JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 2, June 1989

A Study on the Functions of $\kappa\phi$-Bounded Variations

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ABSTRACT. In this paper, we study some properties of generalized function spaces of κ-, ϕ-and $\kappa\phi$- bounded variations and general bounded variations.

In defining a function of bounded variation on the closed interval $[a,b]$ we considered the supremum of $\sum |f(I_n)|$ for every collection $\{I_n\}$ of nonoverlapping subintervals of $[a,b]$ such that $[a,b] = \bigcup I_n$ where $f(I_n) = f(y_n) - f(x_n)$, $I_n = [x_n, y_n]$. A function f is of bounded variation on $[a,b]$ if $V_a^b(f) = \sup \sum |f(I_n)|$ is finite. Equivalently we could say a function is of bounded variation on the closed interval $[a,b]$ if there exists a positive constant C such that for every collection $\{I_n\}$ of subintervals of $[a,b]$, $\sum |f(I_n)| \leq C$. A function f is said to be κ-bounded variation of $[a,b]$ if there exists a positive constant C such that for every collection $\{I_n\}$ of nonoverlapping subintervals of $[a,b]$, $\sum |f(I_n)| \leq C \sum \kappa(|I_n|/(b-a))$ where $|I_n| = y_n - x_n$, $I_n = [x_n, y_n]$. On the other hand, Michael Schramm [4, 5] generalized the above idea by considering a sequence of increasing convex function $\phi = \{\phi_n\}$ defined on $[0,\infty)$; f is of ϕ-bounded variation on $[a,b]$ if $V_\phi(f; a, b) = \sup \sum_n (|f(I_n)|)$ is finite. We are going to combine the above concepts.

The introduction of the function κ can be viewed as a rescaling of lengths of subintervals in $[a,b]$ such that the length of $[a,b]$ is 1 if $\kappa(1) = 1$. We are now requiring through the following that κ has the following properties on $[0,1]$;

(1) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,
(2) κ is concave and strictly increasing, and
(3) $\lim_{x\to 0+} \kappa(x)/x = \infty$.

Received by the editors on 30 June 1989.

1980 *Mathematics subject classifications*: Primary 46F.

Supported by a grant from the Korea Science and Engineering Foundation, 1988-89.

Let $\phi = \{\phi_n\}$ be a sequence of increasing convex functions defined on nonnegative numbers and such that $\phi_n(0) = 0$, $\phi_n(x) > 0$.

Let a real valued function f be defined on the closed interval $[a, b]$. A function f is said to be of $\kappa\phi$-bounded variation on $[a, b]$ if there exists a positive constant C such that for any collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$

$$\sum \phi_n(|f(I_n)|) \leq C \sum \kappa(|I_n|/(b-a))$$

where $[a, b] = \bigcup I_n$ and $|I_n|$ is the length of I_n. The total variation of f over $[a, b]$ is defined by

$$\kappa V_\phi(f) = \kappa V_\phi(f; a, b) = \sup \sum \phi_n(|f(I_n)|) / \sum \kappa(|I_n|/(b-a)),$$

where the supremum is taken over all nonoverlapping subintervals $\{I_n\}$ in $[a, b]$. We denote by $\kappa\phi BV$ the collection of all $\kappa\phi$-bounded variation function on $[a, b]$. We note that if f is of ϕ-bounded variation on a closed interval $[a, b]$, then f is of $\kappa\phi$-bounded variation on $[a, b]$ and ϕBV is included in $\kappa\phi BV$. Let $\kappa\phi BV_0 = \{f \in \kappa\phi BV; f(a) = 0\}$. For f in $\kappa\phi BV_0$, let us define the norm as in the Orlicz spaces;

$$|||f||| = |||f|||_{\kappa\phi} = \inf\{k > 0\ ;\ \kappa V_\phi(f/k) \leq 1\}.$$

Then $(\kappa\phi BV_0, |||\cdot|||)$ is a Banach space and $\kappa\phi BV$ may be a Banach space with the norm $|f(a)| + |||f - f(a)|||$.

Let a function f be defined on the interval $[a, b]$. f is said to be $\kappa\phi$-decreasing on $[a, b]$ if there exists a positive constant C such that for any interval I in $[a, b]$

$$\phi_n(|f(I)|) \leq C \quad \kappa(|I|/(b-a)).$$

If a function f is $\kappa\phi$-decreasing on $[a, b]$, then we have the following properties;

(1) f is of $\kappa\phi$-bounded variation,
(2) $f(x_0^-)$ and $f(y_0^-)$ exist for any $a \leq x_0 < b$ and $a < y_0 \leq b$,
(3) f is continuous on $[a, b]$

(But, κ-decreasing functions need not be continuous). Also, suppose that $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ and $\phi_3 = \{\phi_{3n}\}$ satisfy $\phi_{1n}^{-1}(x)\phi_{2n}^{1}(x) \leq \phi_{3n}^{-1}(x)$ for all n. Then for all $f \in \kappa\phi_1 BV_0$, $g \in \kappa\phi_2 BV_0$, $fg \in \kappa\phi_3 BV_0$ and $|||fg|||_{\kappa\phi_3} \leq 2|||f|||_{\kappa\phi_1}|||g|||_{\kappa\phi_2}$, which is proved by the following.

LEMMA 1. *Suppose that* $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ *and* $\phi_3 = \{\phi_{3n}\}$ *satisfy, for all* n, $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \le \phi_{3n}^{-1}(x)$. *Then* $\phi_{3n}(xy) \le \phi_{1n}(x) + \phi_{2n}(y)$ *for* $x, y \ge 0$.

PROOF: From the definition of ϕ_{1n}^{-1}, we have:

$$\phi_{1n}(\phi_{1n}^{-1}(x)) \le x \le \phi_{1n}^{-1}(\phi_{1n}(x)).$$

Given any $x, y \ge 0$, either $\phi_{1n}(x) \le \phi_{2n}(y)$ or $\phi_{1n}(x) > \phi_{2n}(y)$. If $\phi_{1n}(x) \le \phi_{2n}(y)$ then

$$\begin{aligned} xy &\le \phi_{1n}^{-1}(\phi_{1n}(x))\phi_{2n}^{-1}(\phi_{2n}(y)) \\ &\le \phi_{1n}^{-1}(\phi_{2n}(y))\phi_{2n}^{-1}(\phi_{2n}(y)) \le \phi_{3n}^{-1}(\phi_{2n}(y)). \end{aligned}$$

$$\phi_{3n}(xy) \le \phi_{3n}(\phi_{3n}^{-1}(\phi_{2n}(y))) \le \phi_{2n}(y).$$

If $\phi_{1n}(x) > \phi_{2n}(y)$, a similar argument shows that $\phi_{3n}(x) \le \phi_{1n}(x)$.
Therefore,

$$\begin{aligned} \phi_{3n}(xy) &\le \max(\phi_{1n}(x), \phi_{2n}(x)) \\ &\le \phi_{1n}(x) + \phi_{2n}(y) \qquad \text{for} \quad x, y \ge 0. \end{aligned}$$

By the similar way as Lemma 1, we can prove the following.

LEMMA 2. *Suppose that* $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ *and* $\phi_3 = \{\phi_{3n}\}$ *satisfy* $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \le k\phi_{3n}^{-1}(x)$ *for all* n. *Then there exists a constant* k' *such that* $\phi_{3n}(xy/k') \le \phi_{1n}(x) + \phi_{2n}(y)$ *for any* x, $y \ge 0$.

LEMMA 3. *For* ϕ_1, ϕ_2, *and* ϕ_3 *as the above Lemma 2, the following are equivalent;*

(1) $\lim_{x\to\infty} \sup \phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x)/\phi_{3n}^{-1}(x) < \infty$

(2) *There exists a positive* k *such that, for all* $x, y \ge x_0 \ge 0$,

$$\phi_{3n}(xy/k) \le \phi_{1n}(x) + \phi_{2n}(y).$$

LEMMA 4. *For* ϕ_1, ϕ_2 *and* ϕ_3 *as the above Lemma 2, the followings are equivalent ;*

(1) $\lim_{x\to 0+} \sup \phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x)/\phi_{3n}^{-1}(x) < \infty$,

(2) *There exist numbers* $k > 0$ *and* $x_0 > 0$ *such that for all* $x, y \le x_0$, $\phi_{3n}(xy/k) \le \phi_{1n}(x) + \phi_{2n}(y)$.

THEOREM 5. *For $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ and $\phi_3 = \{\phi_{3n}\}$, the followings are equivalent;*

(1) *There exists $k > 0$ such that $\phi_{1n}^{-1}(x)\phi_{2n}^{-1} \le k\phi_{3n}^{-1}(x)$ for all $x \ge 0$,*

(2) *There exists $k' > 0$ such that, for all $x, y \ge 0$,*

$$\phi_{3n}(xy/k') \le \phi_{1n}(x) + \phi_{2n}(y).$$

PROOF: Combine Lemma 3 and 4, we obtain this result.

THEOREM 6. *Suppose that $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_2\}$ and $\phi_3 = \{\phi_{3n}\}$ satisfy $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \le k\phi_{3n}^{-1}(x)$ for all n. Then for all $f \in \kappa\phi_1 BV_0$ and $g \in \kappa\phi_2 BV_0$, $fg/k \in \kappa\phi_3 BV_0$ and $|||fg|||_{\kappa\phi_3} \le 2k|||f|||_{\kappa\phi_1}|||g|||_{\kappa\phi_2}$.*

PROOF: Given any $I_n \subset [a,b]$, either $\phi_{1n}(|F(I_n)|) \le \phi_{2n}(|g(I_n)|)$ or $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$. If $\phi_{1n}(|f(I_n)|) \le \phi_{2n}(|g(I_n)|)$, then we have the following inequality;

$$\begin{aligned}|f(I_n)g(I_n)/k| &= \frac{1}{k}\phi_{1n}^{-1}(\phi_{1n}(|f(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|))\\ &\le \frac{1}{k}\phi_{1n}^{-1}(\phi_{2n}(|g(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|))\\ &\le \frac{1}{k}\cdot k\phi_{3n}^{-1}(\phi_{2n}(|g(I_n)|))\\ &= \phi_{3n}^{-1}(\phi_{2n}(|g(I_n)|)).\end{aligned}$$

Thus $\phi_{3n}(|f(I_n)g(I_n)|/k) \le \phi_{2n}(|g(I_n)|)$. If $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$, then a similar argument shows that

$$\phi_{3n}(|f(I_n)g(I_n)|/k) \le \phi_{1n}(|f(I_n)|).$$

Therefore we have

$$\begin{aligned}&\sum \phi_{3n}(|f(I_n)g(I_n)|/k)/\sum \kappa(|I_n|/(b-a))\\ &\le \left[\sum \phi_{1n}(|f(I_n)|)/\sum \kappa(|I_n|/(b-a))\right]\\ &\quad + \left[\sum \phi_{2n}(|g(I_n)|)/\sum \kappa(|I_n|/(b-a))\right]\end{aligned}$$

Thus $fg/k \in \kappa\phi_3 BV_0$.

Let $\varepsilon > 0$. Without loss of generality assume $|||f|||_{\kappa\phi_1} = |||g|||_{\kappa\phi_2} = 1$. By the convexity of ϕ_{3n}, we have

$$\begin{aligned}
&\sum \phi_{3n}(|f(I_n)g(I_n)|/2k(1+\varepsilon)^2)/\sum \kappa(|I_n|/(b-a)) \\
&\leq \frac{1}{2}\sum \phi_{3n}(|f(I_n)||g(I_n)|/k(1+\varepsilon)^2)/\sum \kappa(|I_n|/(b-a)) \\
&\leq \frac{1}{2}\sum \phi_{1n}(|f(I_n)|/1+\varepsilon)/\sum \kappa(|I_n|/(b-a)) \\
&\quad + \frac{1}{2}\sum \phi_{2n}(|g(I_n)|/1+\varepsilon)/\sum \kappa(|I_n|/(b-a)) \\
&\leq \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Thus $\kappa V\phi_3(fg/2k(1+\varepsilon)^2) \leq 1$, $|||fg|||_{\kappa\phi_3} \leq 2k(1+\varepsilon)^2$ and the theorem follows by letting $\varepsilon \to 0$.

COROLLARY 7. *Suppose that* $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ *and* $\phi_3 = \{\phi_{3n}\}$ *satisfy, for all* n, $\phi_{2n}(x) \geq \phi_{4n}(x/k)$, *where* $\phi_{4n}(x) = \sup_{y\geq 0} |\phi_{3n}(xy) - \phi_{1n}(y)|$ *for* x, $y \geq 0$ *and* k *constant. Then, for all* $f \in \kappa\phi_1 BV_0$ *and* $g \in \kappa\phi_2 BV_0$, *their product* $fg/k \in \kappa\phi_3 BV_0$ *and* $|||fg|||_{\kappa\phi_3} \leq 2k|||f|||_{\kappa\phi_1}|||g|||_{\kappa\phi_2}$.

PROOF: For all x, $y \geq 0$, $\phi_{3n}(xy) \leq \phi_{4n}(x) + \phi_{1n}(y)$ implies that $\phi_{3n}(xy) \leq \phi_{1n}(y) + \phi_{2n}(kx)$, which implies $\phi_{3n}(xy/k) \leq \phi_{1n}(y) + \phi_{2n}(x)$. By Theorem 5, there exists $k > 0$ such that $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \leq k\phi_{3n}^{-1}(x)$ for all $x \geq 0$. By the same way as Theorem 6, we obtains this corollary.

COROLLARY 8. *For* $\phi_1 = \{\phi_{1n}\}$ *and* $\phi_3 = \{\phi_{3n}\}$, *letting* $\phi_{2n}(y) = \sup_{x\geq 0}(\phi_{3n}(xy) - \phi_{1n}(x))$, $\phi_{4n}(x) = \sup_{y\geq 0}(\phi_{3n}(xy) - \phi_{2n}(y))$ *and* $\phi_{5n}(y) = \sup_{x\geq 0}(\phi_{3n}(xy) - \phi_{4n}(x))$ *for all* n, x, $y \geq 0$, *then for all* $f \in \kappa\phi_1 BV_0$ *and* $g \in \kappa\phi_2 BV_0$ *we have* $\kappa\phi_1 BV_0 \subset \kappa\phi_4 BV_0$, $\kappa\phi_5 BV_0 = \kappa\phi_3 BV_0$ *and* $fg \in \kappa\phi_3 BV_0$ *for fixed* k. *Also*

$$|||fg|||_{\kappa\phi_3} \leq 2k|||f|||_{\kappa\phi_1}|||g|||_{\kappa\phi_2}.$$

PROOF: Note that $\phi_{4n}(x) \leq \phi_{1n}(x)$ and $\phi_{5n} \leq \phi_{3n}(y)$ for x, $y \geq 0$. Thus $\phi_{3n}(xy) \leq \phi_{4n}(x) + \phi_{5n}(y) \leq \phi_{1n}(x) + \phi_{2n}(y)$. By Theorem 5,

$\sup_{x\geq 0}(\phi_{3n}(xy) - \phi_{4n}(x))$ *for all* n, x, $y \geq 0$, *then for all* $f \in \kappa\phi_1 BV_0$ *and* $g \in \kappa\phi_2 BV_0$ *we have* $\kappa\phi_1 BV_0 \subset \kappa\phi_4 BV_0$, $\kappa\phi_5 BV_0 = \kappa\phi_3 BV_0$ *and* $fg \in \kappa\phi_3 BV_0$ *for fixed* k. *Also*

$$|||fg|||_{\kappa\phi_3} \leq 2k|||f|||_{\kappa\phi_1}|||g|||_{\kappa\phi_2}.$$

PROOF: Note that $\phi_{4n}(x) \leq \phi_{1n}(x)$ and $\phi_{5n} \leq \phi_{3n}(y)$ for x, $y \geq 0$. Thus $\phi_{3n}(xy) \leq \phi_{4n}(x) + \phi_{5n}(y) \leq \phi_{1n}(x) + \phi_{2n}(y)$. By Theorem 5, we have $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \leq \phi_{3n}^{-1}(x)$ for all $x \geq 0$, which implies the proof.

REMARK 1: The inequality $\phi_{4n}(x) \leq \phi_{1n}(x)$ may not be replaced by equality.

THEOREM 9. *Suppose that* $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ *and* $\phi_3 = \{\phi_{3n}\}$ *satisfy* $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \leq \phi_{3n}^{1}(x)$ *for all* n, *and there exist* κ-*function* κ_1, κ_2, *and* κ_3 *such that* $\kappa_1^{-1}(x)\kappa_2^{-1}(x) \geq \kappa_3^{-1}(x)$. *Then, for all* $f \in \kappa_1\phi_1 BV_0$ *and* $g \in \kappa_2\phi_2 BV_0$, *the product* $fg/2$ *is in* $\kappa_3\phi_3 BV_0$ *and*

$$|||fg|||_{\kappa_3\phi_3} \leq 4|||f|||_{\kappa_1\phi_1}|||g|||_{\kappa_2\phi_2}.$$

PROOF: If $\kappa_1(|I_n|/b - a) \leq \kappa_2(|I_n)/b - a)$, then we have that

$$\begin{aligned}|I_n|/b - a &\geq (|I_n|/b - a)(|I_n|/b - a)\\ &\geq \kappa_1^{-1}(x_1(|I_n|/b - a))\kappa_2^{-1}(x_2(|I_n|/b - a))\\ &\geq \kappa_1^{-1}(x_2(|I_n|/b - a))\kappa_2^{1}(x_2(|I_n|/b - a))\\ &\geq \kappa_3^{-1}(x_2(|I_n|/b - a)),\end{aligned}$$

which implies that

$$\kappa_3(|I_n|/b - a) \geq \kappa_2(|I_n|/b - a).$$

Also, if $\kappa_1(|I_n|/b - a) \geq \kappa_2(|I_n|/b - a)$, then we have

$$|I_n|/b - a \geq \kappa_3^{-1}(\kappa_1(|I_n|/b - a)),$$

which implies that

$$\kappa_3(|I_n|/b - a) \geq \kappa_1(|I_n|/b - a).$$

Therefore

$$\frac{\sum \phi_{3n}(|f(I_n)g(I_n)/2|)}{\sum \kappa_3(|I_n|/b-a)} \leq \frac{\sum \phi_{1n}(|f(I_n)|) + \sum \phi_{2n}(|g(I_n)|)}{2\sum \kappa_3(|I_n|/b-a)}$$
$$\leq \frac{\sum \phi_{1n}(|f(I_n)|) + \sum \phi_{2n}(|g(I_n)|)}{\sum \kappa_1(|(I_n)|/b-a) + \sum \kappa_2(|I_n|/b-a)}$$
$$\leq \frac{\sum \phi_{1n}(|f(I_n)|)}{\sum \kappa_1(|I_n|/b-a)} + \frac{\sum \phi_{2n}(|g(I_n)|)}{\sum \kappa_2(|I_n|/b-a)} < \infty$$

Thus $fg/2 \in \kappa_3\phi_3 BV_0$. Let $\varepsilon > 0$. Without loss of generality, we may assume $|||f|||_{\kappa_1\phi_1} = 1 = |||g|||_{\kappa_2\phi_2}$. By the convexity of $\phi_{3n}(x)$, we have

$$\frac{\sum \phi_{3n}\left(\frac{|f(I_n)g(I_n)|}{4(1+\varepsilon)^2}\right)}{\sum \kappa_3(|I_n|/b-a)} \leq \frac{\sum \frac{1}{4}\phi_{3n}\left(\frac{|f(I_n)|}{1+\varepsilon}\cdot\frac{|g(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_3(|I_n|/b-a)}$$
$$\leq \frac{\frac{1}{2}\sum \phi_{1n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right) + \frac{1}{2}\sum \phi_{2n}\left(\frac{|g(I_n)|}{1+\varepsilon}\right)}{2\sum \kappa_3(|I_n|/b-a)}$$
$$\leq \frac{d\frac{1}{2}\sum \phi_{1n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right) + \frac{1}{2}\sum \phi_{2n}\left(\frac{|g(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_1(|I_n|)/b-a) + \sum \kappa_2(|I_n|/b-a)}$$
$$\leq \frac{\frac{1}{2}\sum \phi_{1n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_1(|I_n|b-a)} + \frac{\frac{1}{2}\sum \phi_{2n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_2(|I_n|b-a)}$$
$$\leq \frac{1}{2}(|||f|||_{\kappa_1\phi_1} + |||g|||_{\kappa_2\phi_2}) = 1.$$

Thus $\kappa_3 V\phi_3(fg/4(1+\varepsilon)^2) \leq 1$, $|||gf|||_{\kappa_3\phi_3} \leq 4(1+\varepsilon)^2$ and the theorem follows by letting $\varepsilon \to 0$.

COROLLARY 10. *Under the same assumption, if $f \in \kappa_1\phi_2 BV_0$ and $g \in \kappa_2\phi_1 BV_0$, then the product $fg/2$ is in $\kappa_3\phi_3 BV_0$ and*

$$|||fg|||_{\kappa_3\phi_3} \leq 4|||f|||_{\kappa_1\phi_2}|||g|||_{\kappa_2\phi_1}.$$

PROOF:

$$\frac{\sum \phi_{3n}(|f(I_n)g(I_n)/2|)}{\sum \kappa_3(|I_n|/b-a)} \leq \frac{\sum \phi_{1n}(|g(I_n)|) + \sum \phi_{2n}(|f(I_n)|)}{\sum \kappa_1(|I_n|/b-a) + \sum \kappa_2(|I_n|/b-a)}$$
$$\leq \frac{\sum \phi_{1n}(|g(I_n)|)}{\sum \kappa_2(|I_n|/b-a} + \frac{\sum \phi_{2n}(|f(I_n)|)}{\sum \kappa_1(|I_n|/b-a)} < \infty$$

Thus $fg/2 \in \kappa_3\phi_3 BV_0$. Let $\varepsilon > 0$. Without loss of generality, we may assume $|||f|||_{\kappa_1\phi_2} = 1 = |||g|||_{\kappa_2\phi_1}$. By the similar way as the above,

$$\frac{\sum \phi_{3n}\left(\frac{|f(I_n)g(I_n)|}{4(1+\varepsilon)^2}\right)}{\sum \kappa_3(|I_n|/b-a)} \le \frac{\frac{1}{2}\sum \phi_{2n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_1(|I_n|/b-a)} + \frac{\frac{1}{2}\sum \phi_{1n}\left(\frac{|g(I_n)|}{1+\varepsilon}\right)}{\sum \kappa_2(|I_n|/b-a)}$$
$$\le \frac{1}{2}(|||f|||_{\kappa_1\phi_2} + |||g|||_{\kappa_2\phi_1}) = 1.$$

Thus $\kappa_3 V\phi_3(fg/4(1+\varepsilon)^2) \le 1$. $|||fg|||_{\kappa_3\phi_3} \le 4(1+\varepsilon)^2$ and the corollary follows by $\varepsilon \to 0$.

COROLLARY 11. *Suppose that $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\}$ and $\phi_3 = \{\phi_{3n}\}$ satisfy, for all n, $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x) \le k\phi_{3n}^{-1}(x)$, and there exist κ-function κ_1, κ_2 and κ_3 such that $\kappa_1^{-1}(x)\kappa_2^{1}(x) \ge \kappa_3^{-1}(x)$. Then for all $f \in \kappa_1\phi_1 BV_0$ and $g \in \kappa_2\phi_2 BV_0$, the product $fg/2k$ is in $\kappa_3\phi_3 BV_0$ and*

$$|||fg|||_{\kappa_3\phi_3} \le 4k|||f|||_{\kappa_1\phi_1}|||g|||_{\kappa_2\phi_2}.$$

REMARK 2: If κ_1 and κ_2 are κ-function, then the composite function $\kappa_1 \circ \kappa_2$ is a κ-function, which is proved by the definition of κ-function. For example; $\kappa_i \circ \kappa_j$ is κ-function for $i \ne j$, $i = 1, 2, 3$. Here

$$\kappa_1(x) = \begin{cases} x(1-\log x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$
$$\kappa_2(x) = x^{\alpha} \qquad \text{for } 0 < \alpha < 1,$$

and

$$\kappa_3(x) = \left(1 - \frac{1}{2}\ln x\right)^{-1}$$

We now return to the space κBV_0 and BV_0 on the closed interval $[a, b]$. If $f \in BV_0[a, b]$, then f can be decomposed as $f = f_1 - f_2$, where f_1 and f_2 are increasing and $f_1(a) = f_2(a) = 0$. A particular example of such a decomposition is that f_1 and f_2 are the positive and negative variation of f, respectively, which is called the elementary decomposition of f. For any such decomposition of f, $f_1(b) + f_2(b) \ge V_a^b(f)$. If it is the elementary decomposition of f, we have the equality in the above inequality. By these properties, we have an simple proof of elementary theorem as the followings;

THEOREM 12. *Under the concepts of the elementary decomposition, we may have that* $V_a^b(fg) \leq V_a^b(f) \cdot V_a^b(g)$ *for any* f *and* g *such that* $f(a) = g(a) = 0$.

PROOF: Let $f = f_1 - f_2$ and $g = g_1 - g_2$ be the elementary decompositions of f and g, respectively. Then

$$\begin{aligned} fg &= (f_1 - f_2) \cdot (g_1 - g_2) \\ &= (f_1 g_1 + f_2 g_2) - (f_1 g_2 + f_2 g_1). \end{aligned}$$

By the inequality $V_a^b(f) \leq f_1(b{+}f_2(b)$, we have the followings;

$$\begin{aligned} V_a^b(fg) &\leq (f_1 g_1 + f_2 g_2)(b) + (f_1 g_2 + f_2 g_1)(b) \\ &= (f_1(b) + f_2(b))(g_1(b) + g_2(b)) \\ &= V_a^b(f_1 + f_2) V_a^b(g_1 + g_2) \\ &= V_a^b(f) V_a^b(g). \end{aligned}$$

COROLLARY 13. *Under the same condition as the above theorem, we have that* $\kappa V_a^b(fg) \leq \kappa V_a^b(f) \cdot \kappa V_a^b(g)$ *for any* f *and* g *in* κV_a^b *with* $f(a) = g(a) = 0$.

REMARK 3: $BV \subsetneq \phi BV$.

REMARK 4: $\kappa BV \subsetneq \kappa\phi BV$.

REMARK 5: $BV \subsetneq \kappa BV$ in [].

REMARK 6: $\phi BV \subsetneq \kappa\phi BV$ (By Remark 3).

REMARK 7: $\kappa D \subsetneq \kappa BV$.

REMARK 8: $\kappa D \subsetneq \kappa\phi D$ (By Remarks 4 and 7).

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