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Optimum Simple Step–Stress Accelerated Life Tests Under Periodic Observation

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ABSTRACT

This paper presents optimum simple step-stress accelerated life test plans for the case where the test process is observed periodically at intervals of the same length. Two types of failure data, periodically observed complete data and periodically observed censored data, are considered. An exponential life distribution with a mean that is a log-linear function of stress, and a cumulative exposure model for the effect of changing stress are assumed. For each type of data, the optimum test plan which minimizes the asymptotic variance of the maximum likelihood estimator of the mean life at a design stress is obtained and its behaviors are studied.

1. Introduction

Accelerated life tests(ALTs) provide information quickly on the life distribution of the materials or products by testing them at higher-than-usual levels of stress involving high temperature, voltage, pressure, vibration, cycling rate, load, etc. to induce early failures. The results obtained at the accelerated conditions are analyzed in terms of a life test model to relate life length to stress and then extrapolated to the design stress to estimate the life distribution. The model consists of a life distribution and a relationship for the distribution parameters in terms of the accelerating variables. Distributions commonly used

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are the exponential, Weibull, and lognormal distributions and the relationships are the Arrhenius relationship, inverse power law, and Eyring law, etc.

One way of applying stress to the test units is a step stress scheme which allows the stress setting of a unit to be changed at prespecified times or upon the occurrence of a fixed number of failures. The former is called time-step stress test and the latter failure-step stress test. Step-stress ALTs are widely used; for example, life testing of diodes (Bora(1979)), cable insulation(Nelson(1980)), and insulating fluid(Miller and Nelson(1983)) etc. The problems of designing optimum step-stress ALTs and making inferences have been studied by several authors. DeGroot and Goel(1979) studied a partially accelerated life test which combines both ordinary and accelerated life testing in a Bayesian framework. Nelson(1980) optained maximum likelihood estimators(MLES) for the parameters of Weibull distribution under the inverse power law using the breakdown time data of an electrical insulation. Shaked and Singpurwalla(1983) proposed a model based on shock models and wear processes, and obtained nonparametric estimator for the life distribution at use condition. Miller and Nelson(1983) obtained optimum simple step stress ALT plans for the case where test units have exponentially distributed life and are observed continuously until all test units are run to fail. Here the word 'simple' means that only two stress levels are used in a test. Bai et al.(1989) extended the results of Miller and Nelson(1983) to the case of censored data.

Different observation schemes generate different types of accelerated test data, which affect the determination of optimum plan. Four types of data can be considered:

D_1 : continuously observed complete data,
D_2 : continuously observed censored data,
D_3 : periodically observed complete data,
and D_4 : periodically observed censored data.

The purpose of this paper is to extend the results for time-step stress test considered by Miller and Nelson(1983) for D_1 and by Bai et al.(1989) for D_2 to those of D_3 and D_4. The asymptotic variance of the MLE of the mean life at a specified design stress is used as the optimality criterion. It is assumed that the number of test units and the stress levels are given and the test process is observed periodically at intervals of the same length. Exponentially distributed life of test units and a cumulative exposure model are also assumed. For each type of data, optimum test plan is obtained and its behaviors are analyzed. The results of D_1 and D_2 are reviewed in Section 2. In Sections 3 and 4, the optimum designs for D_3 and D_4 are considered, respectively.

Notation

n	number of test units
x_0, x_1, x_2	stresses(design, low, and high)
ξ	extrapolation amount: $\xi=(x_1-x_0)/(x_2-x_1)$
Y	failure time of test unit
θ_i	mean life at stress x_i, $i=0,1,2$
$F_i(y)$	cdf of exponential distribution with mean θ_i
G(y)	cdf of a test unit under simple step-stress
τ_i	stress change point for continuously observed data D_i, $i=1,2$
r_i	stress change point for periodically observed data D_i, $i=3,4$
T	censoring point for continuously observed data D_i, $i=1,2$
l	censoring point for periodically observed data D_i, $i=3,4$
h	observation interval
$V_i(\cdot)$	asymptotic variance of the MLE of the mean life at design stress for D_i, $i=1,2,3,4$

Assumptions

1) Two stresses x_1 and x_2 ($x_1<x_2$) are used.
2) For any level of stress, the life distribution of test unit is exponential.
3) The mean life of a test unit at lower stress is longer than at higher stress, i.e., $\theta_0>\theta_1>\theta_2$.
4) The mean life of a test unit is a log-linear function of stress. That is,

$$\log \theta(x)=\alpha+\beta x, \tag{1}$$

where α and β are unknown parameters depending on the nature of the product and the method of test. If x is the log of voltage stress, then (1) is the inverse power law. If x is the reciprocal of absolute temperature, then (1) is the Arrhenius relationship.
5) Cumulative exposure model holds. That is, the remaining life of a test unit depends only on the exposure it has seen and the unit does not remember how the exposure was accumulated (see Miller and Nelson(1983)).
6) The stress change point is at the end of r th interval, i.e., at time rh.
7) The censoring point is at the end of l th interval, i.e., at time lh.

2. Continuously Observed Complete and Censored Data

Suppose that n test units are initially placed on low stress x_1 and run until time τ, when the stress is changed to x_2 and the test is continued until all units fail. Also, suppose that the test process is observed continuously in time. Then the failure data are of type D_1. If the test is continued until a predetermined censoring time T, $T>\tau$, then the data are of type D_2. In this section, we review the optimum simple time-step stress test plans for D_1 and D_2.

Theorem 1. (i) The asymptotic variance $V_i(\tau)$ of the MLE of mean life at design stress for D_i, $i=1,2$, is given by

$$\begin{aligned} n\cdot V_i(\tau) &= n\cdot \mathrm{Asvar}(\log\hat{\theta}_0) \\ &= n\cdot \mathrm{Asvar}(\hat{\alpha}+\hat{\beta}\cdot x_0) \\ &= (1+\xi)^2/A_{1i}(\tau)+\xi^2/A_{2i}(\tau), \end{aligned} \tag{2}$$

where $A_{11}(\tau)= A_{12}(\tau)= 1-\exp(-\tau/\theta_1)$, $A_{21}(\tau)= \exp(-\tau/\theta_1)$,
and $A_{22}(\tau)=\exp(-\tau/\theta_1)\cdot[1-\exp\{-(T-\tau)/\theta_2\}]$.

(ii) The optimum stress change point τ_i^* for D_i, $i=1,2$, is given by:

a) $\tau^*_1=\theta_1\log[(1+2\xi)/\xi]$.

b) τ_2^* is obtained uniquely by the solution of (3)

$$\left[\frac{A_{12}(\tau)}{A_{22}(\tau)}\right]^2 \frac{A_{22}(\tau)+(\theta_1/\theta_2)\{1-A_{12}(\tau)-A_{22}(\tau)\}}{1-A_{12}(\tau)} = \left(\frac{1+\xi}{\xi}\right)^2 \tag{4}$$

Proof. See Miller and Nelson(1983) and Bai et al.(1989). ■

Note that $A_{ij}(\tau)$, $i,j=1,2$, is the probability that a test unit will fail while testing at stress x_i.

From formula (3) and (4) it can be shown that;

i) τ_1^* increass in θ_1 for given ξ.

ii) τ^*_2 increases in θ_1 for given θ_2, ξ, and T.

iii) τ^*_2 increases in T for given θ_1, θ_2, and ξ

iv) $\lim_{T\to\infty} \tau^*_2=\theta_1\log\{(1+2\xi)/\xi\}=\tau^*_1$.

3. Periodically Observed Complete Data

Suppose that units are tested exactly as described for D_1 in Section 2, but the process is observed periodically at intervals of length h; that is, the data are of type D_3. In this section, we assume that the stress change point is at the end of r^{th} interval, i.e., at time rh.

From the assumptions of cumulative exposure model and exponentially distributed life, the cdf of life of a test unit under simple step stress test is as follows:

$$G(y)= \begin{cases} F_1(y) \text{ for } 0\le y<\tau, \\ F_2(s+y-\tau) \text{ for } \tau\le y<\infty, \end{cases}$$

where s is the solution of $F_2(s)= F_1(\tau)$. Thus the likelihood function from a single observation Y is

$$L(\theta_1,\theta_2)= \prod_{i=1}^{r} \left[\int_{(i-1)h}^{ih}(1/\theta_1)\exp(-y/\theta_1)\, dy \right]^{\delta_i} \cdot \prod_{i=r+1}^{\infty} \left[\int_{(i-1)h}^{ih}(1/\theta_2)\exp(-rh/\theta_1-(y-rh)/\theta_2)dy \right]^{\delta_i}, \tag{5}$$

where, for $i=1,2,\cdots$,

$$\delta_i(y) = \begin{cases} 1 \text{ if}(i-1)h\le y<ih, \\ 0 \text{ otherwise}. \end{cases}$$

Substituting (1) for θ_i, $i=1,2$, in (5) and taking logarithm, we obtain the log-likelihood as a function of the unknown parameters α and β;

$$\begin{aligned} \log L(\alpha,\beta)&=b_1\log(1-g_1(\alpha,\beta))-b_2\exp(-\alpha-\beta x_1) \\ &+ b_3\log(1-g_2(\alpha,\beta))-b_4\exp(-\alpha-\beta x_2) \end{aligned} \tag{6}$$

where $b_1= \sum_{i=1}^{r} \delta_i$, $b_2= h \cdot [\sum_{i=1}^{r}(i-1)\delta_i +\sum_{i=r+1}^{\infty} r\delta_i]$, $b_3= \sum_{i=r+1}^{\infty} \delta_i$, $b_4= h \sum_{i=r+1}^{\infty} (i-r-1)\delta_i$,

and $g_i(\alpha,\beta)= \exp(-h \cdot \exp(-\alpha-\beta x_i))$, $i=1,2$.

From (6), the Fisher information matrix I_1 for a single observation Y is obtained by taking the expectations of the second partial and mixed partial derivatives of $\log L(\alpha,\beta)$ with respect to α and β;

$$I_1= \begin{bmatrix} B_1(r)+B_2(r) & B_1(r)x_1+B_2(r)x_2 \\ B_1(r)x_1+B_2(r)x_2 & B_1(r)x_1^2+B_2(r)x_2^2 \end{bmatrix}, \tag{7}$$

where $\beta_1(r)=\dfrac{(h/\theta_1)^2 e^{-h/\theta_1}(1-e^{-rh/\theta_1})}{(1-e^{-h/\theta_1})^2}$ and $\beta_2(r)=\dfrac{(h/\theta_2)^2 e^{-h/\theta_2} e^{-rh/\theta_1}}{(1-e^{-h/\theta_2})^2}$

Since the information matrix I_n obtained from n independent observations is n times of I_1, the asymptotic variance of the MLE of the mean life at the design stress is then given by

$$\begin{aligned} n\cdot V_3(r) &= n\cdot \text{Asvar}(\log \hat{\theta}_0) \\ &= n\cdot \text{Asvar}(\hat{\alpha}+\hat{\beta}\cdot x_0) \\ &= (1+\xi)^2/B_1(r)+\xi^2/B_2(r), \end{aligned} \tag{8}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are MLE's of α and β. The following theorem gives the optimum stress change point for D_3

Theroem 2. The optimum stress change point r^* for D_3 is obtained uniquely by

$$r^*=\begin{cases}[r_1] & \text{if } V_3([r_1]) \le V_3([r_1+1]) \\ [r_1+1] & \text{if } V_3([r_1]) > V_3([r_1+1]),\end{cases} \tag{9}$$

where [·] is the Gaussian symbol and

$$r_1=\frac{\theta_1}{h}\cdot \log\{1+\frac{(1+\xi)}{\xi}\cdot\frac{\theta_1}{\theta_2}\cdot\frac{\sinh(h/2\theta_1)}{\sinh(h/2\theta_2)}\}$$

Proof. Let

$$K_1=(1+\xi)^2\cdot(\theta_1/h)^2\cdot\exp(h/\theta_1)\cdot\{1-\exp(-h/\theta_1)\}^2$$

and

$$K_2=\xi^2(\theta_2/h)^2\cdot\exp(h/\theta_2)\cdot\{1-\exp(-h/\theta_2)\}^2$$

Then formula (8) can be rewritten as;

$$n\cdot V_3(r)=K_1/[1-\exp(-rh/\theta_1)]+K_2\exp(rh/\theta_1) \tag{10}$$

By differentiating $n\cdot V_3(r)$ with respect to r, we then obtain

$$d[n\cdot V_3(r)]/dr=(h/\theta_1)\cdot\{-K_1\cdot\exp(-rh/\theta_1)/[1-\exp(-rh/\theta_1)]^2+K_2\exp(rh/\theta_1)\}, \tag{11}$$

and $d^2[n \cdot V_3(r)]/dr^2=(h/\theta_1)^2 \cdot \{K_1 \cdot \exp(-rh/\theta_1) \cdot (1+\exp(-rh/\theta_1))/[1-\exp(-rh/\theta_1)]^3+K_2\exp(rh/\theta_1)\}$. (12)

Since K_i's are positive, $d^2[n \cdot V_3(r)]/dr^2 0$, which means that (8) is a convex function of r. Setting (11) to be zero and after some calculation, (11) becomes (9). This completes the proof. ■

From (9), it can be shown that;

i) r^* increases in θ_1 for given θ_2, ξ, and h.

ii) r^* decreases in h for given θ_1, θ_2, and ξ.

iii) $\lim_{\substack{h \to 0 \\ r_1 \to \infty}} h \cdot r_1=\theta_1 \cdot \log\{(1+2\xi)/\xi\}=\tau_1^*$.

Figure 1 shows the effects of θ_1 and h on r^* for values of $\xi=1$, $\theta_2=100$, and h=1, 5, 10. r^* is seen to be approximately linear in θ_1 in the log scale, which can also be shown by Taylor expansion of $\sinh(h/2\theta_1)$.

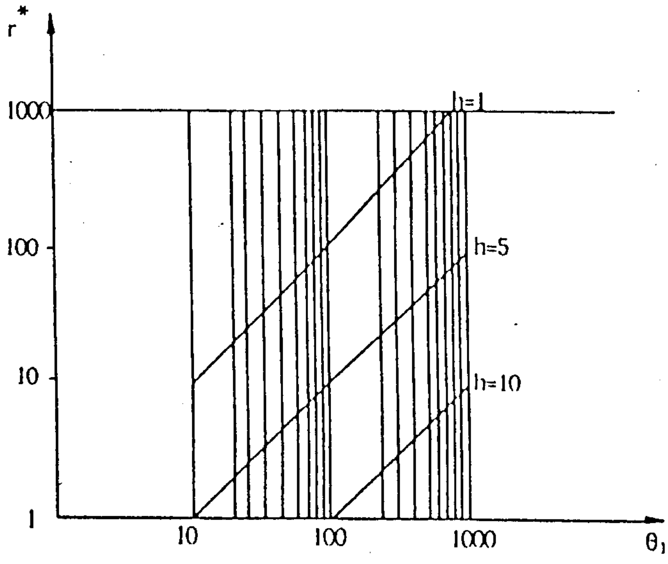

Figure 1. Effect of θ_1 and h on r^* ($\xi=1$, $\theta_2=100$)

Note that the optimum test plan obtained from Theorem 2 depends on θ_1 and θ_2 which are usually unknown. To use the plan, one must have preestimates of θ_1 and θ_2.

Example 1. Suppose that we wish to test the life of diodes by simple time-step stress test, and available preestimates of θ_1 and θ_2 are 1300(min.) and 150(min.), respectively. Also suppose that $\xi=1.5$, n=5, and h=60(min). The optimal stress change point r^* is then obtained as 21 from(9). That is, the failures of the test units are observed at 60 minutes intervals and the stress is elevated at the end of 21th interval, and inspection continues periodically until all test units are found to be dead.

4. Periodically Observed Censored Data

Suppose that units are tested exactly as described in Section 3, but the process is terminated at the end of ℓ^{th} interval, i.e., at time ℓh; that is, the data are of type D_4.

The likelihood function of a single observation Y is

$$L(\theta_1,\theta_2)=\prod_{i=1}^{r}\left[\int_{(i-1)h}^{ih}(1/\theta_1)\cdot\exp(-y/\theta_1)dy\right]^{\delta_i}$$
$$\prod_{i=r+1}\left[\int_{(i-1)h}^{ih}(1/\theta_2)\cdot\exp(-rh/\theta_1-(y-rh)/\theta_2)dy\right]^{\delta_i} \tag{13}$$
$$\left[\exp\{-rh/\theta_1-(\ell-r)h/\theta_2\}\right]^{\delta_{\ell+1}}$$

where, for $i=1,2,\cdots,\ell$,

$$\delta_i=\begin{cases}1 & (i-1)h\leq y<ih,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\delta_{l+1}=\begin{cases}1 & y\geq lh,\\ 0 & \text{otherwise}\end{cases}$$

By substituting (1) for θ_i, $i=1,2$, in (13) and taking logarithm, we obtain the log-likelihood function;

$$\log L(\alpha,\beta)= c_1\log(1-g_1(\alpha,\beta))-c_2\exp(-\alpha-\beta x_1)$$
$$+c_3\log(1-g_2(\alpha,\beta))-c_4\exp(-\alpha-\beta x_2), \tag{14}$$

where $c_1=\sum_{i=1}^{r}\delta_i$, $c_2=h\cdot[\sum_{i=1}^{r}(i-1)\delta_i+\sum_{i=r+1}^{\ell} r\delta_i+r\delta_{\ell+1}]$, $c_3=\sum_{i=r+1}^{\ell}\delta_i$,

$c_4=h\cdot[\sum_{i=r+1}^{\ell}(i-r-1)\delta_i+(\ell-r)\delta_{\ell+1}]$, and $g_i(\alpha,\beta)=\exp(-h\cdot\exp(-\alpha-\beta x_i))$, $i=1,2$.

The Fisher information matix I_1 is

$$I_1=\begin{bmatrix} C_1(r)+C_2(r,\ell) & C_1(r)x_1+C_2(r,\ell)x_2 \\ C_1(r)x_1+C_2(r,\ell)x_2 & C_1(r)x_1^2+C_2(r,\ell)x_2^2 \end{bmatrix}, \tag{15}$$

where
$$C_1(r)=\frac{(h/\theta_1)^2 e^{-h/\theta_1}(1-e^{-rh/\theta_1})}{(1-e^{-h/\theta_1})^2}=B_1(r),$$

$$C_2(r,\ell)=\frac{(h/\theta_2)^2 e^{-h/\theta_2}e^{-rh/\theta_1}(1-e^{-(\ell-r)h/\theta_1})}{(1-e^{-h/\theta_2})^2}=B_2(r)\{1-e^{-(\ell-r)h/\theta_1}\}.$$

Note that $\lim_{\ell\to\infty} C_2(r,\ell)=B_2(r)$. The asymptotic variance $V_4(r,\ell)$ of the MLE of the mean life at the design stress is then given by

$$\begin{aligned} n\cdot V_4(r,\ell) &= n\cdot \mathrm{Asvar}(\log\hat{\theta}_0) \\ &= n\cdot \mathrm{Asvar}(\hat{\alpha}+\hat{\beta}\cdot x_0) \\ &= (1+\xi)^2/C_1(r)+\xi^2/C_2(r,\ell), \end{aligned} \tag{16}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are MLE's of α and β. The optimum stress change point for D_4 is given by the following theorem whose proof is similar to that of Theorem 2 and therefore omitted.

Theorem 3. The optimum stress change point r^* for D_4 is uniquely obtained by

$$r^*=\begin{cases} [r_2] & \text{if } V_4([r_2])\leq V_4([r_2+1]) \\ [r_2+1] & \text{if } V_4([r_2])>V_4([r_2+1]), \end{cases} \tag{17}$$

where $[\cdot]$ is the Gaussian symbol and r_2 satisfies

$$\left[\frac{C_1(r_2)}{C_2(r_2,\ell)}\right][\exp(r_2h/\theta_1)-1]\cdot\left[1+\frac{\theta_1}{\theta_2}\cdot\frac{e^{-(\ell-r_2)h/\theta_2}}{(1-e^{-(\ell-r_2)h/\theta_2})}\right]=\left(\frac{1+\xi}{\xi}\right)^2.$$

The behavior of r^* for D_4 can easily be inferred from the interrelations of D_i's, $i=1,\cdots,4$. Note that:

i) $\lim_{\ell\to\infty} V_4(r,\ell)=V_3(r)$,

ii) $\lim_{h\to 0} V_4(r,\ell)=V_2(\tau)$ if $\lim_{r\to\infty,\,h\to 0} rh=\tau$ (fixed) and $\lim_{\ell\to\infty,\,h\to 0}\ell h=T$(fixed).

Therefore the behavior of r^* for D_4 is similar to that D_2 for small h, D_3 for large ℓ, and D_1 for small h and large ℓ simultaneously.

Example 2. In Example 1, suppose that $\ell=24$. Then we obtain $r^*=14$ from (17) Thus the failure of the test units are observed at 60 minutes intervals and the stress is elevated at the end of 14th interval and inspection continues periodically until 24th interval at which the test is terminated.

5. Concluding Remarks

We have studied simple time-step stress test plans under the periodic observation. Two types of data generated by observation schemes, periodically observed complete data and periodically observed censored data are considered. For each type of data, optimum test plan is obtained and its behaviors are studied. The optimum test plans obtained in Theorems 2 and 3 depend on the values of model parameters θ_1 and θ_2. To use the plan, therefore, these parameters must be approximated from experience, similar data, or a preliminary test data. Since incorrect choice of preestimates gives a test plan different from the optimum plan and may result in poor estimates of the parameters of life distribution at design stress, the effect of preestimates should be in vestigated.

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