_0+n}$ is an algebraic integer for all $n\geqq 1$. As in the previous case, let $f_0(z)$ be the product of $f(z)$ and all its conjugate polynomials, so that $f_0(z)$ is a nontrivial polynomial with rational integral coefficients. Set

$$I=M_0 \text{ and } Q(z)=AB^{k_0+t_1+1}f_0(z).$$

Thus $Q(z)$ is a nontrivial polynomial with rational integral coefficients, $Q(j)\neq 0$ for each integer $j\geq 1$, and

$(I+n)!Q(1)Q(2)\ldots Q(n)a_{I+n}$ is an algebraic integer for all $n\geqq 1$.

In both cases, we have that $y$ is FA–Eisensteinian.

REMARKS. 1) If $a_n$ is rational, then Theorem 2 implies that there exist positive constants $c_1, c_2$ such that the greatest prime factor in the denominator of $a_n$ is $\leqq c_1 n^{c_2}$ for all sufficiently large $n$. An earlier result of this type is due originally to Pincherle [5]. Similar results for algebraic differential equations are due to Hurwitz [1], Kakeya [2] and Popken [8].

2) If, in Theorem 2, all $P_i$, $i=0, 1, \ldots, m$ are FA–Eisensteinian, by exactly the same proof, a much more complicated but much less elegant result can be obtained. Indeed, it can be shown that there exist a nonnegative integer $I$, a nontrivial polynomial $Q(z)$ (possibly depending on $I$) with rational integral coefficients, and with $Q(j)\neq 0$ for each $j\geqq 1$, such that

$$(I+n)!Q(1)^{n-1}Q(2)^{[n/2]-1}\ldots Q(n)^{[n/n]-1}a_{I+n}$$

is an algebraic integer for all $n\geqq 1$, where, for positive real $x$, $[x]$ denotes the least rational integer $\geqq x$.

3) For algebraic $a_n$, let $d_n$ be the least positive integer such that $d_n a_n$ is an algebraic integer. Then Theorem 2 implies that there exists a positive constant $c_0$ such that

$d_n\leqq\exp(c_0 n\log n)$ for all sufficiently large $n$.

This result can be found in Mahler [4, Theorem 18, p. 206] where the case of algebraic differential equations is also discussed.

4) For algebraic $a_n$, another immediate consequence to Theorem 2 is that there exists a positive constant $c_1$ such that

$$\text{either } a_n=0, \text{ or } |\overline{a_n}| \geqq \exp(-c_1 n \log n)$$

for all sufficiently large $n$, where, for nonzero algebraic $x$, $|\overline{x}|$ denotes the maximum absolute value of all conjugates of $x$. For related results, see Popken [8] and Mahler [4, Theorem 19, pp. 206–207].

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