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The Critical Phenomena of a Model for the Metabolic Control System with Positive Feedback

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The static and dynamic phenomena of a model for the metabolic control system with positive feedback are discussed with the static and dynamic renormalization group theory. Then, the explicit results for the static and dynamic exponents are obtained up to the second order of ε-expansion, ε being 4-d, where d is the space dimensionality of the system.

Introduction

One of the most interesting phenomena in a metabolic system is the control mechanism, which regulates the flux of material through the various metabolic pathways. There are two kinds of the metabolic control mechanisms, which are accomplished by negative or positive feedback.[1-4] The most important physical phenomenon by the negative feedback is the biochemical oscillation.[1,3-4] That oscillation can be a sustained oscillation or a limit cycle. The essence in the metabolic control system with positive feedback is the biochemical hysteresis.[1-3] In a real system the mechanism is so complicated that it is necessary to extract the essential physical phenomena of the real system. In fact, there exist simple kinetic models consisted of many dimensional ordinary differential equations for the concentrations of the reactants in the metabolite.[1-4] Even though they are simplified or oversimplified, they are at least compatible with the experimental results qualitatively. Many authors have studied the dynamics of the metabolic control models. However, there are still a lot of works to be investigated. One of them is the critical behavior of the model with positive feedback.

The purpose of the present paper is to investigate the behavior of a model for the metabolic control system with positive feedback near the critical point by using the well-known renormalization group(RG) method.[5-8] As usual, we shall separate the critical behavior into the static and the dynamic behaviors. In both cases the results will be obtained up to the second order of ε in the ε-expansion, ε being 4-d, where d is the space dimensionality of the system.

In section II we discuss general properties of the model given in references 1-3. The deviation of the concentrations of the reactants from the values at the steady state is assumed to be due to the Langevin random forces, which satisfy the Gaussian condition.[9] We obtain a nonlinear equation for the fluctuating concentrations near the critical point, which is very similar to the time-dependent Landau-Ginzburg equation for the classical Ising spin system.[7-8] In the next section the Gaussian approximation is applied to discuss the static critical behavior of the model and then the RG method is used to the nonlinear effect on the critical exponents.[5-7] In section IV the dynamic RG method, which is just the extension of the static RG method, is used to obtain the dynamic exponents. In this case we simplify the dynamic process by assuming that there are very rapid and sufficient energy exchanges between the system and the surroundings. Finally, we discuss the present results and compare these with the numerical values of the exponents from other systems.[10-15]

Theory

Let us consider a kinetic equation for a model of metabolic control system with positive feedback given as[1-3]

$$\dot{X}_1 = f(X_n) - k_1 X_1$$
$$\dot{X}_i = X_{i-1} - k_i X_i, \quad (2 \leq i \leq n) \tag{2.1}$$

where X_i and k_i are the dimensionless concentration of the i-th component and corresponding rate constant, respectively, and

$$f(X_n) = \frac{u + X_n^p}{1 + X_n^p} \tag{2.2}$$

In Eq. (2.2) p is the cooperativity of the system and u is the controllable parameter depending on temperature. Let us define a function, g as
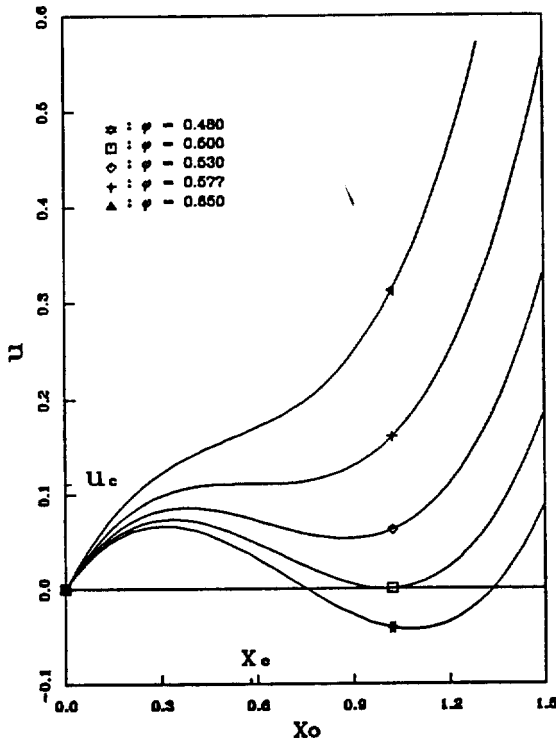
Figure 1. The steady states in (X_o, u) space for various values of ϕ.

$$g(X_n, u, \phi) = f(X_n, u) - \phi X_n \qquad (2.3)$$

with

$$\phi = \prod_{i=2}^{n} k_i.$$

From now on we shall omit n in X_n for simplicity. The steady state of the system is defined as

$$g(X_o, u, p) = 0. \qquad (2.4)$$

The steady state is stable, if one has

$$[\frac{\partial g}{\partial X_o}]_{u,\phi} < 0. \qquad (2.5)$$

In the opposite sign the state is unstable. A marginal stability point is defined as

$$[\frac{\partial u}{\partial X_o}]_{\phi} = [\frac{\partial \phi}{\partial X_o}]_u = 0. \qquad (2.6)$$

From Eq. (2.6) we may obtain the equivalent condition for the marginal stability point

$$[\frac{\partial g}{\partial X_o}]_{u,\phi} = 0. \qquad (2.7)$$

In addition to the above condition given in Eqs. (2.6) and (2.7), we need an additional condition for the critical point

$$[\frac{\partial^2 u}{\partial X_o^2}]_{\phi} = [\frac{\partial^2 \phi}{\partial X_o^2}]_u = 0, \text{ or } [\frac{\partial^2 g}{\partial X_o^2}]_{u,\phi} = 0. \qquad (2.8)$$

The stability condition depends on p and n. For simplicity we shall only consider $n = p = 2$ in the present paper. Then, the detailed analysis of the stability of the system can be done with the aid of the Routh-Hurwitz criteria.[19] The system has three steady states when the following conditions are satis-
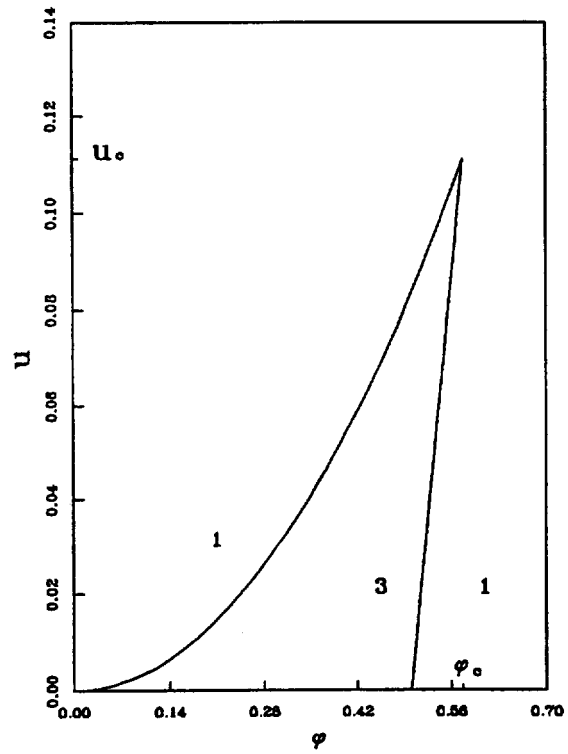

Figure 2. The region of steady states in the positive feedback loop for u and ϕ. The numbers denote the number of steady states.

fied:

$$1/3 > \phi^2 = x, \ 27 x u^2 - 2(2 - 9x) u - x(1 - 4x) < 0. \qquad (2.9)$$

In the case that the above conditions are satisfied, two steady states are stable and the intermediate state is unstable. Otherwise, the system has only one steady state. The stability diagrams are given in Figures 1 and 2. The critical point of the system is give as

$$(X_c, u_c, \phi_c) = (3^{-1/2}, 9^{-1}, 3^{-1/2}). \qquad (2.10)$$

Near the critical point the following relation holds

$$\frac{u}{\phi} - \frac{u_c}{\phi_c} = (X - X_c)^{1/2}. \qquad (2.11)$$

This equation means that near the critical point there is a kind of transition like the second order phase transition in various kinds of system.[7]

Let us assume that the fluctuations occur due to the random forces.[9] Then, expanding Eq. (2.1) with $x_i = X_i - X_{io}$, the Langevin equation is given as

$$\frac{d}{dt}\underset{\sim}{x} = M\underset{\sim}{x} + \sum_{k=2} g^{(k)}(X_o) x_n^k u + \underset{\sim}{\zeta} \qquad (2.12)$$

where

$$M = \begin{pmatrix} -k_1, & 0, & 0, \cdots, & 0, f'(X_o) \\ 1, & -k_2, & 0, \cdots, & 0, & 0 \\ 0, & 0, & 0, \cdots, & 1, & -k_n \end{pmatrix},$$

$$\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)^T$$

$$\underset{\sim}{u} = (1, 0, \cdots, 0)^T$$

$$g^{(k)}(X_o) = \frac{1}{k!}\frac{\partial^k g(X)}{\partial X^k}\big|_{X=X_o} \qquad (2.13)$$

$$\underline{\zeta} = (\zeta_1, \zeta_2, \cdots, \zeta_n)^T$$

It should be noted that the random forces in Eq. (2.12) correspond to the scaled fluctuating concentrations. The forces are assumed to satisfy the Gaussian condition

$$<\zeta_i> = 0, \quad <\zeta_i(t)\zeta_j(t')> = 2\delta_{ij}\delta(t-t'). \qquad (2.14)$$

Let the eigenvalue of **M** be $-\lambda$. Then, the characteristic equation for the eigenvalue is

$$\prod_{i=1}^{n}(\lambda - k_i) = f'(X_o). \qquad (2.15)$$

It is possible to obtain the eigenvalues analytically only in the case that all the k_i's are equal or n is a low integer. Let us take the simplest case, that is, $n = p = 2$. Then, the eigenvalues are

$$\lambda_1 = \alpha + \beta, \quad \lambda_2 = \alpha - \beta \qquad (2.16)$$

where

$$\alpha = -\frac{1}{2}tr(M) = \frac{1}{2}(k_1 + k_2) \text{ and } \beta = [\alpha^2 - g'(X_o)]^{1/2}.$$

Let the right and left eigenvectors of **M** corresponding to λ_i be $\underline{\psi}^i$ and $\underline{\bar{\psi}}^i$, repectively. We may obtain the eigenvectors as[10]

$$\underline{\psi}^i = (\psi_1^i, \psi_2^i)^T = \frac{1}{2\beta}(\lambda_i - k_2, 1)^T,$$

$$\underline{\bar{\psi}}^i = (\bar{\psi}_1^i, \bar{\psi}_2^i) = [1, \frac{-f'(X_o)}{\lambda_i - k_2}]. \qquad (2.17)$$

It can be checked that the eigenvectors satisfy the orthonormal conditions. With the aid of the eigenvectors, the linear equations reduce to

$$\frac{d}{dt}y_i = -\lambda_i y_i + \zeta_i'; \ y_i = x_i - \frac{f'(X_o)}{\lambda_i - k_2}x_2,$$

$$\zeta_i' = \zeta_i - \frac{f'(X_o)}{\lambda_i - \lambda_2}\zeta_2. \qquad (2.18)$$

The solution of the above equation is

$$y_i(t) = y_{io}\exp(-\lambda_i t) = \int \exp[-\lambda_i(t-t')]\zeta_i'(t')dt'. \qquad (2.19)$$

When the system is near the stable steady state, it relaxes to the steady state value. Thus, at the stable steady state far from the instability point or the critical point, the linear equation can be used. For the system at the unstable steady state the fluctuations become divergent due to the mode with λ_2, as $t \to \infty$. Also, the relaxation time of the system near the critical point or the marginal instability point becomes infinite. This means that the system at the unstable steady state or near the critical point and marginal stability point cannot be described properly by the linear equation. Thus, we have to consider the nonlinear equation. In those cases the mode with λ_2 dominates the dynamic behavior of the system. Up to the third order of y_2, we may write[16]

$$\frac{d}{dt}y_2(t) = -\lambda_2 y_2(t) + \frac{g^{(2)}(X_o)}{(k_1 + k_2)^2}y_2(t)^2$$

$$+ \frac{g^{(3)}(X_o)}{(k_1 + k_2)^3}y_2(t)^3 + \zeta_2'. \qquad (2.20)$$

Introducing the variables z and τ as

$$z = y_2/(k_1 + k_2), \quad \tau = t/(k_1 + k_2), \qquad (2.21)$$

we can rewrite Eq. (2.20) as

$$\frac{d}{d\tau}z(\tau) = -g^{(1)}(X_o)z + g^{(2)}(X_o)z^2 + g^{(3)}(X_o)z^3 + \zeta, \qquad (2.22)$$

where for simplicity ζ_2' has been replaced by ζ. The system near the critical point satisfies

$$\frac{\partial}{\partial\tau}z(\underline{r}, \tau) = [D\nabla^2 - q(u_c - u)]z(\underline{r}, \tau)$$

$$- \beta z(\underline{r}, \tau)^3 + \zeta(\underline{r}, \tau), \qquad (2.23)$$

where

$$q = (27/64)^{1/2}, \quad \beta = (27^{1/2}/128)(1 - u). \qquad (2.24)$$

and the spatial inhomogeneity has been added, since the system in the critical region is very sensitive to the gravitational field and thus it cannot be homogeneous. Eq. (2.23) is very similar to the time-dependent Landau–Ginzburg equation for the generalized continuous kinetic Ising model.[7] The equation shows that the system should be controlled by the parameter u, maintaining X and ϕ at X_c and ϕ_c, respectively. The Fokker–Planck equation corresponding to the Langevin equation is

$$\frac{\partial}{\partial\tau}P(z, \tau) = -LP(z, \tau) \qquad (2.25)$$

where

$$LP = \frac{\partial}{\partial z}\{[D\nabla^2 - q(u_c - u)]z - \beta z^3\}P + D\frac{\partial^2}{\partial z^2}P.$$

In order to discuss the critical phenomena of the system we have to separate the problem into two parts, that is, static and dynamic cases. The main idea of the static problem is to investigate how a physical property depends on $|u-u_c|$, when the system approaches to the critical point. The dynamic problem is to obtain the dispersion of the fluctuation mode and determine the effect of the critical phenomena on the noncritical mode. In the next section we shall discuss the static critical phenomena.

The Static Critical Phenomena

Let us define the Fourier transform of a function $f(\underline{r}, \tau)$ with respect to $\underline{r}$ and τ and its inverse Fourier transform as

$$f(\underline{k}, \omega) = \int d\underline{r}\,d\tau\exp[-i(\underline{k}\cdot\underline{r} - \omega\tau)]f(\underline{r}, \tau),$$

$$\qquad (3.1)$$

$$f(\underline{r}, \tau) = \int_{\underline{k}, \omega}\exp[i(\underline{k}\cdot\underline{r} - \omega\tau)]f(\underline{k}, \omega),$$

where

$$\int_{\underline{k}, \omega} \equiv (2\pi)^{-(d+1)}\int d\underline{k}\,d\omega. \qquad (3.2)$$

In Eq. (3.2) d is the space dimensionality of the system. At the steady state the probability distribution is

$$P_{st}(z) = cP_o(z)\exp\left[-\frac{\beta}{4D}\int_{\substack{\underline{k}_i \\ i=1}} z(\underline{k}_1)z(\underline{k}_2)z(\underline{k}_3)z(\underline{k}_4)\right.$$

$$\left.\times \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\right]$$

$$P_o(z) = \exp\left[-\frac{1}{2D}\int_{\underline{k}} z(\underline{k})\left[Dk^2+q(u_c-u)z(-\underline{k})\right]\right]. \quad (3.3)$$

where c is the renormalized constant and P_o denotes the Gaussian probability distribution. In the linear case the correlation function is

$$C_o(\underline{k}) = \langle z(\underline{k})z(-\underline{k})\rangle_o = (k^2+\zeta^{-2})^{-1} \quad (3.4)$$

where

$$\zeta^{-2} = \frac{q}{D}|u_c-u|.$$

The correlation function shows the following limiting properties

$$\lim_{u\to u_c} C_o(\underline{k}) \sim k^{-2} \text{ and } \lim_{k\to 0} C_o(\underline{k}) \sim \zeta^2. \quad (3.5)$$

As the system approaches to the critical point, the correlation function is governed by long wavelength (small k) fluctuations. Also when $k=0$, the function diverges through the correlation length, as $u \to u_c$. Let us define the correlation length and function as

$$\xi \sim \left|\frac{q}{D}(u-u_c)\right|^{-\nu} \text{ and } C_o(k) \sim k^{-2+\eta}. \quad (3.6)$$

The critical exponents for the Gaussian approximation are $\nu=\frac{1}{2}$ and $\eta=0$.

We shall use the RG method to obtain the effect of non-linear term on the critical exponents. The procedure of the RG method is given as follows:[5-7]

(i) We divide the wavelength region into two parts, that is, the long wavelength and the short wavelength regions with the linear size of the system l. Then, we integrate out $z(\underline{k})$ with $\Lambda/l < k < \Lambda$ [denoted by $z_1(\underline{k})$], leaving the component with $0 < k < \Lambda/l$ [denoted by $z_0(\underline{k})$] unintegrated.

(ii) We expand the nonlinear term systematically and integrate out the short wavelength parts.

(iii) Some scaled changes should be performed to make the renormalized L' look like the original L. The scaled variables are

$$k' = lk, \quad z(k) = l^{-[1+(d-\eta/2)]}z_0(k'/l). \quad (3.7)$$

With the aid of the procedures (i) and (ii), we may obtain the probability distribution up to the first order of β as

$$P_{st}(z) = c \exp-\frac{1}{2}\left[\int_{0<k<\Lambda/l} z_0(\underline{k})(k^2+\alpha\right.$$

$$+12\gamma\langle 2z_1^2\rangle)z_0(-\underline{k}) - \gamma\int_{0<\Pi k_i<\Lambda/l} z_0(\underline{k}_1)$$

$$\left.\times z_0(\underline{k}_2)z_0(\underline{k}_3)z_0(\underline{k}_4)\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\right],$$

$$(3.8)$$

where

$$\alpha = \frac{q}{D}(u-u_c), \quad \gamma = \frac{\beta}{4D}.$$

$$\langle z_1^2\rangle = \int_{\Lambda/l<k_1,k_2<\Lambda} \frac{\delta(\underline{k}_1+\underline{k}_2)}{k_1^2+\alpha}. \quad (3.9)$$

In order to obtain the result given in Eq. (3.8) we have used the Gaussian approximation

$$\left\langle \prod_{i=1}^{n} z_i(\underline{k}_i)\right\rangle = \begin{cases} 0, & \text{if } n \text{ is an odd integer,} \\ \text{sum over all the possible} & (3.10) \\ \text{pair products, if } n \text{ is an even} \\ \text{integer.} \end{cases}$$

Using Eq. (3.6), we obtain the renormalized operator L' as

$$P_{st}(z) \propto \exp - L'(z); L' = \frac{1}{2}\left[\int_{0<k<\Lambda} z(\underline{k})(k_l^2+\alpha_l)z(-\underline{k})\right.$$

$$\left.+\gamma_l\int_{0<\pi k_i<\Lambda} z(\underline{k}_1)z(\underline{k}_2)z(\underline{k}_3)z(\underline{k}_4)\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\right],$$

$$(3.11)$$

where

$$k_l^2 = l^{-\eta}k^2, \quad \alpha_l = l^{2-\eta}(\alpha+12\gamma\langle z_1^2\rangle), \quad \gamma_l = l^{\epsilon-2\eta}\gamma;$$

$$\epsilon = 4-d, \quad \langle z_1^2\rangle = (\Lambda^2/2-\alpha\ln l). \quad (3.12)$$

For simplicity we have neglected the angular factor divided by $(2\pi)^d$ for obtaining $\langle z_1^2\rangle$. In Eq. (3.12) we can see that η is always zero so that the first equation may hold for any value of γ. This means that there is no interaction between the hydrodynamic mode and nonlinear term up to first order of γ. In order to obtain the nonzero value of η, we have to consider the higher order terms of γ, which will be discussed later.

It is assumed that the operator L maps onto itself at a fixed point, that is,[5-7]

$$L^*(\alpha_l, \gamma_l) = L^*(\alpha, \gamma) \quad \text{for any } l. \quad (3.13)$$

The sign * denotes a fixed point. Then, from the following relation

$$\frac{d}{d\ln l}\alpha_l|_* = 0, \quad \frac{d}{d\ln l}\gamma_l|_* = 0 \quad (3.14)$$

we may obtain the fixed point as

$$(\alpha^*, \gamma^*) = (0,0). \quad (3.15)$$

The fixed point is called the Gaussian fixed point, since at the fixed point the Gaussian approximation holds. Expanding Eq. (3.12) near the fixed point, we have

$$\frac{d}{d\ln l}(\delta\alpha, \delta\gamma)^T = M'(\delta\alpha, \delta\gamma)^T; M' = \begin{pmatrix} 2, & 12\Lambda^2 \\ 0, & \epsilon \end{pmatrix}, \quad (3.16)$$

where

$$\delta\alpha = \alpha-\alpha^*, \quad \delta\gamma = \gamma-\gamma^*. \quad (3.17)$$

Let $\delta\alpha$ and $\delta\gamma$ be proportional to l as

$$\delta\alpha \sim l^{\mu_1}, \quad \delta\gamma \sim l^{\mu_2}. \quad (3.18)$$

Then, the eigenvalues are

$$\mu_1 = 3, \quad \mu_2 = \epsilon \quad (3.19)$$

For $d<4$, the fixed point is unstable, when $l \to \infty$, while for $d>4$ and $\alpha = \alpha^*$, the fixed point is stable. According to the scaling rule[5-7], the exponent for the correlation length, ν, is

the inverse of the eigenvalue, μ_1. Thus, up to the first order of γ, the Gaussian approximation holds.

In order to obtain the stable fixed point for $4 > d$, let us consider the second order of γ. Following the same procedure as before, we obtain the following equation

$$a_i = l^\varepsilon [a + 6\gamma (\Lambda^2 - 2\alpha \ln l)],$$ (3.20)

$$\gamma_i = l^\varepsilon \gamma (1 - 36 \gamma \ln l).$$

The fixed points are up to the first order of ε

(a) $(a_*^* \gamma^*) = (0, 0)$, (b) $(a_*^* \gamma_*^*) = (-\frac{\varepsilon \Lambda^2}{6}, \frac{\varepsilon}{36})$. (3.21)

Using the same procedure as in the case of the first order of ε, we obtain the eigenvalues for the non-Gaussian case as

$$\mu_1 = 2 - \frac{\varepsilon}{3}, \quad \mu_2 = -\varepsilon.$$ (3.22)

The eigenvalues show the following results:

(1) If $\varepsilon < 0$, the non-Gaussian fixed point is never approached for $l \to \infty$.

(2) When $\varepsilon > 0$ and $u = u_c$, the fixed point is stable for $l \to \infty$.

The correlation length is given as[5-7]

$$\xi \sim |u - u_c|^{-\nu}; \quad \nu = (2 - \frac{\varepsilon}{3})^{-1}.$$ (3.23)

For the system with $d = 3$, the critical exponent is

$$\nu = 0.600.$$ (3.24)

In order to obtain the exponent η, we have to start from the static case of Eq. (2.23). Then, using the iterative method and taking the principal part for the interaction between the hydrodynamic mode and nonlinear term up to the second order of β, we obtain the renormalized static linear response function as

$$G(\underset{\sim}{k})^{-1} = G_0(\underset{\sim}{k})^{-1}$$
$$- 18\beta^2 \int_{k_1, k_2} G_0(\underset{\sim}{k} - \underset{\sim}{k_1} - \underset{\sim}{k_2}) C_0(\underset{\sim}{k_1}) C_0(\underset{\sim}{k_2}),$$ (3.25)

where we have neglected the first and second order correction terms disconnected among wave vectors, since they do not contribute to the coupling between the hydrodynamic mode with $\underset{\sim}{k}$ and the nonlinear term, and $G_0(\underset{\sim}{k})$ is

$$G_0(\underset{\sim}{k})^{-1} = D(k^2 + \xi^{-1}).$$ (3.26)

Let us use the following relations to obtain the result of the integral in Eq. (3.25)[18]

$$\frac{1}{(k_1^2 + m_1^2)^a [(k - k_1)^2 + m_1^2]^b} = B(a, b)^{-1} \int_0^1 dx$$
$$\times \frac{(1 - x)^{a-1} x^{b-1}}{[x(k - k_1)^2 + xm_1^2 + (1 - x)(k_1^2 + m_1^2)]^{a+b}}$$ (3.27a)

$$\times \int_k \frac{1}{[(k - g)^2 + p^2 - r^2]^c} = \frac{1}{2} B(\frac{1}{2} d, c - \frac{1}{2} d)$$
$$\times (p^2 - r^2)^{\frac{d}{2} - c},$$ (3.27b)

where $B(a, b)$ is the beta function defined as .

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}.$$ (3.28)

In Eq. (3.28) $\Gamma(a)$ is the gamma function. Assuming that the renormalized propagator $G(\underset{\sim}{k})$ has the following form

$$\lim_{u \to u_c} G(\underset{\sim}{k})^{-1} \sim D k^{2 - \eta},$$ (3.29)

we can rewrite Eq. (3.25) at the critical point as

$$k^{2 - \eta} = k^2 - \frac{2\varepsilon^2}{9} \int_{k_1, k_2} \frac{1}{(k - k_1 - k_2)^2 k_1^2 k_2^2}.$$ (3.30)

With the aid of Eqs. (3.27) η can be expressed as

$$\eta = \frac{2\varepsilon^2}{9} \frac{d}{d \ln k} g(k, \varepsilon),$$ (3.31)

where

$$g(k, \varepsilon) = \frac{1}{4} B(2 - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) B(1, \frac{\gamma}{2})^{-1} B(2 - \frac{\varepsilon}{2}, \varepsilon - 1) k^{-2\varepsilon}$$
$$\times \int_0^1 \int_0^1 dx dy [x(1 - x)]^{-\varepsilon/2} (1 - y)^{1-\varepsilon} y^{-\varepsilon/2}.$$ (3.32)

Since our calculation is limited to the cases up to ε^2, it is sufficient to take $\varepsilon \to 0$ in calculating $g(k, \varepsilon)$. Then, the result is

$$\eta = \frac{\varepsilon^2}{18}.$$ (3.33)

For $d = 3$, the critical exponent is 0.0556. In the next section let us discuss the critical dynamic behavior of the model.

The Dynamic Critical Phenomena

Let us assume that the energy exchange between the system and the surroundings is sufficiently rapid and large, so that other equations such as heat conduction and etc. except Eq. (2.23) can be neglected. Then let us first consider the linear part of Eq. (2.23)

$$\frac{\partial}{\partial \tau} z(\underset{\sim}{k}, \tau) = -D(k^2 + a) z(\underset{\sim}{k}, \tau) + \zeta(\underset{\sim}{k}, \tau).$$ (4.1)

The dynamic RG method is just the extension of the static RG method by introducing the scaling of time. The dynamic RG method are as follows:

(i) As in the static case we have to eliminate z with short wavelength.

(ii) Let us introduce the following scale of the variable $z(\underset{\sim}{k}, \tau)$ as

$$z(\underset{\sim}{k}, \tau) = l^{1 - \eta/2} z_0(l \underset{\sim}{k}, l^{-z} \tau),$$ (4.2)

where z is the dynamic exponent. Using Eq. (4.2), we obtain

$$\frac{\partial}{\partial \tau} z_0(l \underset{\sim}{k}, l^{-z} \tau) = -D(k^2 + a) z_0(l \underset{\sim}{k}, l^{-z} \tau)$$
$$+ l^{-1 + \eta/2} \zeta(\underset{\sim}{k}, \tau), \quad 0 < k < \Lambda/l.$$ (4.3)

(iii) To make Eq. (4.3) the similar form as Eq. (4.1), let us rescale the following variables

$$\underset{\sim}{k}' = l \underset{\sim}{k}, \tau' = l^{-z} \tau, \quad z(\underset{\sim}{k}, \tau) = z_0(\underset{\sim}{k}', \tau').$$ (4.4)

Then, we may have

$$\frac{\partial}{\partial \tau} z(\underset{\sim}{k}, \tau) = -D'(k^2 + a') z(\underset{\sim}{k}, \tau) + \zeta'(\underset{\sim}{k}, \tau),$$ (4.5)

where

$$D' = l^{z-2}D, \ \zeta'(\underset{\sim}{k}, \tau) = l^{-1+\eta/2}\zeta(\underset{\sim}{k}/l, l^z\tau), \ \alpha' = l^z\alpha. \quad (4.6)$$

From the first equation in Eq. (4.6) we find that the dynamic exponent for the linear case is 2. Also, the fixed point for the parameters α and β is the Gaussian fixed point. The scaled random force has the following property

$$<\zeta'(\underset{\sim}{k}, \omega)> = 0, \ <\zeta'(\underset{\sim}{k}, \omega)\zeta'(\underset{\sim}{k'}, \omega')>$$

$$= 2l^{z-2}D\delta(\underset{\sim}{k} - \underset{\sim}{k'})\delta(\omega - \omega'). \quad (4.7)$$

The time correlation function $C_0(\underset{\sim}{k}, \omega)$ and the linear response function $G_0(\underset{\sim}{k}, \omega)$ are related to each other

$$C_0(\underset{\sim}{k}, \omega) = <z(\underset{\sim}{k}, \omega)z(0, 0)> = \frac{2D}{\omega}Im \ G_0(\underset{\sim}{k}, \omega). \quad (4.8)$$

The linear response function can be written with the aid of Eqs. (4.2) and (4.4)

$$G_0(\underset{\sim}{k}, \omega) = l^{z-\eta}G_0(l\underset{\sim}{k}, l^z\omega). \quad (4.9)$$

Taking $l = k^{-1}$, we have

$$G_0(k, \omega) = k^{-2+\eta}G_0(\omega/\omega_k); \ \omega_k = k^z = k^z. \quad (4.10)$$

At the critical point, $G_0(k,\omega)$ reduces to the static case as $\omega \to 0$ and it is proportional to ω^{-1} as $k \to 0$. Thus, from Eq. (4.10) we may write

$$G_0(0, \omega) \sim \omega^{-1+\Delta}; \ \Delta = \eta/z. \quad (4.11)$$

It is obvious that the linear system has $\Delta = 0$.

Let us obtain the nonzero dynamic exponent. Using the iterative method and taking the part up the second order of β, we have

$$G(\underset{\sim}{k}, \omega) = G_0(\underset{\sim}{k}, \omega) - 18\beta^2 \int_{\underset{\sim}{k_1},\underset{\sim}{k_2}, \omega_1, \omega_2} G_0(\underset{\sim}{k} - \underset{\sim}{k_1} - \underset{\sim}{k_2},$$

$$\omega - \omega_1 - \omega_2)C_0(\underset{\sim}{k_1}, \omega_1) C_0(\underset{\sim}{k_2}, \omega_2). \quad (4.12)$$

The procedure to solve the integral in Eq. (4.12) is very similar to that in obtaining η in the previous section, except that there are two more variables, that is, ω_1 and ω_2. Integrating over ω_1 and ω_2 in the complex plane, we obtain at the critical point

$$G(\underset{\sim}{k}, \omega)^{-1} = G_0(\underset{\sim}{k}, \omega)^{-1} - 18i(\frac{\beta^2}{D})^* \int_{\underset{\sim}{k_1}, \underset{\sim}{k_2}}$$

$$\times \frac{1}{k_1^2 k_2^2[\omega' + (\underset{\sim}{k} - \underset{\sim}{k_1} - \underset{\sim}{k_2})^2 + k_1^2 + k_2^2]} \quad (4.13)$$

with

$$\omega' = -i\frac{\omega}{D}.$$

With the aid of Eq. (3.27) the renormalized response function may be written as

$$G(\underset{\sim}{k}, \omega)^{-1} = G_0(\underset{\sim}{k}, \omega)^{-1} - 18i(\frac{\beta^2}{D})^* h(\underset{\sim}{k}, \omega, \varepsilon), \quad (4.14)$$

where

$$h(\underset{\sim}{k}, \omega, \varepsilon) = \frac{1}{4}B(2 - \frac{\varepsilon}{2}, \frac{\varepsilon}{2})B(1, \frac{\varepsilon}{2})^{-1}B(2 - \frac{\varepsilon}{2}, \varepsilon - 1)$$

Table 1. Comparison of the Present Model with Other Systems

Expo-nent	Liquid-Vapor[12,13]	Magnetic phase transi-tion[12,13]	λ-transi-tion in helium[14]	3-dimen-sional Ising model[15]	The present model	
					Gaussian case	Non-Gaussian case
ν	0.57	0.63	0.666 ±	0.642 ±	0.500	0.600
	0.63	0.72	0.006	0.003		
η		0.05 ±		0.055 ±	0	0.0556
		0.02		0.010		

$$\times \int_0^1 dx \, x^{-\frac{\varepsilon}{2}}(1+x)^{\varepsilon-1}(2+x)^{\frac{1}{2}\varepsilon-1} + \int_0^1 dy \, [(2+x-y)k^2$$

$$+ (2+x)(1+x)\omega']^{1-\varepsilon}y^{-\frac{\varepsilon}{2}}. \quad (4.15)$$

As $k \to 0$, the value of $h(k, \omega, \varepsilon)$ of the zeroth order of can be easily obtained. Substituting the value into Eq. (4.14) and using Eq. (4.11), we obtain the dynamic exponent Δ at the non-Gaussian fixed point as

$$\Delta = -\ln(\frac{4}{3})\eta. \quad (4.16)$$

From the relation of Δ and z, the exponent z is given by

$$z = 2 + [\ln(\frac{4}{3}) - 1]\eta. \quad (4.17)$$

We shall discuss the result in the next section.

Discussions

We have obtained the static and dynamic behaviors for a model of the metabolic control system with positive feed-back[1,2] with the aid of the well-known RG method.[5-8] The results are limited to the second order of $\varepsilon(=4-d)$. It cannot be directly judged whether they are suitable to explain a real system, since the critical behavior of the metabolic system has never been discussed before. Thus we may judge them indirectly by comparing the present results with the results of other systems, such as liquid-vapor and etc.[10-15]

The correlation function in the critical region can be expressed in the Ornstein–Zernike theory as[20]

$$C(r) \propto \frac{\exp(-r/\xi)}{r^{(d-1)/2}}; \ \xi \sim |u - u_c|^{-\nu}, \ (d \geqq 3) \quad (5.1)$$

for a fixed $\xi(>0)$ as r becomes large and

$$C(r) \propto \frac{\exp(-r/\xi)}{r^{d-2}} \quad (d \geqq 3) \quad (5.2)$$

for a fixed r as ξ becomes large. This expression can be reex-pressed in Fischer's modified theory as[21]

$$C(r) \propto r^{-(d-2-\eta)} \quad (d \geqq 3) \quad (5.3)$$

Some experimental and numerical results of the experiments for various systems are given in Table 1. The results except the Gaussian case are in good agreement with each other by considering the experimental errors since the experiments, especially for the liquid–vapor system, are very difficult to perform. This is due to the following reasons: Firstly, a long time is needed to establish an equilibrium and hysteresis

phenomena are difficult to avoid. Secondly, the system is very susceptible to minute amounts of impurities and highly sensitive to the gravitational field due to the large compressibility. The results may indicate that the Gaussian approximation does not hold for the present model of the metabolic control system in the critical region. The present result for the value of the exponent ν in the non–Gaussian case is smaller than that in the other results, since we have considered the terms up to the second order of ε-expansion. Thus, it can be improved by including the higher order terms.

Actually, the dynamic properties of fluctuations in the critical region are more complicated than the static properties, since various couplings between the variables can occur during the dynamic process. However, the dynamic problem has been reduced to the case that there is only one relaxation mode by neglecting the fast decaying mode in Section II and other hydrodynamic modes with the assumption that the system and the surroundings exchange energy very rapidly and sufficiently. Thus, the relaxation time of the model near the critical point is k^z, where z is given as $2 + (\ln \frac{4}{3} - 1)^\eta$.

Let us conclude by referring to some remarks:

(1) The question is open whether the present results for a model of the metabolic control system are applicable to real systems.

(2) The explicit calculation for the critical exponents is limited to the second order of ε in the ε-expansion. As mentioned above, we may take the terms to the infinite order. However, since the concentration variables have a large number of components of the state vector, n, the $1/n$-expansion method may be more effective than the ε-expansion. This problem is under investigation.

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