產業技術硏究(江原大學校 產業技術硏究所 論文集) 第 9 輯(1989)
Research Report, The Institute of Industrial Technology, Kangweon Nat'l Univ., Korea, Vol. 9, 1989

On the Singularity of the Matrix Sign Function Algorithm

Hyoung Joong KIM and Jang Gyu LEE

행렬부호함수의 특이성에 관한 연구

김 형 중* · 이 장 규**

요 약

순허수 고유치를 포함하고 있는 행렬이 행렬부호함수 알고리즘에서 보이는 성질을 규명했다. 역행렬이 존재하는 행렬도 이 알고리즘에서는 행렬의 조건과 무관하게 특이행렬이 될 수 있음을 보였다. 이 특성을 이용해서 이론적으로 모든 고유치를 알아낼 수 있다.

Introduction

Matrix sign function algorithm has been used widely in the various systems engineering fields[1-5]. The standard matrix sign function algorithm proposed by Roberts[1] is represented by the following recursive equation

$$S_{k+1}=\frac{1}{2}(S_k+S_k^{-1}) \qquad (1)$$

Where $S_0=A$. Then this algorithm can compute $sign(A)$. Roberts[1] suggested that convergence of the standard algorithm (1) can be improved by using the recursive equation

$$S_{k+1}=\alpha_k S_k+\beta_k S_k^{-1} \qquad (2)$$

with suitably selected scalars α_k and β_k. Balzer[6] generalized the selection method of α_k and β_k under the constraints that

$$\alpha_k+\beta_k=1 \text{ and } \lim_{k\to\infty}\alpha_k=\lim_{k\to\infty}\beta_k=\frac{1}{2}$$

* 강원대학교 제어계측공학과 전임강사
** 서울대학교 제어계측공학과 부교수

and proposed the optimal values.

It is easy to show that this algorithm is satisfactory for complex and repeated eigenvalues. However, the algorithm is undefined in the case where a matrix A has either zero or purely imaginary eigenvalues. Since a zero or a pair of purely imaginary eigenvalues may not guarantee the algorithm convergence, a property of the matrix sign function algorithm for the matrix having these eigenvalues should be clarified. It can be demonstrated that the matrix sign function algorithm results a singular matrix though an S_0 is not singular, if the matrix contains the purely imaginary eigenvalues. Such singularities are independent of the condition of the matrix A.

Main result

A singular matrix can be easiy identified in the matrix sign function algorithm because its inverse matrix does not exist. A matrix having at least one purely imaginary eigenvalues can also be easily identified in the matrix sign function algorithm due to the methods which will be stated hereafter.

Lemma 1. If an $n \times n$ matrix A has eigenvalue pairs $\pm jm$, then det $(A^2+m^2 I)=0$.

Proof. Assume that A has an eigenvalue pair $\pm jm$. Then its pseudo - Jordan canonical form $M^{-1}AM$ is given by

$$M^{-1}AM = \begin{bmatrix} A_c & 0 \\ 0 & J_c \end{bmatrix}$$

where J_c is an $(n-2) \times (n-2)$ Jordan canonical form and

$$A_c = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix}. \tag{3}$$

Now,

$$M^{-1}(A^2+m^2 I)M = M^{-1}\begin{bmatrix} A_c^2+m^2 I & 0 \\ 0 & J_c^2+m^2 I \end{bmatrix}M,$$

where $A_c^2+m^2 I=0$. Thus, det

$$(A^2+m^2 I) = 0.$$

It equally holds for the matrix having repeated eigenvalue pairs $\pm jm$. It completes the proof. Q. E. D.

Theorem 1. A nonsingular matrix A has at least one eigenvalue pair $\pm jm$ if and only if $(A+m^2 A^{-1})$ is singular.

Proof. (*If part*) Let Λ be a pseudoJordan canonical form of A. Assume that A has at least one real eigenvalue a. Then, $(\Lambda+m^2\Lambda^{-1})$ has an entry $(a^2+m^2)/a$ which cannot be zero unless $a=0$. Since A is nonsingular by the assumption, $a \neq 0$.

Assume that A has an eigenvalue pair $a \pm jb$. Then A has a following pseudoJordan block of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Thus, $(A+m^2A^{-1})$ has a following pseudoJordan block

$$\begin{bmatrix} a+m^2\dfrac{a}{a^2+b^2} & b-m^2\dfrac{b}{a^2+b^2} \\ -b+m^2\dfrac{b}{a^2+b^2} & a+m^2\dfrac{a}{a^2+b^2} \end{bmatrix}$$

The above pseudo-Jordan block becomes singular if the following relation

$$(a+m^2\frac{a}{a^2+b^2})^2+(b-m^2\frac{b}{a^2+b^2})^2=0$$

holds. The pseudo-Jordan block of $(A+m^2A^{-1})$ becomes singular only when $a=0$ and $b=m$ since A is nonsingular. It equally holds for the matrix having repeated eigenvalue pairs $\pm jm$, too.

(*Only if part*) Since A has at least on eigenvalue pair $\pm jm$, its characteristic polynomial $p(s)$ is of the form

$$p(s)=(s^2+m^2)\,q(s).$$

By Cayley - Hamition Theorem,

$$p(A)=(A^2+m^2I)\,q(A)=0.$$

From Lemma 1, $\det(A^2+m^2I)=0$. Since A is nonsingular, we have

$$det\ (A+m^2A^{-1})=0.$$

It equally holds for the matrix having repeated eigenvalue pairs $\pm jm$, too. It completes the proof. Q. E. D.

The above Theorem 1 states that although a matrix A is not singular, S_1 can be singular. That is to say, a nonsigular matrix having at least on purely imaginary eigenvalue pair $\pm j1$ in the standard matrix sign function algorithm (1) or $\pm j\sqrt{\beta_0/\alpha_0}$, in the accelerated matrix sign function algorithm (2) generates a singular matrix S_1.

Assume that A_c has a pair of purely imaginary eigenvalue pair $\pm jm_0$ having the form (3) such that

$$A_c=\begin{bmatrix} 0 & m_0 \\ -m_0 & 0 \end{bmatrix}$$

Then the standard matrix sign function algorithm (1) for the A_c is of the form

$$S_{k+1}=\frac{1}{2}\begin{bmatrix} 0 & m_k-\dfrac{1}{m_k} \\ -m_k+\dfrac{1}{m_k} & 0 \end{bmatrix},S_0=A_c$$

Thus,

$$m_{k+1}=\frac{1}{2}(m_k-m_k^{-1}). \qquad (4)$$

If $m_k=1$, then $m_{k+1}=0$ from (4). That is to say, if a matrix A_c has an eigenvalue pair $\pm j1$, then S_1 of (1) becomes singular. A sequence w_k that drives m_{k+1} with $m_0=w_k$ to be zero can be identified from the inverse mapping of (4) such that

$$w_{k+1}=w_k\pm\sqrt{u_k},\quad w_0=\pm 1 \tag{5}$$

with $u_k=w_k^2+1$. Thus with $S_0=A_c$ having a pair of purely imaginary eigenvalues $\pm jw_k$obtained from the above recursive equation (5), S_{k+1} becomes singular in the standard matrix sign function algorthm (1).

Similarly, the accelerated matrix sign function algorithm (2) for the A_c is of the form

$$S_{k+1}=\frac{1}{2}\begin{bmatrix} 0 & \alpha_k m_k-\frac{\beta_k}{m_k} \\ -\alpha_k m_k+\frac{\beta_k}{m_k} & 0 \end{bmatrix},S_0=A_c$$

Thus,

$$m_{k+1}=\frac{1}{2}(\alpha_k m_k-\beta_k m_k^{-1}) . \tag{6}$$

If $m_k=\pm j\sqrt{\beta_0/\alpha_0}$, then $m_{k+1}=0$ from (6). That is to say, if a matrix A_c has an eigenvalue pair $\pm j\sqrt{\beta_0/\alpha_0}$, then S_1 of (1) becomes singular. A sequence w_k that m_{k+1} with $m_0=w_k$ to be zero can be identified form the inverse mapping of (4) such that

$$w_{k+1}=(w_k\pm\sqrt{u_k})\,\alpha_k^{-1}\ ,$$
$$w_0=\pm\sqrt{\beta_0/\alpha_0} \tag{7}$$

with $u_k=w_k^2+\alpha_k\beta_k$. Thus with $S_0=A_c$ having a pair of purely imaginary eigenvalues $\pm jw_k$ obtained from the above recursive equation (7), S_{k+1} becomes sigular in the accelerated matrix sign function algorithm(2).

The A_c that makes S_{k+1} to be singular is not unique. Since the w_k's have two vlues in the equations (5) and (7), the number of w_k's that drives S_{k+1} into a singular matrix is 2^{k+1}. At any rate, the purely imaginary eigenvalue pair can be identified and located at the $(k+1)$th step in the matrix sign function algorithms (1) and (2) or from the Theorem 1. It can be applied to identify and locate the ordinary eigenvalue pair $\lambda\pm jm$ theoretically by shifting the original eigenvalue pair $\pm jm$ by λ such that

$$A_c+\lambda I$$

where λ is a known scalar.

Conclusion

Some properties concerning the purely imaginary eigenvalues in the matrix sign function algorithm have been explicated. It should be mentioned that a nonsigular matrix can generate a singular matrix in the matrix sign function algorthm independently of the matrix condition. That is to say, even a well-condtioned matrix can generate a singular matrix in the matix sign function algorithms (1) and (2). These properties can be used to identify and locate all the eigenvalues of a matrix theoretically.

REFERENCE

1. ROBERTS, J. D., 'Linear model reduction and solution of the algebraic Riccati equation by use of the sign function,' *Int. J. Control*, 1980, 32, pp. 677~687.
2. DENMAN, E. D. and BEAVERS, Jr. A. N., 'The matrix sign function and computations in systems,' *Appl. Math. Comput.*, 1976, 2, pp. 63~94.
3. MATTHEYS, R. L., 'Stability analysis via the extended matrix sign function,' *Proc. IEE*, 1978, 125, pp. 241~243.
4. SHIEH, L. S., TSAY, Y. T. and WANG, C. T., 'Matrix sector functions and their applications to systems theory,' *IEE Proc. Pt. D*, 1984, 131, pp. 171~181.
5. SHIEH, L. S., DIB, H. M. and YATES, R. E., 'Separation of matrix eigenvalues and structural decomposition of large-scale systems,' *IEE Proc. Pt. D*, 1986, 133, pp. 90~96.
6. BALZER, L. A., 'Accelerated convergence of the matrix sign function method of solving Lyapunov, Riccati and other matrix equations,' *Int. J. Control*, 1980, 32, pp. 1057~1078.