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Convergence of Dual Space Valued Pettis Martingales

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I. Introduction

Vector valued martingales first appeared in the early work of N. Dunford and B.J.Pettis [5] and R.S.Phillips [7]. In 1950, detailed studies of martingales were initiated by J.L.Doob. The subject of convergence of martingales of functions with values in a Banach space was treated by F.S.Scalora [8] and S.D.Chatterji [3] who independently showed that a martingale of functions with values in a reflexive Banach space obeys the same basic convergence theorems as martingales of real or complex valued functions. J.J.Uhl [9] studied mean convergence martingales of measurable Pettis integrable functions.
He proved that for a martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ in $P(\mu, X)$ the following conditions are equivalent:

(a) $\lim_\tau f_\tau$ exists in Pettis norm.
(b) There exists $f \in P(\sigma(\cup_\tau \Sigma_\tau), X)$ such that $(P) - E(f|\Sigma_\tau) = f_\tau$ for all $\tau \in T$.
(c) There exists $f \in P(\sigma(\cup_\tau \Sigma_\tau), X)$ such that

$$\lim_\tau (P) - \int_E f_\tau d\mu = (P) - \int_E f d\mu, \quad E \in \cup_\tau \Sigma_\tau.$$

Recently, the notion of weak* martingale was introduced by E.M.Bator [1,2]. And E.M.Bator studied uniformly bounded X^* valued martingales and various types of convergence of these martingales.
J.Diestel and J.J.Uhl [4] proved that a martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ in $L_P(\mu, X)$ converges in $L_P(\mu, X)$-norm if and only if there exists an $f \in L_P(\mu, X)$ such that for each $E \in \cup_\tau \Sigma_\tau$ one has

$$\lim_\tau (P) - \int_E f_\tau d\mu = (P) - \int_E f d\mu.$$

In this paper, we have some properties of dual space valued martingales using the results of E.M.Bator.

II. Preliminaries

Let (Ω, Σ, μ) be a finite measure space and X a separable Banach space with the successive duals X^*, X^{**}, X^{***}. Let $B(Z)$ be the closed unit ball of any Banach space Z.
A function $f : \Omega \to X^*$ is called simple if there exist $x_1^*, x_2^*, \ldots, x_n^*$ in X^* and $E_1, E_2, \ldots, E_n$ in Σ such that $f = \sum_{i=1}^n x_i^* \chi_{E_i}$ where χ_{E_i} is the characteristic function of E_i.
A function $f : \Omega \to X^*$ is called strongly μ-measurable if f is the limit of a sequence of simple functions almost everywhere, that is, there exists a sequence of simple functions (f_n) with $\lim_n \|f_n - f\| = 0$ almost everywhere.
A function $f : \Omega \to X^*$ is said to be weakly μ-measurable if for each $x^{**} \in X^{**}$ the numerical function $x^{**} f$ is strongly μ-measurable.

A weakly μ-measurable function $f : \Omega \to X^*$ is called uniformly bounded if there exists $M > 0$ such that $|x^{**}f| \leq M\|x^{**}\|$ almost everywhere for each $x^{**} \in X^{**}$.

A strongly μ-measurable function $f : \Omega \to X^*$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0.$$

In this case $\int_E f d\mu$ is defined for each $E \in \Sigma$ by $\int_E f d\mu = \lim_n \int_E f_n d\mu$ where $\int_E f_n d\mu$ is defined in obvious way.

The symbol $L_1(\Omega, \Sigma, \mu, X^*)$, for short $L_1(\mu, X^*)$, will stand for all equivalence classes of Bochner integrable functions $f : \Omega \to X^*$ such that $\|f\|_1 = \int_\Omega \|f\| d\mu < \infty$. Normed by the functional $\| \cdot \|_1$ defined above $L_1(\mu, X^*)$ becomes a Banach space. The symbol $L_1(\mu)$ will always mean $L_1(\mu, X^*)$ for $X^* =$ scalars. In particular L_1 stands for tha space $L_1([0,1], \Sigma, \mu)$ where Σ is the σ-field of Lebesgue measurable subsets of $[0,1]$ and μ is the Lebesgue measure.

If $f : \Omega \to X^*$ is a weakly μ-measurable function such that $x^{**}f \in L_1(\mu)$ for each x^{**} in X^{**}, then f is called Dunford integrable. It can be shown by the closed graph argument [4] that for every $E \in \Sigma$ there exists $x_E^{***} \in X^{***}$ such that $x_E^{***}(x^{**}) = \int_E x^{**} f d\mu$. Hence x^{***} is called the Dunford integral of f over E. And we write

$$x_E^{***} = (D) - \int_E f d\mu.$$

In the case that $(D) - \int_E f d\mu$ is a member of X^* for all $E \in \Sigma$, then f is called Pettis integrable and we write $(P) - \int_E f d\mu$ instead of $(D) - \int_E f d\mu$ to denote the Pettis integral of f over $E \in \Sigma$.

The symbol $P(\Omega, \Sigma, \mu, X^*)$, for short $P(\mu, X^*)$, will denote the space of all weakly equivalence classes of Pettis integrable functions $f : \Omega \to X^*$, endowed with the following norm $\|f\|_P = \sup\{\int_\Omega |x^{**}f| d\mu : x^{**} \in B(X^{**})\}$. We say that the symbol $\| \cdot \|_P$ is the Pettis norm.

Let Σ_0 be a sub-σ-field of Σ and $f \in L_1(\mu, X^*)$. An element g of $L_1(\mu, X^*)$ is called the conditional expection of f relative to Σ_0 if g is strongly μ-measurable with respect to Σ_0 and $\int_E g d\mu = \int_E f d\mu$ for each $E \in \Sigma_0$. In this case g is denoted by $E(f|\Sigma_0)$. Similarly if $f, g \in P(\mu, X^*)$ with g weakly μ-measurable with respect to Σ_0 and $(P) - \int_E g d\mu = (P) - \int_E f d\mu$ for all $E \in \Sigma_0$, then g is said to be the Pettis conditional expectation of f with respect to Σ_0, usually denoted $g = (P) - E(f|\Sigma_0)$.

When $X = X^* = R$, then the above mentioned conditional expectations are the same. In this case we denote the scalar valued conditional expectation of f with respect to a sub-σ-field Σ_0 of Σ as $\widehat{E}(f|\Sigma_0)$. It is easy to show that $\widehat{E}(f|\Sigma_0)$ exists and that $\|\widehat{E}(f|\Sigma_0)\|_1 \leq \|f\|_1$ [4]. Hence $\widehat{E}(\cdot|\Sigma_0)$ is a linear contraction on $L_1(\mu)$.

Whenever we want $(f_\tau, \tau \in T)$ to be a net of uniformly bounded functions $f_\tau : \Omega \to X^*$, we simply assume that each takes its range in $B(X^*)$.

Let $(\Sigma_\tau, \tau \in T)$ be a monotone increasing net of sub-σ-fields of Σ, that is, $\Sigma_{\tau_1} \subset \Sigma_{\tau_2}$ for $\tau_1 \leq \tau_2$ in T. A net $(f_\tau, \tau \in T)$ in $L_1(\mu, X^*)$ over the same directed set T is a (Bochner) martingale if for all $\tau, \tau_1 \in T$,

(a) f_τ is strongly μ-measurable with respect to Σ_τ, and
(b) if $\tau \geq \tau_1$, then $f_{\tau_1} = E(f_\tau|\Sigma_{\tau_1})$.

Now if $f_\tau \in P(\mu, X^*)$ for all $\tau \in T$, then $(f_\tau, \Sigma_\tau, \tau \in T)$ is called a Pettis martingale if for all $\tau, \tau_1 \in T$,

(a) f_τ is weakly μ-measurable with respect to Σ_τ, and
(b) if $\tau \geq \tau_1$, then $f_{\tau_1} = (P) - E(f_\tau|\Sigma_{\tau_1})$.

Usually a martingale of the above mentioned form will be denoted by $(f_\tau, \Sigma_\tau, \tau \in T)$ to display both the functions and sub-σ-fields involved.

We refer to the following classical result [1].

Theorem 1. (a) *If $(f_\tau, \Sigma_\tau, \tau \in T)$ is a uniformly bounded scalar valued martingale, then there exists an $f \in L_1(\mu)$ such that f_τ converges to f in L_1 norm.*
(b) *If $(f_n, \Sigma_n, n \in N)$ is a uniformly bounded scalar valued sequential martingale, then there exists an $f \in L_1(\mu)$ such that f_n converges to f almost everywhere.*

We continue with the following definitions.

A Banach space X^* is said to have the Radon-Nikodym property if for every measure space (Ω, Σ, μ) and every μ-continuous vector measure $G : \Sigma \to X^*$ of bounded variation there exists $g \in L_1(\mu, X^*)$ such that $G(E) = \int_E g d\mu$ for all $E \in \Sigma$. If $g \in P(\mu, X^*)$ such that $G(E) = (P) - \int_E g d\mu$ for all $E \in \Sigma$, then X^* has the weak Radon-Nikodym property.

All notions and notation used in this paper and not defined can be found in [1,4].

III. Convergence of dual space valued Pettis martingales

In [4], the following basic theory of Banach space valued martingales of Bochner integrable functions was discussed.

Lemma 1. *Let Σ_0 be a sub-σ-field of Σ. Then $\widehat{E}(f|\Sigma_0)$ exists for every $f \in L_1(\mu)$, and $||\widehat{E}(f|\Sigma_0)||_1 \leq ||f||_1$. Consequently $\widehat{E}(\cdot|\Sigma_0)$ is a linear contraction on $L_1(\mu)$.*

Theorem 2. *Let Σ_0 be a sub-σ-field of Σ. Then $E(f|\Sigma_0)$ exists for every $f \in L_1(\mu, X)$.*

Theorem 3. *A martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ in $L_1(\mu, X)$ converges in $L_1(\mu, X)$-norm if and only if there exists $f \in L_1(\mu, X)$ such that for each $E \in \cup_{\tau \in T} \Sigma_\tau$ one has* $\lim_\tau \int_E f_\tau d\mu = \int_E f d\mu$.

By using a similar argument we show that a Pettis conditional expectation exists for weakly μ-measurable functions $f : \Omega \to X^*$.

Definition 4: A martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ converges to f in $P(\mu, X^*)$ if there exists $\varepsilon > 0$ such that $||f_\tau - f||_p < \varepsilon$, that is,

$$\lim_\tau ||f_\tau - f||_p = 0.$$

Thus we say f_τ converges to f in $P(\mu, X^*)$ if $x^{**} f_\tau$ converges to $x^{**} f$ in $L_1(\mu)$ for every $x^{**} \in X^{**}$.

Lemma 5. *Let $(f_\tau, \Sigma_\tau, \tau \in T)$ be a Pettis martingale and let f be weakly μ-measurable. If f_τ converges to f in $P(\mu, X^*)$, then $(P) - E(f|\Sigma_\tau) = f_\tau$ for every $\tau \in T$.*

Proof: If f_τ converges to f in $P(\mu, X^*)$, then $x^{**} f_\tau$ converges to $x^{**} f$ in $L_1(\mu)$ for every $x^{**} \in X^{**}$. Thus, by Theorem II-1, a scalar valued martingale $(x^{**} f_\tau, \Sigma_\tau, \tau \in T)$ has an $L_1(\mu)$ limit $x^{**} f$, we have $\widehat{E}(x^{**} f|\Sigma_\tau) = x^{**} f_\tau$ for every $\tau \in T$. This says $(P) - E(f|\Sigma_\tau) = f_\tau$.

Theorem 6. *Let Σ_0 be a sub-σ-field of Σ and let $f : \Omega \to X^*$ be bounded and weakly μ-measurable. Then $(P) - E(f|\Sigma_0)$ exists.*

Proof: Define $\Pi_0 = \{\pi : \pi$ is a partition of Ω into a finite number of elements of $\Sigma_0\}$ and direct Π_0 by refinement. Define for every $\pi \in \Pi_0$ $f_\pi = \sum_{E\in\pi} \frac{(P)-\int_E f d\mu}{\mu(E)} \chi_E$. Then, letting $\Sigma_\pi \sigma$-field generated by $\pi, (f_\pi, \Sigma_\pi, \pi \in \Pi_0)$ is a Pettis martingale, Thus there is a weakly μ-measurable function g such that f_π converges to g in $P(\mu, X^*)$[1]. Hence $f_\pi = (P) - E(g|\Sigma_\pi)$. Let $E \in \Sigma_0$ and $\pi_E = \{E, \Omega \backslash E\}$, then $(P) - \int_E g d\mu = (P) - \int_E f_{\pi_E} d\mu = (P) - \int_E f d\mu$. Therefore $g = (P) - E(f|\Sigma_0)$.

Lemma 7[6]. *Let Σ_0 be a sub-field of Σ such that the σ-field generated by Σ_0 is Σ. Then the linear span of the set $\{x^*\chi_E : x^* \in X^*, E \in \Sigma_0\}$ is dense in $P(\mu, X^*)$.*

Theorem 8. *A Pettis martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ converges in $P(\mu, X^*)$ if and only if there exists a Pettis integrable function $f : \Omega \to X^*$ such that for each $E \in \cup_{\tau\in T}\Sigma_\tau$ one has $\lim_\tau (P) - \int_E f_\tau d\mu = (P) - \int_E f d\mu$.*

Proof: Suppose that $\lim_\tau f_\tau = f$ in $P(\mu, X^*)$. Then, since the operation defined for $g \in P(\mu, X^*)$ by $g \mapsto (P) - \int_E g d\mu$ is a bounded linear operator for each $E \in \cup_{\tau\in T}\Sigma_\tau$, it follows that

$$\lim_\tau (P) - \int_E f_\tau d\mu = (P) - \int_E f d\mu \text{ for all } E \in \cup_{\tau\in T}\Sigma_\tau.$$

For the converse, suppose that there is $f \in P(\mu, X^*)$ with $\lim_\tau (P) - \int_E f_\tau d\mu = (P) - \int_E f d\mu$ for all $E \in \cup_{\tau\in T}\Sigma_\tau$. Since $(\Sigma_\tau, \tau \in T)$ is an increasing net of σ-fields, $\cup_{\tau\in T}\Sigma_\tau$ is a sub-field of Σ. Without loss of generality, it will be assumed that the σ-field generated by $\cup_{\tau\in T}\Sigma_\tau$ is Σ.
Let $\varepsilon > 0$ be given. By Lemma 7, there exists a function $f_\varepsilon = \Sigma x_i^* \chi_{E_i}$, $x_i^* \in X^*$. $E_i \in \cup_{\tau\in T}\Sigma_\tau$, such that $\|f_\varepsilon - f\|_p < \frac{\varepsilon}{2}$. Since $(\Sigma_\tau, \tau \in T)$ is an increasing net, there exists a $\tau_0 \in T$ such that for all $\tau \geqq \tau_0 \{E_i\}_{i=1}^n \subset \Sigma_\tau$.
Hence for $\tau \geqq \tau_0 (P) - E(f_\varepsilon|\Sigma_\tau) = f$. Moreover for $\tau \geqq \tau_1 (P) - \int_E f_\tau d\mu = (P) - \int_E f_{\tau_1} d\mu$ for each $E \in \Sigma_{\tau_1}$. Hence $(P) - E(f|\Sigma_\tau) = f_\tau$. Therefore for $\tau \geqq \tau_0$

$$\begin{aligned} \|f - f_\tau\|_p &\leqq \|f - f_\varepsilon\|_p + \|f_\varepsilon - f_\tau\|_p \\ &= \|f - f_\varepsilon\|_p + \|(P) - E(f_\varepsilon - f|\Sigma_\tau)\|_p \\ &\leqq 2\|f - f_\varepsilon\|_p < \varepsilon. \end{aligned}$$

The next Corollary is simply a translation of Theorem 8 into a form similar to many.

Corollary 9. *A Pettis martingale $(f_\tau, \Sigma_\tau, \tau \in T)$ is convergent in $P(\mu, X^*)$ if and only if there exists an $f \in P(\mu, X^*)$ such that $(P) - E(f|\Sigma_\tau) = f_\tau$ for all $\tau \in T$.*

Recall that f is said to be Σ_0-measurable if $f \in L_1(\Omega, \Sigma_0, \mu|\Sigma_0, X^*)$ where $f \in L_1(\mu, X^*)$ and Σ_0 is a sub-σ-field of Σ.

Corollary 10. *Let X^* have the weak Radon-Nikodym property. If $(f_\tau, \Sigma_\tau, \tau \in T)$ is a uniformly integrable Pettis martingale in $P(\mu, X^*)$ and $\sup \|f_\tau\|_p < \infty$, then $\lim_\tau f_\tau$ exists in $P(\mu, X^*)$.*

Proof: For $E \in \cup_{\tau\in T}\Sigma_\tau$, set $F(E) = \lim_\tau (P) - \int_E f_\tau d\mu$. Since $(f_\tau, \Sigma_\tau, \tau \in T)$ is uniformly integrable, $\lim_{\mu(E)\to 0} F(E) = 0$ on $\cup_{\tau\in T}\Sigma_\tau$. Furthermore if $\pi \subset \Sigma_\tau$ is a partition of Ω, then there exists an index τ_0 such that $\pi \subset \Sigma_{\tau_0}$. Consequently, one has

$$\Sigma_{E\in\pi}\|F(E)\| = \Sigma_{E\in\pi}\|(P) - \int_E f_{\tau_0} d\mu\| \leqq (P) - \int_\Omega \|f_{\tau_0}\| d\mu \leqq \sup \|f_\tau\|_p.$$

Hence F is of bounded variation on $\cup_{\tau\in T}\Sigma_\tau$. An appeal to [4] produces a μ-continuous vector measure G of bounded variation on Σ_0,the σ-field generated by $\cup_{\tau\in T}\Sigma_\tau$, such that $G(E) = F(E)$ for each $E \in \cup_{\tau\in T}\Sigma_\tau$. Since X^* has the weak Radon-Nikodym property, there is $f \in P(\mu|\Sigma_0, X^*)$ such that $G(E) = (P) - \int_E f d\mu$ for each $E \in \Sigma_0$. But if $E \in \cup_{\tau\in T}\Sigma_\tau$, then $\lim_\tau(P) - \int_E f_\tau d\mu = F(E) = G(E) = (P) - \int_E f d\mu$.

References

1. E.M. Bator, "Duals of separable Banach spaces," Ph.D. Thesis, Pennsylvania State University, 1983.
2. ________, *Pettis integrability and the equality of the norms of the weak* integral and the Dunford integral*, Proc. Amer. Math. Soc. **95** (1985), 265–270.
3. S. D. Chatterji, *Martingales of Banach-valued random variables*, Bull. Amer. Math. Soc. **66** (1960), 395–398.
4. J.Diestel and J.J. Uhl, Jr., *Vector measures Math. Surveys, No. 15*, Amer. Math. Soc., Providence, R.I. (1977).
5. N.Dunford and B.J.Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.
6. B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.
7. R.S.Phillips, *Integration in a convex linear topological space*, ibid. **47** (1940), 114–145.
8. F.S.Scalora, *Abstract martingale convergence theorems*, Pacific J. Math. **11** (1961), 347–374.
9. J.J.Uhi, Jr., *Martingales of strongly measurable Pettis integrable function*, ibid **167** (1972), 369–378; *Erratum*, ibid **181** (1973), 507.