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Topological Dynamics on the Circle*

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1. Introduction

Let I be a closed interval, $\mathbf{R}$ the real line, S^1 the circle and X a toplogical space, and $C^0(X)$ denote the set of continuous maps of X into itself. For any $f \in C^0(X)$, let $f^0 : X \to X$ be the identity, and define, inductively, $f^n = f \circ f^{n-1}$ for any positive integer n.
A point $x \in X$ is said to be a *periodic point* of f if for some positive integer n, $f^n(x) = x$. In this case, the least such n is called the *period* of x. A point of period one is called a *fixed point.* We denote the set of periodic points of f by $P(f)$ and the set of periods of f by $\mathrm{Per}(f)$. The orbit of x is the set $\{f^k(x) : k = 0, 1, 2, ...\}$, and denoted by $\mathrm{orb}(x)$. If x is a periodic point of period n, $\mathrm{orb}(x)$ contains exactly n points, each of which is a periodic point of period n. We refer to such an orbit as a periodic orbit of period n.

Recently, there is a growing interest in studying the periodic orbits of maps of one dimensional spaces. One of the most beautiful results in this area is a theorem of A.N.Sarkovskii. It is a theorem motivated by the question: if a map $f \in C^0(\mathbf{R})$ has a periodic point of period n, must f also have a periodic point of some other period k ?

Sarkovskii's theorem is as follows. Arrange the positive integers in the following sequence:

$$3 \to 5 \to 7 \to \cdots \to 2\cdot 3 \to 2\cdot 5 \to 2\cdot 7 \to \cdots \to$$
$$2^2\cdot 3 \to 2^2\cdot 5 \to 2^2\cdot 7 \to \cdots, \cdots, \to 2^3 \to 2^2 \to 2 \to 1.$$

Sarkovskii Theorem [10].

(a) *For any* $f \in C^0(\mathbf{R})$, *if* $n \in \mathrm{Per}(f)$, *then* $k \in \mathrm{Per}(f)$ *for all* k *with* $n \to k$.
(b) *There exists a map* $f \in C^0(\mathbf{R})$ *such that* $n \in \mathrm{Per}(f)$ *but* $k \notin \mathrm{Per}(f)$ *for any* $k \to n$.

Remark. *The above theorem for any* $f \in C^0(I)$ *is true* (see [6]).

Further research starting at this point can go in at least six directions:

1. Replace the real line $\mathbf{R}$ and $C^0(\mathbf{R})$ by another space and another class of maps (see[1] and [6]).
2. Replace periodic orbits by more general orbits (see[11]).
3. Investigate closer the behaviour of periodic orbits (see[4]).
4. Study other orbits in presence of periodic ones (see[2] and [5]).
5. Derive some information about topological entropy (see[2] and [6]).
6. Try to simplify the proofs and to generalize the theorem (see[3]).

There are many attempts to pursue these goals, for instance, see references of [11].

In this paper, we investigate the topological dynamics generated by continuous maps of the circle. In particular, we study some tools, useful for moving in the above fourth direction and use them to get results concerning a twist cycle (see §2 for detailed definitions) of the circle.

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2. Definitions and Preliminaries

We think of a periodic orbit $A = \{a_1 < \cdots < a_n\}$ of period n as a cycle permutation $\alpha : A \to A$ of an n-element set of real numbers, which is called a *cycle* of period n. Two cycles $\alpha : A \to A$ and $\beta : B \to B$ are said to be *equivalent* if $\alpha(a_i) = a_j$ if and only if $\beta(b_i) = b_j$. A set $A \subset S$ is said to be an f-*cycle* if $f|_A : A \to A$ is a cycle.

Formally, we will think of the circle S^1 as $\mathbf{R}/\mathbf{Z}$ and use the natural projection $e : \mathbf{R} \to S^1$ given by the formula $e(X) = \exp(2\pi i X)$. Thus every continuous map f of the circle has countably many liftings, i.e., continuous maps $F : \mathbf{R} \to \mathbf{R}$ satisfying $f \circ e = e \circ F$. We note that F is not defined uniquely, but if F and F' are liftings of the same map f, then $F = F' + k$ for some integer k. There is a unique integer N such that $F(X+1) = F(X) + N$ for all liftings F and all $X \in \mathbf{R}$, which is called the *degree* of f, denoted by $\deg(f)$. Clearly,

$$\deg(f^n) = (\deg(f))^n.$$

In this paper, we will consider maps of the circle of only degree one. From now on, we will denote by $C_1(S^1)$ the set of all continuous maps from S^1 to itself of degree one and by $C_1(\mathbf{R})$ the class of continuous maps $F : \mathbf{R} \to \mathbf{R}$ which are liftings of continuous maps of the circle of degree one.

As is customary in this subject, we deal with liftings of maps and cycles to the reals rather than with maps and cycles themselves. Let $f \in C_1(S^1)$, $F \in C_1(\mathbf{R})$ be a lifting of f and A be an f-cycle of period n. Then, putting $e^{-1}(A) = \{\cdots < X_0 < X_1 < \cdots\}$, we have

(1) $X_{i+n} = X_i + 1$;
(2) $F(X+1) = F(X) + 1$ for all $X \in \mathbf{R}$;
(3) For all $X_i, X_j \in e^{-1}(A)$, there are integers $m > 0$ and s such that $F^m(X_i) = X_j + s$;
(4) There is an integer k such that for all $X \in e^{-1}(A), F^n(X) = X + k$.

(4) follows from (1), (2) and (3). See Lemma 2.1 for the detailed proof.
We abstract this situation, keeping only $F|e^{-1}(A)$.

Difinition: A (degree one) cycle of period n is a map $\varphi : A \to A = \{\cdots < X_0 < X_1 < \cdots\} \subset \mathbf{R}$ such that

(1) $X_{i+n} = X_i + 1$;
(2) For all $X \in A$, $\varphi(X+1) = \varphi(X) + 1$;
(3) For all $X_i, X_j \in A$, there are integers $m > 0$ and s such that $\varphi^m(X_i) = X_j + s$,

We often suppress φ and refer to A as the cycle. The period of A is denoted by $|A|$.

A set $A \subset S^1$ is said to be a *twist periodic orbit* (or *twist cycle*) of f of period n if F is order preserving on the set $e^{-1}(A)$.

For maps of the circle of degree one, one of the main tools is the theory of rotation numbers. In [9], S.Newhouse, J.Palis and F.Takens introduced the concept of rotation interval for endomorphisms of the circle of degree one, which generalizes the notion of the Poincare rotation interval of orientation-preserving homeomorphisms of the circle. In particular, the rotation numbers for periodic points have been defined by [6],[7] and [5]. Since we are interested only in periodic points, we present this theory along the lines of [6] rather than [9].

Lemma 2.1. *Let $f \in C_1(S^1)$, $F \in C_1(\mathbf{R})$ be a lifting of f and A be an f-cycle of period n. Then there is an integer k such that for all $X \in e^{-1}(A)$, $F^n(X) = X + k$.*

Proof: Let $\widetilde{F}(X) = F(X) - [F(X)]$, where $[F(X)]$ denotes the largest integer not greater than $F(X)$. Then $\widetilde{F}$ is a cyclic permutation of the n-element set $e^{-1}(A) \cap [0,1)$. Therefore $\widetilde{F}^n(X) = X$ for all $X \in e^{-1}(A) \cap [0,1)$, and so for each $X \in e^{-1}(A)$, there is an integer k such that $F^n(X) = X + k$.

We shall call the number k/n of Lemma 2.1 the *rotation number* (or F-rotation number, if necessary) of $x = e(X)$ (resp.A) and we denote it by $\rho(x)$ or $\rho(F, x)$ (resp. $\rho(A)$). we denote by $L(f)$ or $L(F, f)$ the set of all rotation numbers of f. The following statements are known (see[2]):

(1) $\rho(x)$ does not depend on the choice of X. Actually, it depends on the periodic orbit.
(2) If $F' = F + m$, then $\rho(F', x) = \rho(F, x) + m$.
(3) $\rho(F^m, x) = m\rho(F, x)$.
(4) If $a < k/n < c$, $a, c \in L(f)$ and $k, n \in \mathbf{Z}$ with $n > 0$, then $k/n \in L(f)$ and $n \in P(f)$.
(5) $L(f) \cap \mathbf{Z} \neq \emptyset$ if and only if $1 \in P(f)$.
(6) If $a_m \in L(f)$ for $m = 1, 2, \ldots$ and $a = \lim_{m\to\infty} a_m \in \mathbf{Q}$, then $a \in L(f)$.
(7) If f has no periodic points, then $\lim_{n\to\infty} \frac{1}{n}(F^n(X) - X)$ exists for all X, it is independent of X and is irrational.

From the above statement, we can write $L(f) = [a, b] \cap \mathbf{Q}$ for some $a, b \in \mathbf{R}$. That is, $L(f)$ is a closed interval (perhaps degenerated to one point) on $\mathbf{Q}$ and it is called the *rotation interval* of f (or more precisely of F). If f has no periodic points, then the situation is very similar to the case of a homeomorphism and every point has the same rotation number. In this case $L(f)$ consists of this number. Now, we give a geometrical interpretation of a twist cycle on the circle.

Proposition 2.2. *Let A be a twist cycle with $\rho(A) = p/q$, $p \in \mathbf{Z}$, $q \in \mathbf{N}$ and* $\mathrm{GCD}(p, q) = 1$. *Let*

$$\cdots X_{-2} < X_{-1} < X_0 < X_1 < X_2 \cdots$$

all be elements of $e^{-1}(A)$. Then for all $i, j \in \mathbf{Z}$, we have

$$X_{i+qj} = X_i + j \text{ and } F(X_i) = X_{i+p}.$$

Proof: Since $\rho(A) = p/q$ and $\mathrm{GCD}(p, q) = 1$, A is a cycle of period nq for some $n \in \mathbf{N}$ and $F^{nq}(X_i) = X_i + np$ for all $i \in \mathbf{Z}$. Hence, for all $k \in \mathbf{Z}$ the number of elements of $e^{-1}(A) \cap [X_k, X_k + 1)$ is nq, and so $X_{k+nq} = X_k + 1$. Inductively, $X_{i+nqj} = X_i + j$ for all $i, j \in \mathbf{Z}$. Since A is a twist cycle, the map

$$F|_{e^{-1}(A)} : e^{-1}(A) \to e^{-1}(A)$$

is an order preserving bijection. Hence there exists $m \in \mathbf{Z}$ such that $F(X_i) = X_{i+m}$ for all $i \in \mathbf{Z}$. Since

$$F^{nq}(X_i) = X_i + np = X_{i+n2pq},$$

we can obtain $m = np$. Since A is a periodic orbit and $X_0, X_1 \in e^{-1}(A)$, for some $k \in \mathbf{N} \cup \{0\}$ and $r \in \mathbf{Z}$ we have $F^k(X_0) = X_1 + r$. Therefore $X_{knp} = X_{1+nqr}$, and hence $1 = n(kp - qr)$. Since $\mathrm{GCD}(p, q) = 1, n = 1$.

3. Forcing and twist cycles on the circle

Definition: (1) A cycle A *forces* a cycle B, denoted by $A \lhd B$, if every map in $C_1(\mathbf{R})$ which has a cycle equivalent to A also has a cycle equivalent to B. (2) Let $F \in C_1(\mathbf{R})$ and let A be an F-cycle. F is *A-monotone* if it is (not necessarily strictly) monotone between adjacent members of A.

Since whether $A \lhd B$ is determined by the Markov graph (see[6] or [4] for the definition) of A and the equivalence classes of A and B, we have

Lemma 3.1. *Let F be A-monotone. Then $A \lhd B$ if and only if F has a cycle equivalent to B.*

Lemma 3.2. *Let $F \in C_1(\mathbf{R})$. If F is piecewise polynomial and each piece is of degree at least two, then F has only finitely many cycles of each period.*

Proof: For given integer $n > 0$, since $\{F^n(X) - X : X \in [0,1]\}$ is compact, there are only finitely many integer k such that $F^n(X) = X + k$ has a solution in $[0,1]$. Fix such an k. Since $F^n - \mathrm{id}$ is piecewise polynomial, and none of its pieces is constatnt, there are only finitely many $X \in [0,1]$ satisfying $F^n(X) = X + k$.

Lemma 3.3. *For every cycle A, there is an A-monotone map $F \in C_1(\mathbf{R})$ which has only finitely many cycles of each period.*

Proof: Let $\varphi : A \to A = \{\cdots < X_{-1} < X_0 < X_1 < \cdots\}$ be a cycle. Let F agree with φ on A and be monotone quadratic on each $[X_i, X_{i+1}]$. By Lemma 3.2, F is an A-monotone map having only finitely many cycles of each period.

Definition: Let $F \in C_1(\mathbf{R})$, let $n > 0$ be an integer, and let $A = \{\cdots < X_0 < X_1 < \cdots\}$ be an F-cycle of period n. The *F-variation* of A is

$$\mathrm{var}_F(A) = \sum_{i=1}^{n} |F(X_i) - F(X_{i-1})|.$$

Note that it does not change if A is renumbered.

Lemma 3.4. *Let $F \in C_1(\mathbf{R})$ and let A be an F-cycle. If $A \lhd B$ but A is not equivalent to B, then F has a cycle B', equivalent to B, such that* $\mathrm{var}_F(B') < \mathrm{var}_F(A)$.

Proof: Let $A = \{\cdots < X_{-1} < X_0 < X_1 < \cdots\}$. $F/A = \varphi$ is not increasing. For otherwise if $\varphi^n(X) = X + k$ for all $X \in A$, then φ is equivalent to $\psi : \mathbf{Z}/n \to \mathbf{Z}/n = \{i/n : i \in \mathbf{Z}\}$, defined by $\psi(y) = y + r/n$, and so A forces only cycles equivalent to itself by Lemma 3.1.

Let $G \in C_1(\mathbf{R})$ be A-monotone. Since $A \lhd B$, G has a cycle $C = \{\cdots < Z_{-1} < Z_0 < Z_1 < \cdots\}$ equivalent to B. As in the proofs of [**3**, Theorem 3.3] and [**1**, Lemma 1.18], F has a cycle $B' = \{\cdots < Y_{-1} < Y_0 < Y_1 < \cdots\}$ equivalent to B, labelled in such a way that for all i, j, k, $X_i < Y_j < X_{i+1}$ if and only if $X_i < Z_j < X_{i+1}$, and $F(Y_j) = Y_k$ if and only if $G(Z_j) = Z_k$. It follows that if $X_i < Y_j < Y_{j+1} < \cdots < Y_{j+s} < X_{i+1}$ for some integer $s \geq 0$, then either

$$F(X_i) < F(Y_j) < \cdots < F(Y_{j+s}) < F(X_{i+1})$$

or

$$F(X_i) > F(Y_j) > \cdots > F(Y_{j+s}) > F(X_{i+1}),$$

because either

$$G(X_i) < G(Z_j) < \cdots < G(Z_{j+s}) < G(X_{i+1})$$

or

$$G(X_i) > G(Z_j) > \cdots > G(Z_{j+s}) > G(X_{i+1}).$$

Thus $\mathrm{var}_F(A) = \mathrm{var}_F(A \cup B')$. On the other hand, since $F|_A$ is not monotone, $\mathrm{var}_F(B') < \mathrm{var}_F(A \cup B')$.

Theorem 3.5. *$\lhd$ is a partial order on equivalence classes of cycles.*

Proof: It is obvious from the definition of $\lhd$ that $\lhd$ is reflexive and transitive on equivalence classes, so we only need to prove that $\lhd$ is antisymmetric.

Suppose that $A \lhd B$ and $B \lhd A$, but A is not equivalent to B. By Lemma 3.3, there is a map $F \in C_1(\mathbf{R})$ for which A is an F-cycle and which has only finitely many cycles of each period. Applying Lemma 3.4 to F inductively, F has cycles $A_0(= A), A_1, A_2, \ldots$, all equivalent to A, and cycles $B_1, B_2, \ldots$, all equivalent to B, such that for each $i \geq 0$,

$$\mathrm{var}_F(A_i) > \mathrm{var}_F(B_{i+1}) > \mathrm{var}_F(A_{i+1}).$$

This violates the choice of F.

Recall that a cycle $\varphi : A \to A$ is a twist cycle if φ is increasing. A twist cycle with period n and rotation number k/n is equivalent to the cycle $\psi : \mathbf{Z}/n \to \mathbf{Z}/n$, defined by $\psi(X) = X + k/n$. If A is a twist cycle of period n and rotation number k/n, then $\mathrm{GCD}(k, n) = 1$. In particular, there exists only one equivalence class of twist cycles with a given rotation number. Therefore, we have

Theorem 3.6. *Every cycle forces a twist cycle with the same rotation number.*

Proof: Let A be a cycle and let F be A-monotone. Then, by [2, Theorem A] (or [8, Theorem B]), F has a twist cycle with the same rotation number as A, which is equivalent to A.

Now, we will characterize a twist cycle on the circle:

Theorem 3.7. *The following properties about a cycle A are equivalent:*

(1) *A is a twist cycle;*
(2) *A is an almost twist cycle;*
(3) *A is a minimal almost twist cycle;*
(4) *Every cycle with rotation number $\rho(A)$ which is forced by A is equivalent to A;*
(5) *Every cycle which is forced by A is equivalent to A;*
(6) *There is a map in $C_1(\mathbf{R})$ whose only cycle is A.*

Proof: (6) $\Rightarrow$ (5): Let B be any cycle forced by A. then by assumption, there is a map $F \in C_1(\mathbf{R})$ whose only cycle is A. On the other hand, since $A \lhd B$, F has a cycle equivalent to B, and hence B is equivalent to A.

(5) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1): Let B be a twist cycle with a rotation number $\rho(A)$. Then, $A \lhd B$ by Theorem 3.6, and so B is equivalent to A by assumption. Since there exists only one equivalence class of twist cycles with the given rotation number, A is a twist cycle.

(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious from the definitions.

(3) $\Rightarrow$ (6): Let $\varphi : A \to A = \{\cdots < X_0 < X_1 < \cdots\}$ have a rotation number r/n. Then by Proposition 2.2, $\varphi(X_i) = X_{i+r}$ for all i. Let F agree with φ on A, be monotone quadratic on $[X_i, X_{i+1}]$ if $i \equiv 0 \pmod n$ and linear on $[X_i, X_{i+1}]$ if $i \not\equiv 0 \pmod n$. Then F is a map in $C_1(\mathbf{R})$ and its only cycle is A.

References

1. L.Alseda, J. Llibre and M.Misiurewicz, *Periodic orbits of maps of Y*, Trans. Amer. Math. Soc. 649 (1989), 475–538.
2. L.Alseda, J.Llibre, M.Misiurewicz and C.Simo, *Twist periodic orbits and topological entropy for continuous maps of the circle of degree one which have a fixed point*, Ergod. Th. & Dynam. Sys. 5 (1985), 501–517.
3. S.Baldwin, "Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions," preprint, Auburn University, 1985.
4. C.Bernhardt, *Simple permutations with order a power of two*, Ergod.Th. & Dynam. Sys. 4 (1984), 179–186.
5. L.Block, E.M.Coven, L.Jonker and M.Misiurewicz, "Primary cycles on the circle," preprint, University of Florida, Gainesville, 1989.
6. L.Block, J.Guckenheimer, M.Misiurewicz and L.S.Young, Lecture Notes in Mathematics 819, "Periodic points and topological entropy of one dimensional maps," Springer, New York, 1980, pp. 18–34.
7. M.Misiurewicz, *Periodic points of maps of degree one of a circle*, Ergod. Th. & Dynam. Sys. 2 (1982), 221–228.
8. M.Misiurewicz, *Twist sets for maps of the circle*, Ergod. Th. & Dynam. Sys. 4 (1984), 391–404.
9. S.Newhouse, J.Palis and F.Takens, *Bifurcations and stability for families of diffeomorphisms*, Publ. IHES 57 (1983), 5–72.
10. A.N.Sarkovskii, *Co-existence of cycles of a continuous mappings of the line into itself*, Ukrain.Mat. Zh. 16 (1964), 61–71.
11. S.K.Yang, "Topological dynaimcs on one dimensional spaces," Ph. D. thesis., Chungnam National University, Daejeon, Korea, 1988.