韓國數學教育學會誌<數學教育>
1989. 12, Vol. XXVIII, No. 2

Journal of the Korea Society of Mathematical Education
Dec. 1989. Vol. XXVIII, No.2

Characterizations of Pettis Integrable Functions

Dong-IL Rim

ChungBuk National University, Cheong Ju, Korea

1. Introduction

In this paper we shall give some characterizations of Pettis integrable functions. We used properties of the operator $T_f : X^* \to L^1(\mu)$ is defined by $T_f(x^*) = x^*f$ to give several equivalent relations for the Pettis integrable functions.

E.M.Bator, P.Lewis and D.Race [2] and R.E.Huff [14] used properties of operator T_f to study the Pettis integrable functions. R.E.Huff [14] gave a simple proof of the G.A.Edgar Pettis integrability criteria [8] in terms of the (w^*, w)-continuity of T_f and the action of T_f on the sets $K(F, \varepsilon)$.

R.F. Geitz and J.J.Uhl, Jr.[11] gave characterizations of the family $\{x^*f : x^* \in B_{x^*}\}$ for a bounded weakly measurable $f : \Omega \to X$.

M.Talagrand [16] gave some necessary and sufficent condition for Pettis integrable functions.

In this paper we shall modify the E.M.Bator [2] and R.E.Huff [14] as the following equivalent relations:

(1) f is Pettis integrable.
(2) The canonical map $\{x^*f : x^* \in B_{x^*}\} \to L^1(\mu)$ is pointwise to weak continuous.
(3) The map $x^* \to \int_A \langle f, x^* \rangle \, d\mu$ on B_{x^*} is weak* continuous for each $A \in \Sigma$.
(4) The above map is continuous at 0.
(5) $\{x^*f : x^* \in B_{x^*}\}$ is relatively weakly compact in $L^1(\mu)$ and $\{0\} = \{T(K(F,\varepsilon)) | F \subset X, F : \text{finite and } \varepsilon > 0\}$.

2. Preliminaries

Throughout this paper (Ω, Σ, μ) is a complete probability space, and let X be a Banach space with dual X^*.

A function $f : \Omega \to X$ is μ-measurable if it is the a.e. limit of a sequence of simple functions.

A function $f : \Omega \to X$ is Dunford integrable provided the composition

$$T(x^*) = x^*f \text{ is in } L^1(\mu) \text{ for every } x^* \text{ in } X^*.$$

In this case it follows from the closed graph theorem that $T : X^* \to L^1(\mu)$ is a bounded linear operator.

A function $f : \Omega \to X$ is a μ- Pettis integrable if T_f^* maps $L^\infty(\mu)$ into the canonical image of x in X^{**}.

An operator $T : X^* \to L^1(\mu)$ is said to be (w^*, w)- continuous provided that $(T(x^*_\alpha))$ converges to $T(x^*)$ in the weak topology of $L^1(\mu)$ whenever (x^*_α) is a net which converges to x^* in the weak* topology of X^*.

$B(\Sigma)$ is the space of all bounded mesurable functions on Σ equipped with the suprimum norm.

Let Σ be field of subsets of the set Ω and $F : \Sigma \to X$ be a vector measure. F is said to be strongly additive whenever given a sequence (A_n) of pairwise disjoint sets of Σ, the series $\sum_{n=1}^{\infty} F(A_n)$ converges in norm.

Let X be a Banach space and $f : \Omega \to X$ a function, then we say that f is weakly measurable with respect to μ if x^*f is μ- measurable for $x^* \in X^*$, we say weak $L^1(\mu)$ if $x^*f \in L^1(\mu)$ for all $x^* \in X^*$, and we say $\{x^*f : x^* \in B_{x^*}\} \subset L^1(\mu)$ is uniformly integrable if for each $\varepsilon > 0$ there is $\alpha > 0$ such that $\mu(A) < \alpha$ implies

$$\sup \int_A |x^*f| d\mu \leq \varepsilon.$$

When f is weak $L^1(\mu)$, the Dunford operator of f is the operator

$$U : L^\infty(\mu) \to X^{**} \text{ defined by}$$
$$\langle U_\phi, x^* \rangle = \int_\Omega \phi \langle f, x^* \rangle \, d\mu, \qquad \forall x^* \in X^*.$$

The mapping $x^* \to x^*f$ defines another useful operator, $T : X^* \to L^1(\mu)$. We see that U and T satisfy the equation $\langle U_\phi, x^* \rangle = \langle \phi, Tx^* \rangle$.
For f as above and $A \in \Sigma$ the Dunford integral of f over A is defined to be the element

$$m(A) = \int_A f d\mu = U(\chi_A) \in X^{**}.$$

The Dunford integral $A \to m(A)$ defines a finitely additive vector measure $m : \Sigma \to X^{**}$. We recall that the semivariation of m on $A \in \Sigma$ is defined

$$\|m\|(A) = \sup |x^*m|(A), \quad x^* \in B_{x^*},$$
$$B_{x^*} \text{is the unit ball in } X^*.$$

For $\phi \in M(\mu)$, the space of measurable functions on Ω, we denote the equivalence class

$$\{f \in M(\mu) : f = \phi \quad \mu - a.e.\} \text{ by } [\phi].$$

The map $\phi \to [\phi]$ defines a function $M(\mu) \to M^0(\mu)$ called the canonical injection.
Here $M^0(\mu)$ denotes the space of equivalence classes of functions $\phi \in M(\mu)$. The integral $\int_\Omega f d\mu$ of a measurable function $f \in L^1(\mu)$ is called the expectation of f and denoted by $E(f)$.
Our notation follows N.Dunford and J.T.Schwartz[6].

3. The Main Theorem

Proposition 1. *A Dunford integrable function f is Pettis integrable if and only if T is (w^*, w)- continuous.*
In particular, if f is Pettis integrable then T is necessary a weakly compact operator.

Proof: Suppose T is (w^*, w)- continuous. Let $A \in \Sigma$. Then $\chi_A \in L^\infty(\mu) = (L^1(w))^*$, so the linear functional $\alpha \in X^{**}$ defined by

$$< x^*, \alpha >= \int_A < f, x^* > d\mu = \int T_f(x^*)\chi_A d\mu =< T_f(x^*), \chi_A >$$

is weak* continuous, so there is $x_A \in X$ with $\langle x_A, x^* \rangle = \int_A \langle f, x^* \rangle \, d\mu$ for all $x^* \in X^*$. Thus f is Pettis integrable.

Conversely, suppose f is Pettis integrable. Then for $A \in \Sigma$ there exists $x \in X$ with

$$\langle x_A, x^* \rangle = \int_A \langle f, x^* \rangle \, d\mu \text{ for all } x^* \in X^*.$$

Thus

$$\langle T_f(x^*), \chi_A \rangle = \int_A \langle f, x^* \rangle \, d\mu = \langle x_A, x^* \rangle$$

is a weak*-continuous function of x^*. Hence $\langle T_f(x^*), h \rangle$ is a weak* continuous function of x^* for any simple function h. It follows that $\langle T_f(x^*), h \rangle$ is a weak* continuous function of x^* for any $h \in L^\infty(\mu)$. Hence T is (w^*, w)-continuous in $L^1(\mu)$.

Theorem 2. *Let $f : \Omega \to X$ be bounded and weakly measurable. Then f is Pettis integrable.*

Proof: Let H be any bounded linear operator on $L^1(\mu)$. Thus $S = T^*(H)$ given by $S(x^*) = H(T(x^*))$ is a bounded linear operator on X^*, as

$$|S(x^*)| = |H(T(x^*))| \leq \|H\| \|T\| \|x^*\|.$$

Let $x^* \in X^*$ be arbitrary and let (x_n^*) converge weakly to x_0^* in X^*.

$$\lim_{n\to\infty} H(T(x_n^*)) = \lim_{n\to\infty} S(x_n^*) = S(x_0^*) = H(T(x_0^*)).$$

Hence $T(x_n^*)$ converges weakly to $T(x_0^*)$ in $L^1(\mu)$. And so $\lim_{n\to\infty} \langle x_n^*, f \rangle = \langle x_0^*, f \rangle$ weakly in $B(\Sigma)$. We have $\lim_{n\to\infty} \int_A \langle x_n^*, f \rangle \, d\mu = \int_A \langle x_0^*, f \rangle \, d\mu$.
This proves that the functional $x^* \to \int_A \langle x^*, f \rangle \, d\mu$ is weak* continuous on X^*. Hence f is Pettis integrable.

Theorem 3. *Let f be μ- measurable Dunford integrable functions. Then f is Pettis integrable if and only if T is a weakly compact operator.*

Proof: Suppose f is a Pettis integrable. By proposition 1, T is (w^*, w)-continuous. Therefore T is weakly compact operator. Conversely, if a sequence (x_n^*) convergers weakly x_0^* in X^*, by processing in proposition 1. $T(x_n^*) \to T(x_0^*)$ weakly in $L^1(\mu)$. It follows that T is (w^*, w) continuous. Hence f is Pettis integrable.

Theorem 4. *Let $f : \Omega \to X$ be μ- measurable Dunford integrable with respect to μ. Then the following are equivalent:*

(a) *$T^*(y^*) \in X$ for every $y^* \in L^\infty(\mu)$.*
(b) *f is Pettis integrable.*
(c) *$\{x^* f : x^* \in B_{x^*}\}$ is a relatively weakly compact set of $B(\Sigma)$.*
(d) *T is (w^*, w)- continuous.*
(e) *T is weakly compact operator and*
$\{0\} = \cap\{T(K(F, \varepsilon)) | F \subset X, F : \text{finite and } \varepsilon > 0\}$.

Proof: (a) $\Rightarrow$ (b). Since $T^*(Y^*) \in X$, by definition of Pettis integrable, f is Pettis integrable.

(b) $\Rightarrow$ (c). Let $(x^*_\alpha f)$ be a net in $\{x^* f : x^* \in B_{x^*}\}$ and select a weak*-convergent subset of (x^*_n) of (x^*_α). Let x^*_0 be the weak*- limit of (x^*_n).
Since f is Pettis integrable,

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \langle x^*_n, f\rangle \, d\mu &= \lim_{n\to\infty} \left\langle x^*_n, p - \int_\Omega f d\mu \right\rangle \\ &= \left\langle x^*_0, p - \int_\Omega f d\mu \right\rangle \\ &= \int_\Omega \langle x^*_0, f\rangle \, d\mu.\end{aligned}$$

This $(x^* f)$ has a weakly convergent subnet of $L^\infty(\mu)$. Hence $(x^* f)$ has a weakly convergent subset of $B(\Sigma)$. Therefore $\{x^* f : x^* \in B_{x^*}\}$ is a relatively weakly compact set of $B(\Sigma)$.

(c) $\Rightarrow$ (d). Let (x^*_α) be a net in B_{x^*}. The net $x^*_\alpha f$ converges pointwise to $x^* f$. Since $x^*_\alpha f$ is a net in a weakly compact subset of $B(\Sigma)$, it follows that

$$\lim_\alpha x^*_\alpha f = x^*_\alpha f \text{ weakly in } B(\Sigma).$$

Then we have

$$\lim_\alpha \int_A \langle x^*_\alpha, f\rangle \, d\mu = \int_A \langle x^*, f\rangle \, d\mu$$

Hence T is (w^*, w)-continuous in $L^1(\mu)$.

(d) $\Rightarrow$ (e). Since B_{x^*} is w^*-compact by Banach-Alaoglu Theorem, and T is (w^*, w)-continuous, $T(B_{x^*})$ is weakly compact, i.e., T is weakly compact operator. Define $K(F, \varepsilon) = \{x^* \in B_{x^*} : |x^*(x)| \leq \varepsilon \text{ for } x \in F, F \subset X, F : \text{finite } \}$. Then $K(F, \varepsilon)$ is w^*-compact for each (F, ε) and $(x^*_{F,\varepsilon})$ is a net in B_{x^*} which converges weak* to 0. Therefore $T(x^*_{(F,\varepsilon)})$ converges weakly to 0 and so

$$\{0\} = \cap\{T(K(F, \varepsilon)) | F \subset X, F : \text{finite and } \varepsilon > 0\}.$$

(e) $\Rightarrow$ (a). Suppose $x^*_\alpha \to x^*$ in B_{x^*}. Then $x^*_\alpha - x^* \in B_{x^*}$ for each α, and $(x^*_\alpha - x^*)_\alpha$ is in $K(F, \varepsilon)$ for each pair (F, ε).
Since $(T(x^*_\alpha - x^*))_\alpha \subseteq T(B_{x^*})$, $(T(x^*_\alpha - x^*))_\alpha$ is a relatively wakly compact subset of $L^1(\mu)$. Since (x^*_i) is a bounded sequence in B_{x^*} such that $x^*_i \to 0$ for every $x \in F \subset X$, $T(x^*_\alpha) \xrightarrow{w} T(x^*)$. Now suppose that $y^* \in L^\infty(\mu)$, and consider y^*T. Since $\text{Ker}(y^*T) \cap \frac{1}{2} B_{x^*}$ is w^*-closed, then $\text{Ker}(y^*T) \cap \alpha B_{x^*}$ is w^*-closed for each $\alpha > 0$. Therefore by the Krein-Smulin Theorem ([4], p.429), y^*T is w^*-continuous, and it follows that $T^*(y^*) \in X$.

Next we shall give another properties for Pettis integrable function. The following Lemma is very useful. It's proof is omitted.

Lemma 5. *If $m : \Sigma \to X$ is a countably additive vector measure and (A_n) is a sequence of measurable sets decreasing to ϕ, then $\|m\|(A_n) \to 0$.*

Finally we point to a specil feature of X^{**}-valued measures. The result can also be deduced from a more general result of Dinculeanu ([5], p.55).

Lemma 6. *Let $m : \Sigma \to X^{**}$ be a vector measure and $A \in \Sigma$. Then*

$$\|m\|(A) = \sup_{\|x^*\| \leq 1} |x^* m|(A), \qquad x^* \in B_{x^*}.$$

Proof: In the following calculations we denote arbitrary elements from the unit balls of X^{***} and X^* by x^{***} and x^*. By α_i we denote complex numbers and $\Pi = (A_i)$ stands for an arbitrary partition of A into a finite number of disjoint measurable subsets.

$$\begin{aligned}\|m\|(A) &= \sup_{x^{***}} |x^{***} m|(A) = \sup_{x^{***}} \sup_{\Pi} \sum_i |x^{***} m(A_i)| \\ &= \sup_{x^{***}} \sup_{\Pi} \sup_{|\alpha_i| \leq 1} \left| \left\langle x^{***}, \sum_i \alpha_i m(A_i) \right\rangle \right| \\ &= \sup_{x^*} \sup_{\Pi} \sup_{|\alpha_i| \leq 1} \left| \left\langle x^*, \sum_i \alpha_i m(A_i) \right\rangle \right| \\ &= \sup_{x^*} |x^* m|(A).\end{aligned}$$

Here the fourth equalty is true because for each element $x^{**} \in X^{**}$

$$\begin{aligned}\sup_{\|x^{***}\| \leq 1} |\langle x^{***}, x^{**} \rangle| &= \sup_{\|x^*\| \leqq 1} |\langle x^*, x^{**} \rangle| \\ &= \|x^{**}\|.\end{aligned}$$

The following propositions due to P.E.Kopp will be need in the sequel proof.

Proposition 7 ([15], p.30). *If K is a weakly compact set of a separable Banach space, then K is metrizable in the weak topology and hence weakly sequentilly compact.*

Proposition 8 ([15], p.32). *If the sequence (f_n) in $L^1(\mu)$ is such that for every $A \in \Sigma$ the sequence $(\int_A f_n d\mu)$ converges, then (f_n) is uniformly integrable, and converges weakly to some $f \in L^1(\mu)$.*

Lemma 9. *For a bounded subset K of $L^1(\mu)$. The following are equivalent:*

1. *K is uniformly integrable.*
2. *K is relatively weakly compact in $L^1(\mu)$.*
3. *K is weakly sequentially compact in $L^1(\mu)$.*

Proof: (1) $\Rightarrow$ (2). Suppose K is bounded and uniformly integrable. Let (f_n) be a sequence in K. Then there is a countable field $\mathcal{F}$ such that f_n is measurable relative to the σ-field, Σ_1 generated by $\mathcal{F}$. By a diagonal procedure, select a subsequence (f_{nj}) such that

$$\lim \int_A f_{nj} d\mu = m(A)$$

exists for all $A \in \mathcal{F}$. Also since K is uniformly integrable, it follows that m is μ-continuous. Thus there exists $f \in L^1(\Sigma_1, \mu)$ such that

$$\lim_j \int_A f_{nj} d\mu = \int_A f d\mu, \text{ for each } A \in \Sigma_1.$$

From this point, it is a simple argument to verify that

$$\lim_j \int_\Omega f_{nj} g d\mu = \int_\Omega f g d\mu, \text{ for each } g \in L^\infty(\Sigma_1, \mu).$$

Hence $f_{nj} \to f$ weakly in $L^1(\Sigma_1, \mu)$. But $L^1(\Sigma_1, \mu)$ is a closed linear subspace of $L^1(\mu)$. Hence $f_{nj} \to f$ weakly in $L^1(\mu)$, and K is relatively weakly compact.

(2) $\Rightarrow$ (3). First note that any sequence $(f_n) \subset L^1(\Sigma)$ induce a countably generated (e.g.by $\{f_n \leq r\}$) such that each f_n is Σ_0-measurable.
Since the countable set of Q-valued Σ_0-simple functions in dense in $L^1(\Sigma_0)$, this is a separable Banach space, and we can apply proposition 7 to weakly compact subsets of $L^1(\Sigma_0)$. Now if $K \subset L^1(\Sigma)$ is relatively weakly compact and (f_n) is a sequence in K, we can construct $L^1(\Sigma_0)$ as above.
The operator $f \to E(f|\Sigma_0)$ is weakly continuous by ([15], p.40). So the set $\{E(f|\Sigma_0) : f \in K\}$ is relatively weakly compact in $L^1(\Sigma_0)$. By proposition 7 some sequence (g_{nk}) of $(E(f|\Sigma_0))_n$ converges weakly to an element $g \in L^1(\Sigma_0)$. But since each f_n is Σ_0-measurable, $g_{nk} = f_{nk}$ for all k. Hence for all $h \in L^\infty(\Sigma)$, since $E(\cdot|\Sigma_0)$ is self-adjoint,

$$\int_\Omega f_{nk} h d\mu = \int_\Omega f_{nk} E(h|\Sigma_0) d\mu \Rightarrow \int_\Omega g E(h|\Sigma_0) d\mu = \int_\Omega g h d\mu \text{ as } k \to \infty$$

so (f_{nk}) converges weakly in $L^1(\mu)$.

(3) $\Rightarrow$ (1). If $KL^1(\mu)$ is weakly sequentially compact, K is norm-bounded.
If K were not uniformly integrable, we would be able to find $\varepsilon > 0$ and a sequence (f_n) in K and a sequence (A_n) in Σ such that $\mu(A_n) < \frac{1}{n}$ but $\int_{A_n} |f_n| d\mu \geq \varepsilon$ for all $n \geq 1$. This means that no subsequence of (f_n) is uniformly integrable. On the other hand, some subsequences (f_{nk}) must converge weakly. In particular, for each $A \in \Sigma$, $\int_A f_{nk} d\mu = \int_\Omega f_{nk} \chi_A d\mu$ defines a convergent sequence in $\mathbf{R}$. But by the Vitali-Hahn-Saks theorem (proposition 8.), (f_{nk}) is then uniformly integrable. This contradiction shows that K must itself be uniformly integrable.

Proposition 10. *Let $f : \Omega \to X$ be weak $L^1(\mu)$. Then the following are equivalent:*

1. *U is weakly compact.*
2. *T is weakly compact.*
3. *$\{x^* f : x^* \in B_{x^*}\}$ is uniformly integrable.*
4. *The Dunford integrable $m(A) = U(\chi_A)$ is σ-additive.*

Proof: (1) $\Rightarrow$ (2). The operator $U^* : X^{***} \to L^\infty(\mu)^*$ is weakly compact by the Gantmacher theorem ([6],p.485). Let $B_{x^{***}}$ denote the unit ball in X^{***} and let $W \subset L^\infty(\mu)^*$ be a weakly compact set such that $U^*(B_{x^{***}}) \subset W$. Because $U^* = T$ on X^* and $T(X^*) \subset L^1(\mu)$, we have $T(B_{X^{***}} \cap X^*) \subset W \cap L^1(\mu)$.
Now $W \cap L^1(\mu)$ is weakly compact and so T is a weakly compact operator.

(2) $\Rightarrow$ (1). In this case of non-σ-finite μ, the dual of $L^1(\mu)$ need not be $L^\infty(\mu)$ ([**12**], p.349) and so the operator U need not be the adjoint of T.
We define a linear map

$$\phi : L^\infty(\mu) \to L^1(\mu)^*$$

as follows:

$$\langle f, \phi_\phi \rangle = \int_\Omega \phi f d\mu \tag{1}$$

then $\|\phi\| \leq 1, U = T^* \circ \Phi$ and T^* is weakly compact by Gantmacher theorem. And so U is weakly compact.

(2) $\Rightarrow$ (3). This is trivial by Lemma 9.

(3) $\Rightarrow$ (4). Let $A_n \in \Sigma$ be a sequence decreasing to ϕ and let $\varepsilon > 0$.
Because $\{x^*f : x^* \in B_{x^*}\}$ is uniformly integrable, there is an $n \in N$ for which

$$\sup_{\|x^*\| \leq 1} \int_{A_n} |x^* f| d\mu \leq \varepsilon$$

this means that $|m(A_m)| \leq \varepsilon$ when $m \geq n$.

(4) $\Rightarrow$ (3). Let $A_n \in \Sigma$ be a sequence decreasing to ϕ and let $\varepsilon > 0$.
Then, by countable additive and Lemma 5,

$$\sup_{\|x^*\| \leq 1} \int_{A_n} |x^* f| d\mu = \|m\|(A_n) \to 0$$

and so $\{x^*f : x^* \in B_{x^*}\}$ is uniformly integrable.

Theorem 11. *Let $f : \Omega \to X$ be weak $L^1(\mu)$. Then the following are equivalent:*

(1) *T is weakly compact.*
(2) *$\{x^*f : x^* \in B_{x^*}\} \subset L^1(\mu)$ is relatively weakly compact.*
(3) *$m(\Sigma)$ is relatively weakly compact.*

Proof: (1) $\Rightarrow$ (2). By Lemma 9.

(2) $\Rightarrow$ (3). Since $\{x^*f : x^* \in B_{x^*}\}$ is relatively compact then f is Pettis integrable by Theorem 4. Hence $m(\Sigma)$ is relatively weakly compact by ([4], p.56).

(3) $\Rightarrow$ (1). If $\phi \in L^\infty(\mu)$ is a measurable function with finite many value and $0 \leq \phi \leq 1$ μ-a.e., then $T_\phi \in c0(m(\Sigma))$ [**13**]
By proposition 1land proposition 10, we get the following Theorem.

Theorem 12. *If f is Pettis integrable, the map $A \to U(\chi_A)$ from Σ to X is σ-additive, or equivalently $\{x^*f : x^* \in B_{x^*}\}$ is uniformly integrable, of equivalently $m(\Sigma)$ is reatively weakly compact.*

Theorem 13. *Let $f : \Omega \to X$ be weak $L^1(\mu)$. Then the following are equivalent.*

(1) *f is Pettis integrable.*
(2) *The canonical map $\{x^*f : x^* \in B_{x^*}\} \to L^1(\mu)$ is pointwise to weak continuous.*
(3) *The map $x^* \to \int_A \langle f, x^* \rangle d\mu$ restricted on B_{x^*} is weak* continuous for each $A \in \Sigma$.*
(4) *The above map is continuous at 0.*
(5) *$\{x^*f : x^* \in B_{x^*}\}$ is relatively weakly compact in $L^1(\mu)$, and $\{0\} = \{T(K(F, \varepsilon)) | F \subset X, F : \text{finite and } \varepsilon > 0\}$.*

Proof: (1) $\Rightarrow$ (2). Suppose $x^*_\alpha \in B_{x^*}$ is a net and $y^* \in B_{x^*}$ is an element such that $x^*_\alpha f \to y^* f$ pointwise. This means $x^*_\alpha \to y^*$ pointwise on span $f(\Omega)$.
The boundedness of B_{x^*} implies that the subspace $\{x \in X : x^*_\alpha x \to y^* x\}$ is norm closed, and so we see that $x^*_\alpha \to y^*$ pointwise on $\overline{\text{span}}\ f(\Omega)$.
It follows from the Hahn-Banach theorem that for each $\phi \in L^\infty(\mu)$ we have

$$T_\phi = \int_\Omega \phi f d\mu \in \overline{\text{span}} f(\Omega)$$

and so

$$\left\langle \int_\Omega \phi f d\mu, x^*_\alpha \right\rangle \to \left\langle \int_\Omega \phi f d\mu, y^* \right\rangle$$

This shows that $x^*_\alpha f \to y^* f$ in weak topology.

(2) $\Rightarrow$ (3). Since $x^*_\alpha f \to y^* f$ in weak topology, then above map is weak*-continuous for each $A \in \Sigma$.

(3) $\Rightarrow$ (4). If a linear map is continuous on X, then that map is continuous at one point of X, the above map is continuous at 0.

(4) $\Rightarrow$ (1). Since a linear functional on X^* which is weak*-continuous on B_{x^*} is weak*-continuous, f is Pettis integrable.

(1) $\Leftrightarrow$ (5). Since f is Pettis integrable, then T is weakly compact operator. Hence $\{x^* f : x^* \in B_{x^*}\}$ is relatively weakly compact in $L^1(\mu)$ by Theorem 11 and by Theorem 4, $\{0\} = \cap\{T(K(F,\varepsilon))|F \subset X, F : \text{finite and } \varepsilon > 0\}$.
Conversely, by Theorem 4, trivial.

References

1. E.M.Bator, *Pettis integrability and the equality of the norms of the weak* integral and the Dunford integral*, Proc. Amer. Math. Soc. **95** (1985), 265–270.
2. E.M.Bator, P.Lewis and D.Race, *Some connections between pettis integration and operator theory*, Rocky Mountain J. Math. **17** (1987), 683–695.
3. J.Diestel, "Sequences and series in Banach spaces," Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1984.
4. J.Diestel and J.J.Uhl,Jr., *Vector measures, math surveys*, Amer. Soc., Providence, RI (1977).
5. N.Dinculeanu, "Vector measures," Pergammon Press, Oxford, 1967.
6. N.Dunford and J.T.Schwartz, "Linear operators, part I," Interscience, New York, 1958.
7. G.A.Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), 663–677.
8. G.A.Edgar, *Measurability in a Banach space II*, Indiana Univ. Math. J. **28** (1979), 559–580.
9. D.Fremlin and M.Talagrand, *Adecomposition theorem for additive set functions and applications to pettis integral and ergodic means*, Math. Z. **168** (1979), 117–142.
10. R.F.Geitz, *Pettis integration*, Proc. Amer. Math. Soc. **82** (1981), 81–86.
11. R.F.Geitz and J.J.Uhl,Jr., *Vector valued functions as families of scalars-valued functions*, Pacific J. Math. **95** (1981), 75–83.
12. E.Hewitt and K.Stromberg, "Real and abstract analysis," Spring-Verlag, Berlin-Heideberg New York, 1969.
13. Hoffman-J.Jorgensen, *Vector measure*, Math. Scand. **28** (1971), 5–32.
14. R.E.Huff, *Some remarks on the Pettis integral*, Proc. Amer. Math. Soc. **96** (1986), 402–404.
15. P.E.Kopp, "Martingales and stochastic integrals," Cambridge University Press, 1984.
16. M.Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **307** (1984).
17. A.Wilansky, "Mordern methods in topological vector spaces," McGraw-Hill, New York, 1978.