⟨Research Paper⟩

Journal of the Korean Statistical Society
Vol. 18, No. 2, 1989

Exact Variance of Location Estimator in One-Way Random Effect Models with Two Distinct Group Sizes.

Youngjo Lee* and Han Yeong Chung*

ABSTRACT

In the one−way random effect model, we often estimate the variance components by the ANOVA method and then estimate the population mean. When there are only two distinct group sizes, the conventional mean estimator is represented as a weighted average of two normal means with weights being the function of variance component estimators. In this paper, we will study a method which can compute the exact variance of the mean estimator when we set the negative variance component estimate to zero.

1. Introduction

In the one−way random effect model, if the values of the variance components are known, we can obtain the best linear unbiased estimator(BLUE) of the population mean. However, the values of the variance components are usually unknown. In this case, we often estimate the variance components by the ANOVA method, and then, regarding the estimates as the true values, we estimate the populaton mean. For the one−way random effect model, Swallow(1981)and Swallow and Monahan(1984)conducted the Monte Carlo comparisions of the ANOVA estimators and other variance components estimators and found that the ANOVA estimators are adequate unless the inter−class correlation coefficient $\rho > 0.5$. In addition to this, they have other appealing properties: they are familiar, easy to compute, and are unbiased. Hence, it would be interesting to develop a method

+ Department of Statistics, Hallym University, Chunchon, Korea

with which we can compute the exact variance of the mean estimator with the variance components being estimated by the ANOVA method.

Khatri and Shah(1975) introduced a method for computing the exact variance of the combined inter—and intra—block estimates of treatment effects in incomplete block designs. Their method could be used for computing the exact variance of the estimator of the mean in the one—way random effect model when there are only two distinct group sizes. However, their expression is the infinite sum of the complicated terms involving incomplete Beta functions and one dimensinonal integrations. Seely, El—Arish and Lee (1989) represents the exact variance as a finite sum of one—dimensional integrations. One disavantage of both of these methods is that it cannot be used when the variance components are truncated. However, in an experimental situation, it is common practice to set the negative variance component estimate to zero. Hence, in this paper, we will develop a method which can be used for computing the exact variance of the mean estimator in the case of truncating of the variance component estimates when there are two distinct group sizes.

2. The Problem Setting

Consider the one—way random effects model with two distinct group sizes. Suppose that there are t different groups separated into two classes in such a way that the ith class consists of m_i groups, and that each group in the class has n_i observations. Thus, we have $t=m_1+m_2$ groups and $n=m_1n_1+m_2n_2$ observations. We can express this model as,

$$y_{ijk}=\mu+a_{ij}+e_{ijk},\ i=1,2,\ j=1,\ \ldots,\ m_i\ k=1,\ldots,n_i,$$

where the a_{ij} and e_{ijk} are independent random variables with $a_{ij}\sim N(0,\sigma_a^2)$ and $e_{ijk}\sim N(0,\alpha_e^2)$. Let $\pi=\alpha_a^2+\alpha_e^2$ and $\rho=\alpha_a^2/\pi$. When ρ is known, the BLUE can be written as

$$\hat{\mu}(\rho)=\sum_i w_i(\rho)\bar{y}_i, \tag{1.1}$$

where $w_i(\rho)=[m_in_i/\{n_i\rho+(1-\rho)\}]/[\sum_i m_in_i/\{n_i\rho+(1-\rho)\}]$ and $\bar{y}_i=\sum_j\sum_k y_{ijk}/(m_in_i)$. In practice, ρ is usually unkown. The ANOVA estimatiors of variance components are obtained from the ususal between and within the sum of squares,

$$SSA=\sum_i\sum_j n_i(\bar{y}_{ij}-\bar{y})^2=\sum_i\sum_j n_i(\bar{y}_{ij}-\bar{y}_i)^2+\sum_i m_in_i(\bar{y}_i-\bar{y})^2,$$

and $$SSE=\sum_i \sum_j \sum_k (y_{ijk}-\bar{y}_{ij})^2,$$

where $\bar{y}_{ij}=\sum_k y_{ijk}/n_i$, $\bar{y}=\sum_i \sum_j \sum_k y_{ijk}/n$. Let $S1=\sum_j n_1(\bar{y}_{1j}-\bar{y}_1)^2$, $S2=\sum_j n_2(\bar{y}_{2j}-\bar{y}_2)^2$ and $S3=\sum_i m_i n_i(\bar{y}_i-y)^2=(m_1n_1m_2n_2/n)(\bar{y}_1-\bar{y}_2)^2$. Note that SSA=S1 +S2 +S3 . Then it can be shown that SSE, S1,S2 and S3 are independent random variables with distributions;

$$SSE\sim\pi(1-\rho)\chi^2_{n-t},$$
$$S1\sim\pi\{n_1\rho+(1-\rho)\}\chi^2_{m_1-1},$$
$$S2\sim\pi\{n_2\rho+(1-\rho)\}\chi^2_{m_2-1},$$
and $$S3\sim\pi\{(n_1n_2t/n)\rho+(1-\rho)\}.$$

The ANOVA estimator of σ_e^2 and σ_a^2 is (see Searle, 1971, page 474),

$$s_e^2=SSE/(n-t),$$
and $$s_a^2=\{SSA-(t-1)s_e^2\}/\{n-(m_1n_1^2+m_2n_2^2)/n\}. \qquad (1.2)$$

Note that the equation(1.1) is equivalent to

$$\hat{\mu}(\rho)=w_1(\rho)(\bar{y}_1-\bar{y}_2)+\bar{y}_2,$$

where $w_1(\rho)=(m_1n_1n_2\sigma_a^2+m_1n_1\sigma_e^2)/(n_1n_2t\sigma_a^2+n\sigma_e^2)=1-m_2/t+\{m_1m_2(n_1-n_2)\sigma_e^2\}/\{t(n_1n_2t\sigma_a^2+n\sigma_e^2)\}$. Hence, if we replace the variance components by their estimates, we have

$$\hat{\mu}(\hat{\rho})=W(S1,S2,(\bar{y}_1-\bar{y}_2)^2,s_e^2)(\bar{y}_1-\bar{y}_2)+\bar{y}_2, \qquad (1.3)$$

where $W(S1,S2,(\bar{y}_1-\bar{y}_2)^2,s_e^2)=1-m_2/t+os_e^2/(n_1n_2ts_a^2+ns_e^2)$
$=1-m_2/t+os_e^2/(pS1+pS2+q(\bar{y}_1-\bar{y}_2)^2+rs_e^2)$, $o=m_1m_2(n_1-n_2)/t$, $p=n_1n_2t/\{n-(m_1n_1^2+m_2n_2^2)/n\}$, $q=n_1^2n_2^2m_1m_2t/\{n^2-(m_1n_1^2+m_2n_2^2)\}$ and $r=n-n_1n_2t(t-1)/\{n-(m_1n_1^2+m_2n_2^2)/n\}$.

From the above expression, $\hat{\mu}(\hat{\rho})$ is represented as the combined estimator of two normal means with the weight of W being the function of the quadratic forms of observations. Hence, Khatri and Shah's method(1975) is applicable to compute the exact variance of $\hat{\mu}(\hat{\rho})$ when there are two distinct group sizes. Seely et al.(1989) derived another representation for the exact variance of $\hat{\mu}(\hat{\rho})$ using the canonical forms. Their method can be applicable to general cases without any restriction on group sizes.

The ANOVA estimate s_a^2 can take a negative value. In this case, it is a common practice to truncate a negative estimate to zero. However, both of the above two methods cannot be used when variance component estimates are truncated. In this paper, we will develop a method to overcome this problem. However, one restriction of the new method is that it can be used only when there are two distinct group sizes.

The problem of combining two normal means is formulated as(1.3), and therefore, the new method can be applicable to compute the exact variance of the combined estimator even in the case of truncating a negative variance component estimate at zero.

3. The Main Result

Let Y be a p−dimensional random vector with distribution,

$$Y \sim N(\theta, I_p),$$

where I_p denotes the p−dimensional identity matrix. Let $U=Y'Y$. Lee(1987) showed the following Theorem:

Theorem 1. Suppose we have two estimatiors for θ, $\hat{\theta}_1=Y+g_1(U)Y$ and $\hat{\theta}_2=Y+g_2(U)Y$. If functions g_1 and g_2 are differentiable and the estimators, $\hat{\theta}_1$ and $\hat{\theta}_2$, have finite mean squared error risks, then the estimator of the form

$$\hat{\theta}=I_{(U\leq u)}\hat{\theta}_1+I_{(U>u)}\hat{\theta}_2,$$

where $I_{(\ )}$ is an indicator function and u is a positive constant, has the risk,

$$E\{(\hat{\theta}-\theta)'(\hat{\theta}-\theta)\}=E[I_{(U\leq u)}\phi_1(U)+I_{(U>u)}\phi_2(U)]$$
$$+4u(g_2(u)-g_1(u)f(u),$$

where $\phi_i(U)=p+g_i(U)^2U+2pg_i(U)+4g_i'(U)U$, g_i' is the derivative of g_i and f is the density function of U.

Let Y be a p−dimensional random vector with the distribution $Y \sim N(\theta, \sigma^2 I_p)$ and $V=(Y-A)'(Y-A)$, where A is a p−dimensional constant vector. Suppose we have two estimatiors for θ, $\hat{\theta}_1=(1+g_1(V))(Y-A)+A$ and $\hat{\theta}_2=(1+g_2(V))(Y-A)+A$. Then, from Theorem 1, we have the following Corollary.

Corollary 1. Let $\hat{\theta}=I_{(V\leq v)}\,\hat{\theta}_1+I_{(V>v)}\,\hat{\theta}_2$. If g_1 and g_2 are differentiable and the estimators, $\hat{\theta}_1$ and $\hat{\theta}_2$, have finite mean squared error risks, then,

$$E\{(\hat{\theta}-\theta)'(\hat{\theta}-\theta)\}= E[1_{(V\leq v)}(V)+1_{(V>v)}\,\varphi_2(V)]$$
$$+4\sigma^2 v(g_2(v)-g_1(v))f(v),$$

where $\varphi_i(V)= p\sigma^2+g_i(V)^2V+2pg_i(V)\sigma^2+4g_i'(V)V\sigma^2$ and f is the density function of V.

Proof. Let $Z=(Y-A)/\alpha_a$. Then Z is a random vector with the distribution $Z\sim N(\nu,I_p)$, where $\nu=(\theta-A)/\alpha$. By applying Theorem 1 to Z, we can have the desired result. ∎

Now consider the estimator of the form(1.3),

$$\hat{\mu}(\hat{\rho})=W(S1,\ S2,\ (\bar{y}_1-\bar{y}_2)^2,\ s_e^2)(\bar{y}_1-\bar{y}_2)+\bar{y}_2$$

Note that the condition $s_a^2\leq 0$ is equivalent to $(\bar{y}_1-\bar{y}_2)^2\leq D(S1,S2,s_e^2)$, where $D(S1,S2,s_e^2)=\{(t-1)s_e^2-S1-S2\}n/(n_1n_2m_1m_2)$. For notational convenience, we will denote functions W( ) and D( )as W and D without specifying the arguments. If we set the negative estimate s_a^2 to zero, the estimator $\hat{\mu}(\hat{\rho})$ becomes

$$\hat{\mu}(\hat{\rho})=I_{(V\leq D)}[(1-m_2n_2/n)(\bar{y}_1-\bar{y}_2)+\bar{y}_2]+I_{(V>D)}[W(\bar{y}_1-\bar{y}_2)+\bar{y}_2],$$

where $V=(\bar{y}_1-\bar{y}_2)^2$. Note that when $V\leq D$, $\hat{\mu}(\hat{\rho})$ becomes the over-all mean $\bar{y}$. Since $\bar{y}_1\sim N(\mu,\sigma^2)$, where $\sigma^2=\sigma_a^2/m_1+\sigma_e^2/m_1n_1$, we have the following Theorem.

Theorem 2. $Var(\hat{\mu}(\hat{\rho}))=\{I_{(V\leq D)}\varphi_1+I_{(V\geq D)}\varphi_2\}$,

where $\varphi_1=(m_2n_2/n)^2V+(m_1n_1-m_2n_2)\sigma^2/n$, $\varphi_2=(m_1-m_2)\sigma^2/t+(m_2/t)^2V+2(\sigma^2-m_2V/t)os_e^2/(pS1+pS2+qV+rs_e^2)+os_e^2V(os_e^2-4q\sigma^2)/(pS1+pS2+qV+rs_e^2)^2$ and o, p, q and r are defined in equation(1.3).

Proof. Seely and Hogg (1982) showed that $\hat{\mu}(\hat{\rho})$ is an unbiased estimator. Hence, the mean squared error risk is same as the variance. Set $g_1(V)=-m_2n_2/n$ and $g_2(V)=-m_2/t+os_e^2/(pS1+pS2+qV+rs_e^2)$. Note tat $g_1(D)=g_2(D)$ Since S1, S2, s_e^2, $\bar{y}_1$ and $\bar{y}_2$ are independent random variables and $Var(\hat{\mu}(\hat{\rho}))=E[E\{\hat{\mu}(\hat{\rho})-\mu)^2 \mid S1,\ S2,\ s_e^2,\ \bar{y}_2\}]$, by applying corollary 1 to the conditional expectation we have the desired result. ∎

4. An Example

In this section, we will show how the Theorem 2 can be used to compute the exact variance of $\hat{\mu}(\hat{\rho})$. For the sake of simplicity, we will show the case for $m_1=m_2=1$. In this case, Sl=S2=0 and the $\hat{\mu}(\hat{\rho})$ becomes

$$\hat{\mu}(\hat{\rho})=I_{(V\leq D)}\bar{y}+I_{(V>D)}\,W(\bar{y}_1-\bar{y}_2)+\bar{y}_2,$$

where $D=ns_e^2/(n_1n_2)$, $W=1-1(1-C/V)/2$ and $C=(n_1-n_2)s_e^2/(n_1n_2)$. Hence, by Theorem 2, we have

$$Var(\hat{\mu}(\hat{\rho}))=\int_0^\infty\int_0^D\varphi_1\,dF_V dF_{s_e^2}+\int_0^\infty\int_n^\infty\varphi_2\,dF_V\,dF_{s_e^2}\quad,$$

where $\varphi_1=(n_2/n)^2V+(n_1-n_2)\sigma^2/n$ and $\varphi_2=V/4-C/2+C(C-4\sigma^2)/(4V)$. Note that $V/\pi\{2\rho+(1/n_1+1/n_2)(1-\rho)\}\sim\chi_1^2$ and $(n-2)s_e^2/\pi(1-\rho)\sim\chi_{n-2}^2$. Hence, the density functions, dF_V and $dF_{s_e^2}$, can be obtained. Alternatively, we can campute the above integration by first applying the integral by parts. In this case, the distribution function of χ^2 random variables can be directlv obtained from IMSL subroutine MDCM.

Under the normality, the BLUE, $\hat{\mu}(\rho)$, is the uniformly minimum variance unbiased estimator and its variance attains the Cramer-Rao lower bound, In the table below, we compute the efficiency of $\hat{\mu}(\hat{\rho})$, which is $Var_\rho(\hat{\mu}(\rho))/Var_\rho(\hat{\mu}(\hat{\rho}))$ can be obtained by computing the above intergration. This was conducted by Micro VAX using IMSL subroutine DBLIN. From the table, we can see the efficiency of $\hat{\mu}(\hat{\rho})$ decreases as the difference between n_1 and n_2 increases.

Table 1. Efficiencies of $\hat{\mu}(\hat{\rho})$ when $n_1=2$

n_2 \ ρ	0.05	0.25	0.50	0.75	0.95
4	0.9624	0.9837	0.9926	0.9972	0.9995
8	0.8792	0.9554	0.9807	0.9929	0.9988
16	0.7845	0.9316	0.9714	0.9898	0.9983

5. Conclusion

In this paper, we showed a method which can be used for computing the exact variance of the location estimator for the one-way random effect model when there are two distinct group sizes. Allowing the truncation of the variance components estimators makes this problem quite complicated. However, it would be interesting to develop a method which can be applicable to general group sizes. As in Khatri and Shah(1975), the combined estimator of the inter—and intra—block estimators is formulated as (1.3). But, their method could not be applicable to compute the exact variance of the combined estimator if we set a negative variance component estimate to zero. The new method can be used for this case.

Acknowledgements

This research is supported by a research grant from Hallym University.

References

(1) Khatri, C.G. and Shah, K.R.(1975). Exact variance of Combined Inter and Intra—Block Estimators in Incomplete Block Designs. *The Journal of the American Statistical Association,* 70, 402—406

(2) Lee, Y. (1987). The Exact Mean Squared Error Risks of Preliminary Test Type Estimators. *Communications in Statistics,* Theory and Method. 16(7), 1965—1975.

(3) Searle, S.R.(1971). *Linear Model.* John Wiley & Sons.

(4) Seely, J. El-Arishy, S. and Lee, Y.(1989). An Efficient Estimator of the Mean in One—Way Random Effect Models, manuscript prepared for submission.

(5) Seely, J. and Hogg, R.V.(1982). Symmetrically Distributed and Unbiased Estimators in. Linear Models. *Communications in Statistics.* Theory and Method, A11(7), 721-729.

(6) Swallow, W.H.(1981). Variances of Locally Minimum Variance Quadratic Unbiased Estimators("MIVQUE's") of Variance Components. *Technometrics,*23, 272—283.

(7) Swallow, W. H. and Monahan, J.F.(1984). Monte Carlo Comparision of ANOVA, MIVQUE, REML, and ML Estimators of Variance Components. *Technometrics,* 26, 47—57.