〈Research Paper〉

Journal of the Korean
Statistical Society
Vol. 18, No.1, 1989

A Test Procedure for Change in Level Occurring at Unknown Points+

Jae Chang Lee*, Il Seong Song**

ABSTRACT

A procedure is considered to the problem of testing whether there exist changes in location at possibly two points in a sequence of independent random variables which are successively drawn from normal population. A test statistic based on modified likelihood ratio is proposed and its asymptotic null distribution is derived through the stochastic process representation. A small sample power comparison is made by Monte Carlo method.

1. Introduction

Consider a specific statistical problem where independent observations are successively generated and the distributions of the individuals in this random sequence are subject to change at possibly two unknown points. The following applications are a few of such examples which can be expressed by this change-points problem:

i) stablity of some industrial output over time within a statistical quality control setting,

ii) variation of share prices on the stock exchange,

iii) time-series of meteorological data such as annual rainfall at a certain spot or annual temperature variation at a fixed time in season, and

iv) movement of population from one location to another over a period of time.

+ Research supported by Korea Science and Engineering Foundation under Grant 881-0105-002-1.

* Department of Statistics, Korea University, Seoul.

** Department of Statistics, Sungshin Women's University, Seoul.

We investigate a testing procedure for problems of this kind. We are concerned with the model that deals with possible changes in location-shift parameter. To describe the model more specifically, let $X_1, X_2, \cdots, X_N$ be independent random variables that are successively drawn from normal populations with means $\mu_1, \mu_2, \cdots, \mu_N$, respectively and common known variance σ^2. We consider the problem of testing the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_N = \mu(\text{unknown})$$

against the alternative

$$\begin{aligned} H_1 : \mu_1 &= \mu_2 = \cdots = \mu_{\tau_1} = \mu \\ \mu_{\tau_1+1} &= \cdots = \mu_{\tau_2} = \mu + \Delta_1 \\ \mu_{\tau_2+1} &= \cdots = \mu_N = \mu + \Delta_2 \ , \end{aligned} \tag{1.1}$$

where τ_1, τ_2 $(1 \leqq \tau_1 < \tau_2 < N)$ are unknown, and Δ_1, Δ_2 are also unknown but at least one of them is not zero. Then this sequence of random variables is said to have so-called change-points τ_1, τ_2. In this paper we refer to this statistical setting as at most two change-points (AMTC) problem, which is an extension of two-sided at most one change-point(AMOC) problem. We propose a normal theory test based on modified likelihood ratio for the AMTC problem and derive its asymptotic null distribution through the stochastic process representation.

Change-point problems for AMOC model was first considered by Page(1954,1955) and thereafter extensively studied by various authors, including testing and estimation of the change-point or estimation of the magnitude of change. These studies lead to estimators and test statistics in parametric and nonparametric, classical and Bayesian frameworks. Gardner(1969) considered estimation of current mean and testing problem following Bayesian approach. Hawkins(1977) suggested the likelihood ratio test statistic for normal random variables and found its null distribution in both cases of σ^2 known and unknown. But Worsley(1979) pointed out incorrectness in Hawkins' result for σ^2 unknown case and found its correct null distribution. For the AMTC problem, Lombard(1985) proposed a quadratic rank statistic which is distribution-free type.

In Section 2, a test procedure is developed for the AMTC problem and asymptotic null distribution of the proposed test statistic is derived. Section 3 is devoted to brief review of nonparametric rank test by Lombard(1985) for power comparison with our normal theory test, and gives the simulation result.

2. The Proposed Test Statistic and Asymptotic Distribution

Consider the AMTC problem of (1.1). If the change-points $\tau_1 = t_1$, $\tau_2 = t_2$ are fixed, this is nothing more than a one-way analysis of variance problem and the likelihood ratio statistic is monotone increasing function of

$$\sum_{j=1}^{3} \Big[\sum_{i=t_{j-1}+1}^{t_j} X_i - (t_j - t_{j-1}) \overline{X}_N \Big]^2 / (t_j - t_{j-1}),$$

where $t_0 = 0$, $t_3 = N$ and $\overline{X}_N = \sum_{i=1}^{N} X_i / N$. We propose a sum-type test statistic based on the modified likelihood ratio statistic as defined by

$$U_N = \sum_{1 \leq t_1 < t_2 < N} \sum_{j=1}^{3} \Big[\sum_{i=t_{j-1}+1}^{t_j} X_i - (t_j - t_{j-1}) \overline{X}_N \Big]^2 \tag{2.1}$$

with large values being significant in favor of the alternative(1.1).

We now show the derivation of the asymptotic null distribution of U_N, which can be expressed as functional of stochastic process. Using empirical central limit theorem and integration of Gaussian process, we will prove that $N^{-3}\sigma^{-2}U_N$ converges in distribution to an integral of Brownian bridge process.

Denote by C the space of continuous functions and D the space of Cadlag functions both defined on the closed interval [0,1]. Let $\{B(u),\ 0 \leq u \leq 1\}$ and $\{W(u),\ 0 \leq u \leq 1\}$ be the Brownian bridge process and the Wiener process (or Brownian motion process) respectively both defined on [0,1]. We also refer to W as Wiener measure on the space C.

Theorem 2.1. Let $X_1, X_2, \cdots, X_N$ be i.i.d. with mean μ and finite variance σ^2. Then

$$N^{-3}\sigma^{-2} U_N \xrightarrow{d} Q = 2\int B^2(u)du - \Big(\int B(u)du\Big)^2.$$

Proof. The methods used in the proof are partial adaptations of Theorem 4.3.1 in Praagman(1986). Define stochastic process $Z_N = \{Z_N(u),\ 0 \leq u \leq 1\}$ by

$$Z_N(u) = N^{-1/2}\sigma^{-1} \sum_{i=1}^{[Nu]} (X_i - \mu).$$

Then for every N, Z_N belongs to the space D equipped with the Skorohod topology, and $N^{-3}\sigma^{-2} U_N$ can be expressed as

$$N^{-3}\sigma^{-2}U_N=N^{-2}\sum_{t_1<t_2}\sum \{(Z_N(u)-\frac{[Nu]}{N}Z_N(1))^2+(Z_N(v)-Z_N(u)-\frac{[Nv]-[Nu]}{N}Z_N(1))^2 + (Z_N(1)-Z_N(v)-\frac{N-[Nv]}{N}Z_N(1))^2\}$$

where $\frac{t_1}{N}\leqq u<\frac{t_1+1}{N}$ and $\frac{t_2}{N}\leqq v<\frac{t_2+1}{N}$ for $1\leqq t_1<t_2<N$. From Donsker Theorem, $Z_N\xrightarrow{d}W$. Define the mappings h and h_N: $D\rightarrow R$ by

$$h(z)=\int\int_{u<v}\{(z(u)-uz(1))^2+(z(v)-z(u)-(v-u)z(1))^2 +(z(1)-z(v)-(1-v)z(1))^2\}\,dudv,$$

and

$$h_N(z_N)=N^{-3}\sigma^{-2}U_N.$$

Let E be the set of $z\in D$ such that $h_N(z_N)\rightarrow h(z)$ fails to hold for some sequence $\{z_N\}$ approaching to z. We can restrict ourselves to continuous z, For continuous z, however, convergence in the Skorohod topology implies convergence in the uniform topology. Hence, for each positive ε an N_0 can be found such that $|Z_N(u)-Z(u)|<\varepsilon$ for all $N\geq N_0$, and uniform in u. Then $W(E)=0$. We invoke Theorem 5.5 in Billingsley (1968, p.34) to obtain $h_N(Z_N)\xrightarrow{d}h(W)$. ∎

3. Power Comparison with Nonparametric Rank Test

First, we present a brief review of the quadratic rank statistic proposed by Lombard (1985). For the AMTC problem, he considered the following rank statistic

$$Q_N=N^{-3}A^{-2}\sum_{1\leqq t_1<t_2<N}\sum [(\sum_{i=1}^{t_1}a_N(R_i))^2+(\sum_{i=t_1+1}^{t_2}a_N(R_i))^2+(\sum_{i=t_2+1}^{N}a_N(R_i))^2], \tag{3.1}$$

with $\sum_{i=1}^{N}a_N(R_i)=0$ and $A^2=\sum_{i=1}^{N}a_N^2(R_i)/(N-1)$, where $a_N(R_i)$, $i=1,\cdots,N$ are appropriate scores associated with the rank R_i of X_i. Lombard(1985) showed that under H_0

$$Q_N\xrightarrow{d}Q=2\int B^2(u)du-(\int B(u)du)^2,$$

where B is a standard Brownian bridge process, and found that the tail probability of Q is given by

$$\Pr(4.116\, Q > x) \doteqdot 1.028\, x^{-1/2} \exp(-x),$$

which can be used for size α test of U_N approximately.

We were not able to find a workable expression for the asymptotic efficiency of the proposed test statistic. Therefore, a small sample Monte Carlo study was performed to investigate the relative performance of the tests U_N (2.1) and Q_N(3.1). We will investigate the empirical power of the tests for the sample size N=30 when the alternative is specified by the change-points τ_1 and τ_2, and amount of shifts Δ_1 and Δ_2 for the AMTC problem of (1.1). We choose the change-points τ_1 and τ_2 such that

$$\tau_i / N = .1(.2).9,\ i=1,2, \qquad \tau_1 / N < \tau_2 / N$$

and $\Delta_1 = .5\sigma$ and $\Delta_2 = 1.0\sigma$, where σ is the standard deviation of the underlying distribution. We take normal or logistic for the underlying distribution. Both procedures were run at a nominal significance level of $\alpha = .05$. The critical values of the test were empirically estimated based on 3,000 replications, and the power estimates were obtained based on 1,000 replications for each alternative through the Monte Carlo method.

FORTRAN program has been prepared for conducting the Monte Carlo study, and the simulation was done on CYBER 170 / 835 system. The uniform random numbers were generated by RANF function of FORTRAN IV. The Box-Müller method was used to generate normally distributed random variables, and inverse transform method to generate logistic random variables.

Table 3.1 gives the empirical power for the parametric test U_N and quadratic rank tests Q_N with Wilcoxon scores(denoted by $Q_N^{(W)}$) and quantile normal scores (denoted by $Q_N^{(N)}$). As expected, we can find that the power of the normal theory test U_N based on modified likelihood ratio is always better than that of the rank analogue $Q_N^{(W)}$ or $Q_N^{(N)}$ when we restrict the underlying distribution to normal. It is noted that the power of quantile normal scores test $Q_N^{(N)}$ is slightly lower than, but very close to, that of U_N in each case.

Table 3.1. Empirical power for the tests U_N and Q_N for the AMTC problem of (1.1) when the underlying distribution is normal with variance 1, Δ_1=.5, Δ_2 =1.0, and α=.05. The estimations are based on 1,000 repetitions.

N	$(\tau_1/N, \tau_2/N)$	U_N	$Q_N^{(W)}$	$Q_N^{(N)}$
30	(.1,.3)	.301	.247	.281
	(.1,.5)	.355	.335	.338
	(.1,.7)	.282	.256	.282
	(.1,.9)	.151	.112	.119
	(.3,.5)	.623	.546	.593
	(.3,.7)	.525	.484	.501
	(.3,.9)	.309	.266	.287
	(.5,.7)	.643	.573	.592
	(.5,.9)	.412	.353	.389
	(.7,.9)	.392	.281	.351

$Q_N^{(W)}$: Q_N with Wilcoxon scores
$Q_N^{(N)}$: Q_N with quantile normal scores

Table 3.2 compares the empirical power for the tests U_N and Q_N when the underlying distribution is logistic. In this situation the powers of the rank tests $Q_N^{(W)}$ and $Q_N^{(N)}$ are higher than that of the parametric test U_N.

Table 3.2. Empirical power for the tests U_N and Q_N for the AMTC problem of (1.1) when the underlying distribution is logistic with $\sigma^2 = \pi^2/3$, Δ_1= .5σ, Δ_2 =1.0σ, and α=.05. The estimations are based on 1,000 repetitions.

N	$(\tau_1/N, \tau_2/N)$	U_N	$Q_N^{(W)}$	$Q_N^{(N)}$
30	(.1,.3)	.252	.277	.264
	(.1,.5)	.331	.366	.354
	(.1,.7)	.254	.287	.272
	(.1,.9)	.096	.107	.102
	(.3,.5)	.589	.632	.597
	(.3,.7)	.462	.491	.486
	(.3,.9)	.263	.295	.277
	(.5,.7)	.589	.636	.608
	(.5,.9)	.350	.399	.390
	(.7,.9)	.326	.355	.336

$Q_N^{(W)}$: Q_N with Wilcoxon scores
$Q_N^{(N)}$: Q_N with quantile normal scores

From the simulation study, we recommend the normal theory test U_N when we are confident that the underlying distribution is normal. However, using the normal theory test U_N has disadvantages that it requires the knowledge of variance σ^2 of the underlying distribution and the power depends sensitively on the type of the underlying distributions. On the contrary, it is well known that the null distribution of the rank test does not depend on σ^2 and the type of the underlying distribution, and for the rank tests we can choose appropriate scores so as to optimize the power according to the type of the underlying distribution.

4. Conclusions

We have proposed the parametric test U_N assuming that the variance of the underlying distribution is known. We could also construct the corresponding test statistic using appropriate estimates for the variance in case of variance unknown, even if it is difficult to find its distributional properties. Max-type test statistic can also be developed if we replace the summation, $\sum\sum_{t_1<t_2}$, by maxima, $\max_{t_1<t_2}$, for the proposed sum-type test statistic for the AMTC problem.

Finally, we mention that there remain yet several unsolved problems for the AMTC problem, namely, testing procedures for the possible changes in scale parameter, and estimation of the change-points or the magnitudes of shifts at the suspected change-points.

Acknowledgement

The authors would like to thank the referees for their valuable comments.

References

(1) Billingsley, P. (1968). *Convergence of Probability Measures,* Wiley, New York.

(2) Gardner, L.A.(1969). On detecting changes in the mean of normal variates. *The Annals of Mathematical Statistics* 40, 116-126.

(3) Hawkins, D.W.(1977). Testing a sequence of observations for a shift in location. *Journal of the American Statistical Association* 72, 180-186.

(4) Lombard, F.(1985). Rank tests for changepoint problems. Research Report 85/2, *University of South Africa.*

(5) Page, E.S.(1954). Continuous inspection schemes. *Biometrika* 41, 100-115.
(6) Page, E.S.(1955). A test for change in a parameter occurring at an unknown point. *Biometrika* 42, 523-526.
(7) Praagman, J.(1986). *Efficiency of Change-point Tests*, Ph. D. thesis, Eindhoven University of Technology, the Netherlands.
(8) Worsley, K.J.(1979). On the likelihood ratio test for a shift in location of normal populations. *Journal of the American Statistical Association* 74, 365-367.