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—Research Paper—

Asset Selling Problem With Beta Distributed Price Offers
재산매도 결정문제 : 호가가 베타 분포를 따를때

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ABSTRACT

This practical paper puts existing optimal stopping rules for various asset selling situations into a coherent perspective, using simple non-measure theoretical terms.

It also provides analytical or numerical solutions when the price offers are beta distributed.

1. Introduction

The asset selling problem is an optimal stopping policy problem like the well-known secretary problem (see [7, 8]). Consider a person having an asset to sell. The seller receives independent and indentically distributed price offers. The only decision the seller must make is when to stop receiving offers and accept an offer. The objective is to maximize the seller's expected net return.

This paper considers various asset selling situations with regard to the arrival pattern of offers, and to the length of the planning horizon. Regarding the arrival pattern of offers, the following two cases are considered :

1) Poisson arrival of offers :

This case is commonly termed as a semi-Markov decision problem in "continuous time".

2) Non-random arrival of offers :

This case is termed as a "discrete" Markov decision problem.

Regarding the length of the planning horizon, the following two cases are considered :

1) Finite planning horizon :

In a continuous time problem the seller is subject to a fixed amount of time, whereas, in a discrete

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time problem the seller may consider up to a fixed number of offers. In either case, continuous or discrete, it is assumed that once an offer is rejected it is immediately lost and cannot be recalled.

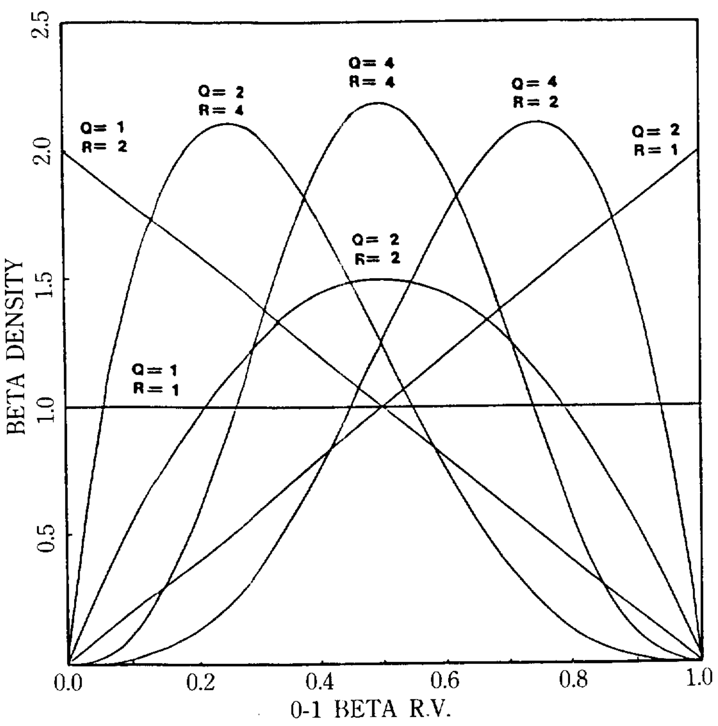
2) Infinite planning horizon :

In a continuous time problem the seller is allowed an unlimited amount of time, whereas, in a discrete time problem the seller is allowed an unlimited number of offers. However, delaying the decision incurs a cost. A continuous type cost is a cost per unit time, and a discrete type cost is a cost to receive another offer.

The basic principles of the optimal stopping problem can be found in [1] and [4]. Since Karlin's [8] extensive study on the asset selling problem, various authors considered further developments of the subject and further applications to specific cases :

1) The case of fixed number of offers (discrete, finite horizon case) can be found in some textbooks such as [2].
2) The case of an unlimited number of offers (discrete, infinite horizon case) can be found in [9] and [11].
3) Cowan and Zabczyk [5] consider the case of a fixed amount of time (continuous, finite horizon case). However, their objective is to maximize the chance of selecting the best choice, as is the objective of the secretary problem.

Figure 1 : Examples of Beta Distributions



4) Zuckerman [12] investigates, among other things, the case of an unlimited amount of time (continuous, infinite horizon case), within the context of job search.

The purpose of this paper is twofold. First, various cases are put into a coherent perspective. In addition, for the sake of readability, optimal stopping rules are derived and stated in simple non-measure theoretical terms. Rigorous formulation and complete proofs can be found in the literature referred to earlier. Second, instead of providing users with only general optimality conditions and characteristics, this paper provides numerical solutions and some analytical solutions when the price offers are beta distributed. (Karlin [8] provides solutions for the cases of exponential and uniform distributed price offers).

The choice of a beta distribution is primarily due to the nature of the price offers which have definite maximum (asking price) and non-negative minimum values. Furthermore, the parametric family of beta distributions provides the seller with flexibility in choosing the distribution which most closely suits his specific situation. Some examples of beta distributions are presented in Figure 1.

Additional variations briefly investigated include :

1) non-homogeneous Poisson arrival of offers (see [3]) ;
2) infinite sequence of buy-sell processes (see [6]) ; and
3) selling several identical assets by a deadline (see [3, 6, 8, 10]).

2. The Case Of Fixed Number Of Offers
(Discrete Time, Finite Horizon Policy)

We will first consider a simple example. A person has a used car for sale, and there are two potential customers. Whenever the seller visits a customer, he receives an offer which is uniformly distributed ranging from $5,000 to $10,000 (the asking price). The seller, however, will not consider any offer below the trade-in value (salvage value) of $6,000. Each offer must be accepted or rejected right away. The seller intends to maximize the expected price.

Suppose the seller has rejected the first offer and is now at his second customer. It is obvious, then, that he should sell the car if the second customer's offer is greater than (or equal to) the salvage value of $6,000. The expected price the seller can obtain at this stage is given by the following conditional expectation :

$$\int_{5000}^{6000} (6000)(1/5000)\,dx + \int_{6000}^{10000} x\,(1/5000)\,dx = \$7600.$$

Suppose the seller is at his first customer. In this case, the principle of dynamic programming (section 4) suggests that he should sell the car if the first customer's offer is greater than (or equal to) $7,600. The expected price he can obtain at this stage is :

$$\int_{5000}^{7600} (7600)(1/5000)\,dx + \int_{7600}^{10000} x\,(1/5000)\,dx = \$8176.$$

If the seller had three potential customers instead of two, then $8,176 would, of course, serve as the optimal threshold value when the seller is at his first customer.

The general case of the problem will be developed using the following notation :

B = maximum price (asking price)

A = minimum price

S = salvage value (trade-in value), $B>S \geqq A \geqq 0$

Note : We separate the less likely case that "$S<A$", and discuss it in the Appendix.

X = price offer (independent and indentical)

f(x) = density function of beta distributed X

$= K(q,r)[1/(B-A)]^{q+r-1}(x-A)^{q-1}(B-x)^{r-1}$, $B \geqq x \geqq A$

where $q>0$, $r>0$, and $K(q,r)=\Gamma(q+r)/[\Gamma(q)\cdot\Gamma(r)]$.

V_n = maximum expected value (price) when the seller has n offers to receive where $n=0, 1, 2, \cdots$ and $V_0=S$.

Note that, in the example presented earlier, $B=\$10,000$, $A=\$5,000$, $S=\$6,000$, $q=r=1$, $V_0=\$6,000$, $V_1=\$7,600$, and $V_2=\$8,176$.

In general, V_n serves as the optimal threshold value when the seller has $(n+1)$ offers to receive. The maximum expected price he can obtain at this stage is :

$$V_{n+1}=\int_A^{V_n} V_n f(x)\,dx+\int_{V_n}^{B} x\, f(x)\,dx$$

$$=V_n+\int_{V_n}^{B}(x-V_n)\,f(x)\,dx \qquad (1)$$

The integral in (1) can be carried out in a straightforward manner by using the binomial expansion and then by integrating the resulting power series term by term. For simplicity, we transform X to the zero-one beta random variable Y, $Y=(X-A)/(B-A)$, whose density function is $g(y)=K(q,r)y^{q-1}(1-y)^{r-1}$. Note that the binomial expansion of $(1-y)^{r-1}$ is an infinite series unless r is a positive integer, and so is the integral in (1). For this reason, we consider, in this paper, positive integer valued r only. (One exception is the case when $q=1$).

For a positive integer valued r, (1) becomes

$$V_{n+1}=V_n+(B-A)K(q,r)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\,\frac{(q+i)-(q+i+1)E_n+E_n^{q+i+1}}{(q+i)(q+i+1)} \qquad (2)$$

where $K(q,r)=(q+r-1)(q+r-2)\cdots[(q+r)-r]/(r-1)!$,

$\binom{r-1}{i}=(r-1)!/[(r-1-i)!\,i!]$, and

$E_n=(V_n-A)/(B-A)$ is the standardized expected value.

When r is not a positive integer, "$\sum_{i=0}^{r-1}\binom{r-1}{i}$" in (2) should be replaced by

"$\sum_{i=0}^{\infty}(r-1)(r-2)\cdots(r-i)/i!$".

When $q=1$, however, it can be shown, by using binomial indentities, that the infinite series mentioned above (or the non-integer-r version of (2)) becomes

$$V_{n+1}=V_n+(B-V_n)^{r+1}/[(r+1)(B-A)^r], \qquad (3)$$

which is valid for any $r>0$. Note that (3) includes the case of uniform destribution ($q=r=1$).

3. The Case Of Unlimited Number Of Offers
(Discrete Time, Infinite Horizon Policy)

In this case, we assume that the previous seller is allowed to receive an unlimited number of offers and that the seller has to pay a cost of C monetary units each time he receives an offer.

Including the cost factor of C into (1), we get

$$V_{n+1} = V_n + \int_{V_n}^{B} (x - V_n)\, f(x)\, dx - C \qquad (4)$$

Here, V_n is interpreted as the maximum "net" expected value. In addition, it is assumed that

Table 1 : Roots of Equation (6)

E_c	Q=2 R=2	Q=2 R=3	Q=2 R=4	Q=3 R=1	Q=3 R=2	Q=3 R=3	Q=3 R=4	Q=4 R=1	Q=4 R=2	Q=4 R=3	Q=4 R=4
1.00	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
0.98	.000008	.000000	.000000	.000592	.000016	.000000	.000000	.000784	.000026	.000001	.000000
0.96	.000063	.000002	.000000	.002337	.000123	.000006	.000000	.003075	.000201	.000012	.000001
0.94	.000210	.000012	.000001	.005187	.000407	.000030	.000002	.006781	.000658	.000058	.000005
0.92	.000492	.000039	.000003	.009098	.000944	.000093	.000009	.011816	.001512	.000177	.000019
0.90	.000950	.000094	.000009	.014025	.001806	.000221	.000026	.018098	.002863	.000416	.000057
0.88	.001524	.000192	.000023	.019924	.003056	.000447	.000063	.025546	.004796	.000830	.000136
0.86	.002552	.000352	.000049	.026752	.004752	.000807	.000133	.034085	.007382	.001480	.000281
0.84	.003768	.000592	.000094	.034468	.006944	.001341	.000251	.043642	.010680	.002430	.000525
0.82	.005307	.000936	.000166	.043030	.009678	.002093	.000440	.054148	.014735	.003744	.000905
0.80	.007200	.001408	.000277	.052400	.012992	.003104	.000722	.065536	.019584	.005486	.001466
0.78	.009477	.002033	.000440	.062538	.016920	.004424	.001128	.077743	.025251	.007719	.002256
0.76	.012165	.002840	.000669	.073405	.021490	.006097	.001690	.090711	.031753	.010504	.003330
0.74	.015291	.003857	.000982	.084966	.026725	.008169	.002443	.104380	.039096	.013894	.004743
0.72	.018879	.005114	.001400	.097183	.032644	.010686	.003428	.118698	.047282	.017942	.006556
0.70	.022950	.006642	.001944	.110025	.039258	.013689	.004686	.133614	.056304	.022692	.008827
0.68	.027525	.008472	.002640	.123453	.046578	.017222	.006262	.149079	.066148	.028181	.011617
0.66	.032622	.010637	.003514	.137437	.054607	.021323	.008202	.165047	.076797	.034443	.014983
0.64	.038258	.013168	.004595	.151943	.063348	.026027	.010552	.181475	.088228	.041502	.018983
0.62	.044446	.016097	.005916	.166941	.072795	.031369	.013361	.198323	.100413	.049376	.023666
0.60	.051200	.019456	.007509	.182400	.082944	.037376	.016677	.215552	.113323	.058075	.029082
0.58	.058530	.023275	.009410	.198291	.093784	.044074	.020545	.233127	.126923	.067603	.035271
0.56	.066444	.027586	.011654	.214586	.105301	.051484	.025011	.251015	.141179	.077957	.042269
0.54	.074949	.032417	.014280	.231258	.117481	.059622	.030118	.269183	.156053	.089126	.050104
0.52	.084050	.037796	.017327	.248279	.130304	.068500	.035906	.287604	.171507	.101094	.058797
0.50	.093750	.043750	.020833	.265625	.143750	.078125	.042411	.306250	.187500	.113839	.068359
0.48	.104050	.050304	.024840	.283271	.157796	.088500	.049666	.325096	.203993	.127334	.078797
0.46	.114949	.057481	.029387	.301194	.172417	.099622	.057697	.344119	.220947	.141546	.090104
0.44	.126444	.065301	.034513	.319370	.187586	.111484	.066528	.363298	.238321	.156439	.102269
0.42	.138530	.073784	.040257	.337779	.203275	.124074	.076175	.382614	.256076	.171973	.115271
0.40	.151200	.082944	.046656	.356400	.219456	.137376	.086647	.402048	.274176	.188105	.129082
0.38	.164446	.092795	.053746	.375213	.236097	.151369	.097948	.421585	.292583	.204790	.143666
0.36	.178258	.103348	.061561	.394199	.253168	.166027	.110074	.441209	.311262	.221981	.158983
0.34	.192622	.114607	.070131	.413341	.270637	.181323	.123015	.460909	.330180	.239630	.174983
0.32	.207525	.126578	.079482	.432621	.288472	.197222	.136753	.480671	.349306	.257691	.191617
0.30	.222950	.139258	.089637	.452025	.306642	.213689	.151263	.500486	.368611	.276115	.208827
0.28	.238879	.152644	.100616	.471537	.325114	.230685	.166513	.520344	.388066	.294857	.226556
0.26	.255291	.166725	.112430	.491142	.343857	.248169	.182466	.540238	.407649	.313872	.244743
0.24	.272165	.181490	.125086	.510829	.362840	.266097	.199075	.560159	.427336	.333118	.263330
0.22	.289477	.196920	.138584	.530586	.382033	.284424	.216291	.580103	.447106	.352557	.282256
0.20	.307200	.212992	.152917	.550400	.401408	.303104	.234057	.600064	.466944	.372151	.301466
0.18	.325307	.229678	.168069	.570262	.420936	.322092	.252315	.620038	.486833	.391868	.320905
0.16	[illegible]43768	.246944	.184013	.590164	.440592	.341341	.271001	.640021	.506760	.411680	.340525
0.14	[illegible]2552	.264752	.200716	.610096	.460352	.360807	.290052	.660011	.526715	.431561	.360281
0.12	[illegible]624	.283056	.218129	.630052	.480192	.380447	.309402	.680005	.546690	.451492	.380136
0.10	.40[illegible]	.301806	.236196	.650025	.500094	.400221	.328987	.700002	.566676	.471455	.400057
0.08	.42049[illegible]	.320944	.254845	.670010	.520039	.420093	.348748	.720001	.586670	.491437	.420019
0.06	.440210	.340407	.273991	.690003	.540012	.440030	.368630	.740000	.606667	.511431	.440005
0.04	.460063	.360123	.293534	.710001	.560002	.460006	.388583	.760000	.626667	.531429	.460001
0.02	.480008	.380016	.313359	.730000	.580000	.480000	.408572	.780000	.646667	.551429	.480000
0.00	.500000	.400000	.333333	.750000	.600000	.500000	.428571	.800000	.666667	.571429	.500000

$C<\int_{V_0}^{B}(x-V_0)\,f(x)\,dx$, otherwise it is not worth it to pay C to receive an offer.

Let V_c be the limiting value of V_n as n approaches infinity. Then (4) becomes

$$C=\int_{V_c}^{B}(x-V_c)\,f(x)\,dx. \tag{5}$$

The optimality condition (5) can be interpreted in the following terms of marginal analysis: For a maximum expected net return to occur, marginal cost, or the cost to receive another offer, must equal the expected marginal revenue from receiving another offer.

For a positive integer valued r, (5) becomes

$$C=(B-A)\;K(q,r)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\;\frac{(q+i)-(q+i+1)E_c+E_c^{\,q+i+1}}{(q+i)(q+i+1)} \tag{6}$$

just like the way (1) becomes (2),

where $K(q,r)=(q+r-1)(q+r-2)\cdots q/(r-1)!$ and $E_c=(V_c-A)/(B-A)$.

Likewise, when $q=1$, (5) becomes

$$C=(B-V_c)^{r+1}/[(r+1)(B-A)^r]. \tag{7}$$

From (7), we obtain the following closed form solution

$$V_c=B-[(r+1)(B-A)^r C]^{1/(r+1)} \tag{8}$$

which is valid for any $r>0$.

As an example, if the seller in the previous case (where $q=r=1$) was allowed to receive an unlimited number of offers, but had to pay $C=\$576$ to receive an offer, then from (8)

$$V_c=10000-[(1+1)(10000-5000)^1(576)]^{1/(1+1)}=\$7600$$

Therefore, the seller would take the first offer that is at least \$7,600.

When $q\neq 1$, it is usually difficult to obtain analytical solutions. One way to compute V_c when $q\neq 1$ is to use the iterative equation (4). Unlike (1), (4) quickly converges to a steady state (with a four digit accuracy after approximately ten iterations).

We can also compute V_c using (6) for the case of positive integer valued r. This time, finding V_c given C usually requires one-dimensional search. When both q and r are positive integer valued, (6) becomes a $(q+r)^{th}$ degree polynomial equation, thus, at least in principle, we can obtain roots of the equation analytically up to $q+r=4$. For example, for $q=2$ and $r=1$, $E_c=2\cos(\theta+4\pi/3)$, where $\theta=(1/3)\cos^{-1}(1.5C-1)$. On the other hand, finding C given V_c is straightforward. Table 1 lists the values of $C/(B-A)$ thus computed given various values of $E_c=(V_c-A)/(B-A)$, for some selected beta parameters ($q, r=2, 3, 4$).

Suppose, in the previous example, the beta parameter values were $q=r=4$ (instead of $q=r=1$). Then, an approximate optimal threshold value can be obtained as follows: In the last column of Table 1, the value, nearest to $C/(B-A)=576/(10000-5000)=.1152$, is .115271 which corresponds to $E_c=.42$, which, in turn, yields $V_c\doteq\$7,100$.

4. The Case Of Fixed Amount Of Time
(Continuous Time, Finite Horizon Policy)

In this case, the seller is allowed a fixed amount of time during which he receives a stream of offers determined by a Poisson process with a rate λ. Like the case of a fixed number of offers (section 2), each offer must be accepted or rejected immediately, and the seller intends to maximize the expected return.

The optimal stopping policy based on the principle of dynamic programming is quite intuitive. An offer is acceptable if the price offered meets or exceeds what the seller expects to get in case of rejecting the offer just made, assuming the seller will follow the same decision rule during the remaining time. (In section 8, the optimal stopping rule is mathematically proved for an extended case of this problem).

Let $V(t)$ be the maximum expected return at t, i.e., $V(t)$ is the expected return under the optimal policy described above when the remaining time until a fixed deadline is t. Then, $V(t+\delta t)$, the maximum expected return at $t+\delta t$, is given by the following conditional expectation :

$$V(t+\delta t)=(1-\lambda\delta t)\ V(t)+\lambda\delta t\ \left[\int_A^{V(t)} V(t)\ f(x)\ dx+\int_{V(T)}^{B} xf(x)\ dx\right]+O(\delta t) \tag{9}$$

The conditions are whether or not an offer arises during $[t+\delta t, t]$, with the probability of one arrival being $\lambda\delta t$, and, if an offer does arise, whether or not the offered price exceeds the optimal threshold value at that time. If no offer or an unacceptable offer is made during $[t+\delta t, t]$, the maximum expected return at $t+\delta t$ is $V(t)$; if an acceptable offer of size x, $x \geq V(t)$, is made, then x is the return at $t+\delta t$. (The event of more than one arrival during $[t+\delta t, t]$ is reflected by $O(\delta t)$, where $O(\delta t)/\delta t$ approaches zero as δt approaches zero according to the Poisson axiom).

Converting (9) to a differential equation, we get

$$dV(t)/\lambda dt=\int_{V(t)}^{B}[x-V(t)]\ f(x)\ dx \tag{10}$$

Note that we could have obtained (10) by simply replacing V_n and $(V_{n+1}-V_n)/[(n+1)-n]$ in (1) with $V(t)$ and $dV(t)/\lambda dt$, respectively.

As before, for a positive integer valued r, (10) becomes

$$\frac{dV(t)}{\lambda dt}=(B-A)\ K(q,r)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\ \frac{(q+i)-(q+i+1)E(\tau)+E(\tau)^{q+i+1}}{(q+i)(q+i+1)} \tag{11}$$

where $t \geqq 0$, $V(0)=S$, $K(q,r)=(q+r-1)(q+r-2)\cdots q/(r-1)!$, and $E(\tau\equiv\lambda t)=[V(t)-A]/(B-A)$.

Also, when $q=1$, (10) becomes the following equation which is similar to (3) and (7) :

$$dV(t)/\lambda dt=[B-V(t)]^{r+1}/[(r+1)(B-A)^r]. \tag{12}$$

Solving this differential equation (12) we obtain the following closed form solution for the case of $q=1$:

$$V(t)=B-\{\lambda t\ [r/(r+1)]\ (B-A)^{-r}+(B-S)^{-r}\}^{-1/r} \tag{13}$$

As an example, suppose the seller in section 2 is allowed a fixed amount of time instead of having 2 potential customers. Specifically, he expects to receive offers at a rate of 2 per week during a 4-week period of waiting for his new car on order. As before, price offers are uniformly distributed ($q=r=1$) ranging from \$5,000 to the asking price of \$10,000. In the case of no acceptable offer, he can still trade

Table 2 : Roots of Equation(11)

τ	Q=2 R=1	Q=2 R=2	Q=2 R=3	Q=2 R=4	Q=3 R=1	Q=3 R=2	Q=3 R=3	Q=3 R=4	Q=4 R=1	Q=4 R=2	Q=4 R=3	Q=4 R=4
0.0	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000
0.1	.063444	.047584	.038058	.031723	.071372	.057098	.047582	.040784	.076130	.063442	.054379	.047581
0.2	.120876	.090571	.072544	.060458	.135955	.108767	.090541	.077694	.145016	.120847	.103584	.090636
0.3	.172920	.129750	.103831	.086543	.194409	.155543	.129630	.111118	.207350	.172795	.148113	.129602
0.4	.220154	.165274	.132299	.110295	.247339	.197925	.164972	.141427	.263762	.219819	.188429	.164885
0.5	.263100	.197652	.158284	.131995	.296306	.236376	.197062	.168964	.314832	.262410	.224960	.196857
0.6	.332229	.227249	.182081	.151892	.338821	.271320	.226261	.194042	.361088	.301020	.258100	.225897
0.7	.337959	.254382	.203947	.170201	.378349	.303141	.252894	.216942	.403014	.336066	.288213	.252298
0.8	.370660	.279332	.224104	.187106	.414309	.332182	.277250	.237915	.441051	.367926	.315629	.276361
0.9	.400657	.302342	.242747	.202769	.447079	.358750	.299586	.257179	.475598	.396940	.340644	.298347
1.0	.428239	.323622	.250041	.217326	.476997	.383117	.320128	.274928	.507019	.423417	.363523	.318488
1.1	.453657	.343356	.276131	.230896	.504362	.405520	.339071	.291330	.535638	.447630	.384502	.336991
1.2	.477135	.361706	.291142	.243583	.529443	.426171	.356591	.306531	.561747	.469822	.403786	.354034
1.3	.498866	.378812	.305184	.255475	.552477	.445254	.372837	.320658	.586608	.490211	.421560	.369777
1.4	.519025	.394795	.318352	.266651	.573672	.462931	.387941	.333823	.607453	.508985	.437984	.384359
1.5	.537764	.409764	.330730	.277178	.593216	.479346	.402019	.346123	.627491	.526313	.453199	.397900
1.6	.555216	.423813	.342389	.287116	.611273	.494624	.415173	.357643	.645905	.542345	.467330	.410570
1.7	.571501	.437026	.353395	.296517	.627988	.508874	.427491	.368458	.662858	.557210	.480485	.422274
1.8	.586726	.449477	.363804	.305428	.643492	.522196	.439052	.378633	.678497	.571025	.492760	.433283
1.9	.600984	.461231	.373668	.313890	.657899	.534674	.449952	.388226	.692951	.583890	.504239	.443606
2.0	.614360	.472347	.383030	.321939	.671311	.546386	.460172	.397289	.706334	.595898	.514998	.453306
2.2	.638756	.492868	.400409	.336927	.695506	.567775	.478998	.413999	.730286	.517648	.534609	.471057
2.4	.660426	.511395	.416212	.350612	.716691	.586820	.495897	.429072	.751041	.636814	.552038	.486915
2.6	.679783	.528214	.430551	.363476	.735359	.503888	.511163	.442752	.769148	.653823	.567636	.501181
2.8	.697165	.543561	.443938	.374765	.751907	.619275	.525032	.455237	.785047	.669014	.581685	.514095
3.0	.712847	.557629	.456190	.385501	.766656	.633221	.537699	.466691	.799090	.682563	.594412	.525851
3.2	.727050	.570578	.457544	.395486	.779870	.645923	.549322	.477247	.811562	.694997	.606002	.536609
3.4	.739993	.582544	.478102	.404805	.791764	.657544	.560034	.487015	.822699	.706196	.616909	.546498
3.6	.751807	.593639	.487954	.413531	.802518	.668221	.569944	.496089	.832591	.716411	.626358	.555628
3.8	.762636	.603961	.497175	.421725	.812281	.578068	.579145	.504547	.841597	.725770	.635355	.564089
4.0	.772595	.613591	.506829	.429442	.821177	.687180	.587719	.512456	.849849	.734377	.643688	.571959
4.2	.781783	.622602	.513974	.436726	.829314	.695640	.596129	.519874	.857256	.742322	.651434	.579303
4.4	.790283	.631504	.521656	.443618	.836781	.703519	.603236	.526849	.864012	.749681	.658655	.586177
4.6	.798168	.639001	.528928	.450153	.843654	.710876	.610288	.533425	.870196	.756517	.665408	.592628
4.8	.805501	.646490	.535799	.456362	.849999	.717765	.616930	.539638	.875873	.762886	.671739	.598698
5.0	.812336	.653562	.542329	.462271	.855872	.724230	.623199	.545521	.881102	.768837	.677690	.604423
5.2	.818722	.660252	.548538	.467906	.861323	.730311	.629128	.551103	.885931	.774410	.683296	.609835
5.4	.824700	.666594	.554452	.473287	.866394	.736043	.634748	.556410	.890403	.779641	.688689	.614962
5.6	.830307	.672615	.560094	.478434	.871123	.741457	.640083	.561462	.894555	.784564	.693597	.619827
5.8	.835577	.678340	.565485	.483363	.875542	.746580	.645158	.566281	.898419	.789204	.698344	.624452
6.0	.840538	.683793	.570642	.488090	.879579	.771436	.649991	.570884	.902022	.793587	.702851	.628858
6.2	.845216	.688994	.575582	.492630	.883560	.756046	.654603	.576288	.905390	.797735	.707138	.633059
6.4	.849634	.693960	.580320	.496994	.887208	.760431	.659009	.579505	.908544	.801567	.711222	.637073
6.6	.853813	.698710	.584879	.501195	.890642	.754606	.663224	.583551	.911503	.805400	.715118	.640913
6.8	.857771	.703257	.589245	.505241	.893880	.768587	.667262	.587435	.914284	.808950	.718840	.644592
7.0	.861526	.707616	.593454	.509145	.896938	.772389	.671134	.591170	.916903	.812330	.722401	.648119
7.5	.870117	.717764	.603323	.518328	.903887	.781188	.680162	.599911	.922823	.820119	.730668	.656344
8.0	.877724	.726974	.612362	.526781	.909983	.789108	.688369	.607901	.927982	.827088	.738143	.663825
8.5	.884505	.735378	.620684	.534601	.915371	.796282	.695873	.616245	.932515	.833366	.744945	.670669
9.0	.890587	.743085	.628380	.541865	.920166	.802815	.702771	.622028	.936527	.839058	.751168	.676963
9.5	.896071	.750185	.635527	.548639	.924460	.808796	.709139	.628321	.940102	.844245	.756891	.682780
10.0	.901039	.756751	.642187	.554978	.928325	.814297	.715044	.634180	.943305	.848996	.762177	.688177
11.0	.909594	.768523	.654258	.566532	.935001	.824084	.725669	.644789	.948806	.857405	.771642	.697903
12.0	.916976	.778797	.664935	.576825	.940569	.832548	.734989	.654166	.953354	.864630	.779895	.706452
13.0	.923184	.787859	.674472	.586081	.945258	.839955	.743252	.662540	.957175	.870919	.787175	.714049
14.0	.928536	.795928	.683063	.594470	.949276	.846504	.750549	.670084	.960428	.876453	.793661	.720853
15.0	.933201	.808171	.690858	.602125	.952767	.852347	.757321	.676980	.963229	.881368	.799490	.727024
16.0	.937299	.809717	.697977	.609153	.955797	.857389	.763382	.683185	.966666	.885770	.804765	.732532
17.0	.940928	.815670	.704513	.615639	.958471	.862350	.768921	.688932	.967804	.889740	.809571	.737770
18.0	.944163	.821114	.710544	.621663	.960845	.866675	.774011	.694238	.969696	.893343	.813974	.742501
19.0	.947065	.826115	.716133	.627251	.962966	.870634	.778710	.699160	.971380	.896631	.818028	.746877
20.0	.949683	.830731	.721334	.632483	.964871	.874273	.783067	.730744	.972889	.899647	.821778	.750944
30.0	.966373	.863139	.759141	.671214	.976844	.899457	.814305	.737234	.982287	.920329	.848419	.780389
40.0	.947743	.882248	.782699	.696047	.982755	.914041	.833411	.758323	.986869	.932146	.864513	.798707
50.0	.979825	.895179	.799304	.713923	.986270	.923791	.846725	.773335	.989575	.939995	.875645	.811650
60.0	.983195	.904660	.811880	.727690	.988599	.930886	.856729	.784806	.991359	.945679	.883968	.821489
70.0	.985602	.911989	.821867	.738776	.990254	.936339	.864627	.793988	.992624	.950032	.890514	.829335
80.0	.987407	.917870	.830068	.747987	.991490	.940596	.871083	.801582	.993567	.953501	.895848	.835803
90.0	.988810	.922722	.836973	.755822	.992449	.944279	.876498	.808018	.994296	.956347	.900313	.841272
100.0	.989933	.926814	.842900	.762610	.993214	.947292	.881132	.813575	.994877	.958737	.904126	.845987
1000.0	.998999	.977387	.929105	.870286	.999332	.983929	.947197	.899817	.999499	.987516	.957820	.918108

in his car at a salvage value of \$6,000 when he picks up his new car after the 4-week period. Substituting these values of the decision parameters into (13), we get

$$V(4) = 10000 - \{(2)(4)\ [1/(1+1)]\ (10000-5000)^{\ 1} + (10000-6000)^{\ 1}\}^{1/1}$$
$$= 9047.62$$

Consequently, the optimal threshold value, when the time remaining is 4 weeks, is \$9,047.62.

When $q \neq 1$, however, we compute approximate numerical solutions to (11). For some selected beta distribution parameters (combinations of $q=2$, 3, or 4 and $r=1$, 2, 3, or 4), numerical solutions are computed using the Taylor series expansion, (14). The results are presented in Table 2.

Here is how the values in Table 2 are generated. We first rewrite (11) in terms of τ and $E(\tau)$. Here, $\tau \equiv \lambda t$ is a standard time measure representing the expected number of offers to arise at a rate of λ during the remaining time t, and $E(\tau) = [V(t=\tau/\lambda) - A]/(B-A)$ is the standardized maximum expected return at $\tau = \lambda t$. Rewriting (11) is equivalent to replacing $dV(t)/(B-A)$ and λdt by $dE(\tau)$ and $d\tau$, respectively.

Next, obtain the second and the third order derivatives of $E(\tau)$ by differentiating the rewritten equation twice with respect to τ, and then include them in the following approximate Taylor series of $E(\tau+h)$ with center at τ:

$$E(\tau+h) \doteq E(\tau) + h\ dE(\tau)/d\tau + (h^2/2)\ d^2E(\tau)/d\tau^2 + (h^3/6)\ d^3E(\tau)/d\tau^3 \qquad (14)$$

As an initial increment, h is set to .0001 when $\tau=0$. Then at each iteration, h is increased in proportion to $[E(\tau+h) - E(\tau)]^{\ 1}$. It takes about 10,000 iterations to compute up to $E(\tau=1000)$ with a six digit accuracy. (Accuracy of numerical solutions is based on the case $q=1$ where analytical solutions are available).

Note that in Table 2, $E(0)$ is set to zero, that is, $V(0)=S$ is assumed to be equal to A. Thus, when $S>A$, we need to shift (reset) the time origin by an appropriate amount. (Shifting the time origin is the only necessary adjustment since $V(t+\delta t)$ in (9) is determined by $V(t)$ and δt but not directly by t).

Suppose that the seller in the previous example had no acceptable offer during the first week, and that he now has a better idea about the price offer distribution. Accordingly he updates his decision parameters to $t=3$, $q=3$, and $r=2$. This time the seller wants $V(t=3) = A + (B-A)E(\tau=\lambda t=6)$, but, as before, $V(0) = S =$ \$6,000 > \$5,000 $= A$ and $E(0) = (S-A)/(B-A) = 1/5$. According to the unadjusted time measure τ in Table 2, it takes approximately .4 units to reach the seller's standardized salvage value of .2 (.197925 when $\tau=.4$), and an additional $\lambda t=6$ units of τ brings the standard expected value to .760431. Therefore, the seller's approximate threshold value in this case is:

$$V(t=3) \doteq A + (B-A)(.760431) = \$8,802.16.$$

(More accurate values, computed by rerunning the numerical integration procedure with an initial value of $E(0)=.2$, are $E(6) \doteq .760541$ and $V(3) \doteq$ \$8,802.71).

It is noted that even when $q \neq 1$ we can still solve (11), at least for some beta parameters, and convert the differential equation to a non-differential equation of $V(t)$. For example, when $q=2$ and $r=1$, solving (11), we get

$$\lambda t = \frac{[V(t)-S](B-A)}{(B-V)(B-S)} + \frac{1}{3} \log \frac{[2B+V(t)-3A]/(2B+S-3A)}{[B-V(t)]/(B-S)}$$

5. Non-Homogeneous Poisson Arrival Of Offers

Consider the case of a fixed amount of time as in section 4. This time, however, we allow a time dependent rate of offer arrivals. For example, more offers arise just after a house is listed for sale, and gradually fewer offers arise as time elapses.

Define

$m(t)$ = expected number of offers to arise during the remaining time t, and

$\lambda(t) = dm(t)/dt$

= time dependent arrival rate of offers.

We require $m(t)$ to be non-negative, continuous, and differentiable with a finite first derivative.

All that is needed for the non-homogeneous extension is to replace λdt in (10) and (11) with $dm(t) = \lambda(t)dt$. When $q=1$, this replacement leads to an analytical solution like (13), with λt replaced by $m(t)$. Likewise, when $q \neq 1$, use $\tau = m(t)$ as the standard time measure instead of using its homogeneous counterpart, $\tau = \lambda t$. (Actually, we could have formulated (9) using $m(t)$ or $\lambda(t)$ to begin with like Blum [3] did. Then, in the special case of time independent (homogeneous) Poisson arrival processes, $m(t)$ and $dm(t)$ simply reduce to λt and λdt, respectively).

Back to the example in section 4. Suppose the seller anticipated $m(t) = t^2/2$ or $\lambda(t) = t$, instead of $\lambda(t) = 2$. In other words, the rate of offer arrivals is expected to decrease from 4 to 0 offers per week as the remaining time decreases from 4 to 0 weeks. In the first part of the example, where $t=4$, replacing $\lambda t = (2)(4) = 8$ with $m(t) = 4^2/2 = 8$ makes no difference, and does not alter the threshold value. However, in the second part of the example, where $t=3$, $\tau = m(3) = 3^2/2 = 4.5$ is no longer the same as the previous $\tau = \lambda t = (2)(3) = 6$. The seller's approximate threshold value this time is :

$$V(t=3) \doteq A + (B-A)(.721047) = \$8,605.24.$$

(More accurate values, computed by rerunning the numerical procedure with an initial value of $E(0) = .2$, are $E(4.5) \doteq .721214$ and $V(3) \doteq \$8,606.07$).

6. The Case Of Unlimited Amount Of Time
(Continuous Time, Infinite Horizon Policy)

In this case, the seller has no deadline by which his asset must be sold, but there is a cost of C' monetary units per unit time instead. As in section 4, we assume Poisson arrival of offers with a constant rate λ.

Now we modify (9) by subtracting $C'\delta t$ from the right hand side. This leads to the following :

$$dV(t)/\lambda dt = \int_{V(t)}^{B} [x - V(t)] \, f(x) \, dx - C'/\lambda. \tag{15}$$

As was in the discrete counterpart (section 3), it is assumed that $C'/\lambda < \int_{V(0)}^{B} [x - V(0)] \, f(x) \, dx$, otherwise it is not worth it to pay the cost to receive offers.

Let $C = C'/\lambda$ and denote the limit of $V(t)$ as t approaches infinity by V_c. Then, (15) becomes

$$C = \int_{V_c}^{B} (x - V_c) \, f(x) \, dx \tag{16}$$

Note that (16) is exactly indentical to (5). (C in (16), though, is the "average" cost to receive another offer.) This implies that, when q=1, (8) is the closed form solution to (16) as well as to (5) ; for q,r=2, 3 and 4, Table 1 still provides approximate numerical solutions to (16) as well ; and when q=2 and r=1, $V_c = A+(B-A)2\cos(\theta+4\pi/3)$, where $\theta=(1/3)\cos^{-1}(1.5C-1)$.

For the rest of the beta parameter combinations, perhaps the easiest way to find V_c is to use (4) and to peform about 10 iterations, as suggested in section 3. (Since (16) is identical to (5), the limit of V(t) in (15) is the same as the limit of V_n in (4) when the value of C'/λ in (15) equals the value of C in (4).)

For example, if C' = $1,152/week and λ=2 offers/week, then $C=C'/\lambda=$ $576/offer, which is the same value for C as in section 3's example. As a result, the seller ends up with the same maximum net expected values as before : V_c = $7,600 if q=r=1, and $V_c \doteqdot$ $7,100 if q=r=4.

7. The Case With Both Deadline And Cost

Consider the case with both a deadline (section 4) and a cost C' per unit time (section 6). In this case, the only analytical solution we found thus far is for q=r=1 (uniform), which is :

$$V(t)=B-C^* \coth \{\frac{1}{2}[\lambda t \sqrt{\frac{2C}{B-A}} - \log(\frac{B-S-C^*}{B-S+C^*})]\} \quad (17)$$

where $C^* = \sqrt{2(B-A)C}$ and $C=C'/\lambda$.

For the rest of the beta parameter combinations, one can rely on the numerical procedure similar to that in section 4, but, this time, including the cost factor. In other words, derivatives to be included in the approximate Taylor series, (14), should be based on (15).

It should be noted that Mamer's [10] recent paper deals with the case having both a deadline and a cost. However, he suggests a discrete time approximation of the continuous time decision problem, perhaps as a result of mixing a discrete type cost with a continuous time decision process.

8. Extensions

We conclude this paper by briefly indicating two variations of the preceding models. We first relate the infinite horizon, single-asset liquidation problem (sections 3 and 6) to the problem of an infinite sequence of buy-sell processes. (A similar problem can be found in [6].) Then, we examine the problem of selling several indentical assets by a fixed deadline (cf. section 4). (Various different settings of the multi-asset case can be found in [3, 8, 10].)

8.1. Infinite Horizon Buy-Sell Sequence

The infinite horizon case, discussed in sections 3 and 6, is a onetime asset liquidation problem. A natural variation of this is the problem of an infinite sequence of repeated buy-sell processes.

Suppose the seller, immediately after selling an asset, can now buy an identical asset at a cost of S. (In other sections, S is referred to as the salvage value.) The objective in this case is to maximize the expected profit per unit time.

The optimal solution to this problem turns out to be trivial, though intuitively appealing : The seller should accept any price offer which is at least S since the seller can immediately buy another identical asset for S.

As before, it is assumed that $C<\int_S^B (x-S)\ f(x)\ dx$. As a matter of fact, the difference, $\int_S^B (x-S)\ f(x)\ dx-C$, between the maximum expected marginal revenue from receiving another offer and the cost (average cost in the continuous case) to receive another offer, is the maximum expected marginal profit from receiving another offer. Furthermore, in the continuous case, $\lambda\int_S^B (x-S)\ f(x)\ dx-C'$ is the maximum expected profit per unit time.

8.2. Selling Several Indentical Assets by Deadline

Suppose the seller in section 4 had several indentical assets to sell by a fixed deadline. Let $V_m(t)$ be maximum expected value of m assets when the remaining time is t, where m is a finite non-negative integer, $V_0(t)=0$, and $V_m(0)=mS$.

We will formulate the equation for $V_m(t)$, which reduces to (9) when m=1. This time, however, a semi-formal proof of optimality condition is provided as follows : Suppose y is the seller's current threshold, then

$$V_m(t+dt)=(1-\lambda dt)\ V_m(t)+\lambda dt\ \{\int_A^y V_m(t)\ f(x)\ dx+\int_y^B [x+V_{m-1}(t)]\ f(x)\ dx\}$$

$$dV_m(t)/dt=\lambda\int_y^B \{x-[V_m(t)-V_{m-1}(t)]\}\ f(x)\ dx \tag{18}$$

Here, we are seeking the optimal threshold, y^*, which maximizes $V_m(t+\delta t)$, given the maximum expected values at t, $V_m(t)$ and $V_{m-1}(t)$. In other words, we want y^* such that $dV_m(t)/dt$ is maximized. We set the first derivative of (18), with respect to y, at zero, and solve for y.

$$-\lambda\{y-[V_m(t)-V_{m-1}(t)]\}\ f(y)=0$$
$$y^*=V_m(t)-V_{m-1}(t)$$

This y^* maximizes $dV_m(t)/dt$ since the second derivative of (18), with respect to y, is negative when $y=y^*$, that is, $-\lambda f(y^*)<0$.

Note that in section 4, where m=1 and $V_1(t)=V(t)$, $y^*=V_1(t)-V_0(t)=V(t)-0=V(t)$.

APPENDIX : What if S < A?

Thus far the salvage value (S) is assumed to be greater than or equal to the minimum price offer (A). However, it is also possible, though less likely, that $S<A$. The only difference now when $S<A$ is that V_n in (1), (4) or V(t) in (10), (15) can be less than A. In fact, when we carried out the integrals in (1), (4), (10), (15) using the binomial expansion, we implicitly assumed that $V_n\geq A$ or $V(t)\geq A$, because $S\geq A$. This implicit assumption is no longer valid when (but only when) $V_n<A$ or $V(t)<A$.

If $S<A$, it is obvious that the seller in section 2 will never reject the last offer. According to (1), V_1 in this case is simply the expected value of X. Denote the expected value of X by E(X). Then, $V_1=E(X)=(Bq+Ar)/(q+r)$. However, the rest of V_n, n=2, 3, ⋯, can now be obtained from (2) or (3), since $V_1\geq A$.

Likewise, according to (4), $V_1=E(X)-C$, when $V_0=S<A$. As in section 3, it is assumed that $C<E(X)$

$-S$ in order to guarantee $V_1 > S$. Moreover, if $C \leqq E(X) - A$, then $V_1 \geqq A$ and, once $V_1 \geqq A$, the rest of V_n, $n=2, 3, \cdots$, can now be obtained by the usual methods used in section 3. In this case, it makes no difference whether $S \geqq A$ or $S < A$, as long as the limiting value V_c is concerned. If, however, $C > E(X) - A$ (but $C < E(X) - S$, as assumed), then $A > V_1$ ($> S$). Since $V_1 < A$ this time, according to (4), $V_2 = E(X) - C$, which is the same as V_1. In fact, when $C > E(X) - A$, $V_1 = V_2 = \cdots V_c = E(X) - C$.

The general equation for all continuous-time problems is (15), where it is assumed that $C < E(X) - S$ in order for a positive rate of change of $V(t)$ at $t=0$. When $V(t) < A$, (15) simply reduces to $dV(t)/\lambda dt = E(X) - V(t) - C$, and the solution of that is $V(t) = [E(X) - C] - [E(X) - S - C]\exp(-\lambda t)$. Now, if $C > E(X) - A$, $V(t)$ will never reach A, and the limiting value in this case is the same as that of the discrete-time case, that is, $V_c = E(X) - C$. On the other hand, if $C < E(X) - A$, $V(t)$ reaches A at $t = (1/\lambda)\log\{[E(X) - S - C]/[E(X) - A - C]\}$, and at this point we switch to the old equations and formulas derived for $V(t) \geqq A$. When we switch, however, we need to reset the time origin in such a way that the threshold value at the new time origin equals A. In other words, every t in (11), (12), (13), (17) should be replaced by $(t - \triangle t)$, where $\triangle t = (1/\lambda)\log\{[E(X) - S - C]/[E(X) - A - C]\}$, and every S in (13), (17) by A. Similarly, we replace τ in Table 2 with $(\tau - \triangle \tau)$, where $\triangle \tau = \lambda \triangle t$. Finally, no adjustment is necessary as long as the limiting value V_c is concerned.

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