醫工學會誌：第10巻, 第3號, 1989
J. of KOSOMBE : Vol. 10, No. 3

研究論文
89-3-15 *

Error Analysis Casued by Using the Dftin Numerical Evaluation of Rayleigh's Integral

Sun I. Kim

－Abstract－

Large bias errors which occur during a numerical evaluation of the Rayleigh''s integral is not due to the replicated source problem but due to the coincidence of singularities of the Green's function and the sampling points in Fourier domain. We found that there is no replicated source problem in evaluating the Rayleigh's integral numerically by the reason of the periodic assumption of the input sequence in DFT or by the periodic sampling of the Green's function in the Fourier domain. The wrap around error is not due to an overlap of the individual adjacent sources but because of the undersampling of the Green's function in the frequency domain. The replicated and overlapped one is inverse Fourier transformed Green's function rather than the source function.

1. Introduction

Several problems arise when we evaluate analog signals numerically. Frequently, unexpected results may occur unless one thoroughly understands the basics of the discrete Fourier transform (DFT) and pays attention to several things. These include aliasing due to undersampling, circular convolution and windowing effect. Also, some confusion may easily arise when evaluating continuous convolution integrals numerically by using the DFT, especially, when the integral kernels are given in both time (or spatial) and frequency (or Fourier) domains in analytical form. The evaluation of Rayleigh's integral is a good example of such confusion. Earlier, Williams and Maynard[1] analyzed bias errors associated with evaluation of the Rayleigh integral for planar radiators using the FFT. Based on theoretical consideration, We propose an alternate explanation for interpreting the bias error analysis.

〈접수：1989년 12월 1일〉
본 연구는 한국 과학재단 신진연구비 893-0807-010-2로 이루워 졌음.

Department of Medical Information and Management
College of Medicine, Hanyang University

2. Analysis

By the use of the Green's function technique, the Helmholtz equation can be used to calculate a complex field from pressure or pressure gradient data over a closed surface[2]. Rayleigh's integral is a special case of the Helmholtz integral equation which is the foundation of the theory of the radiation from wave sources. Rayleigh's integral can be represented in terms of pressure or pressure gradient fields as source functions. The solution of either of the Rayleigh integrals equals to the solution of scalar wave equation in a half space bounded by an infinite plane with known boundary condition[3].

One of the Rayleigh's integrals which relates the field pressure distribution with source pressure gradient, for the geometry shown in Fig.1, is given by,

$$p(x,y,z)=\iint_{-\infty}^{\infty} dx'dy' \; v(x',y',0) \; \frac{-Exp(jkr)}{2\pi R} \qquad \text{for } z\geq 0, \qquad (1)$$

where $R=[(x-x')^2+(y-y')^2+z^2]^{1/2}$, the wave number $k=2\pi/\lambda$ and $p(x,y,z)$, $v(x,y,z)$ represent pressure and pressure gradient at position (x,y,z) at time t, respectively. While the form of the integral is simple, the detailed computation of the field remains a complicated one. It is time－consuming to evaluate this integral by standard numerical techniques. the fast Fourier transform algorithm can be used to compute this integral at least 400 times faster than numerical calculation using Simpson's rule[1]. To fully utilize the FFT algorithm, we need to represent Eq.(1) in terms of Fourier transform relations. Rewriting Eq.(1) in convolution integral form at plane $z=d$, $d\geq 0$

$$p(x,y,d)=v(x,y,0) \;**\; g(x,y,d), \qquad (2)$$

where the symbol ＊＊ represents 2－dimensional convolution and the Green's function $g(x,y,d)$ is,

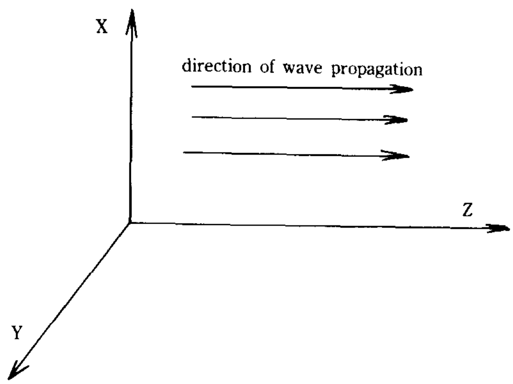

Fig. 1 Coordinate system for the Rayleigh's integral calculation.

$$g(x,y,d)= - Exp(jkr)/(2\pi r) \qquad (3)$$

and $r=(x^2+y^2+d^2)^{1/2}$. Taking the Fourier transform of each side of Eq.(2) and using the convolution theorem yields,

$$P(f_x,f_y,d)=V(f_x,f_y,0)\; G(f_x,f_y,d) \qquad (4)$$

Where the capital letter means the function is in the Fourier domain. By Weyl's intergral identity[4],

$$\frac{Exp(jkr)}{r}=\frac{j}{2\pi}\iint_{-\infty}^{\infty} df_x' df_y' \frac{Exp[j(f_x'x+f_y')+jd(k^2-f_x'^2x-f_y'^2y)^{1/2}]}{(k^2-f_x'^2-f_y'^2)^{1/2}} \qquad (5)$$

we obtain the Green's function in Fourier domaitn,

$$G(fx,fy,d)=\begin{cases} \dfrac{Exp[jd(k^2-k_x^2-k_y^2)^{1/2}]}{j(k^2-k_x^2-k_y^2)^{1/2}} & \text{for } k^2<k_x^2+k_y^2 \\ \dfrac{Exp[-d(k_x^2+k_y^2-k^2)^{1/2}]}{(k^2-k_{xy}^{22})^{1/2}} & \text{for } k^2<k_x^2+k_y^2 \end{cases} \qquad (6)$$

where $k_x=2\pi f_x$, $k_y=2\pi f_y$ When $k^2<k_x^2+k_y^2$, the exponent gives strong attenuation and these waves are called evanescent waves[5]. Multiplying the Green's function with Fourier transformed source function $V(f_x,f_y,0)$ and applting inverse Fourier transform, one obtains,

$$p(x,y,d)=FT^{-1}[V(f_x,f_y,0)\; G(f_x,f_y,0)] \qquad (7)$$

Eq.(7) can be evaluated numerically by using the discrete Fourier transform (DFT),

$$p_d(mx_0,ny_0,d)=DFT^{-1}[V(pf_{x0},qf_{y0},0)\; G(pf_{x0},qf_{y0},d)\; W(pf_{x0},qf_{y0})] \qquad (8)$$

where n,m,p and q＝0,1,2 ···, N-1, and N is number of samples. x_0 and y_0 are sampling intervals in spatial domain on x and y axis, and f_{x0} and f_{y0} are sampling intervals in Fourier domain on f_x and f_y axis, respectively. W is a window function. We use

^ to distinguish discrete Fourier transformed sequences from the sampled sequences that are formed by digitizing the continuous function. As we see in Eq.(8), the function $G(f_x, f_y, d)$ is sampled in the fourier domain with a sampling interval of f_{xo} and f_{yo}. The sampling intervals are determined by the spatial data length (or the width of spatial windows) T_x and T_y by,

$$f_{xo}=1/T_x,\ f_{yo}=1/Ty. \tag{9}$$

the width of the sampling window W in Fourier domain is thus given by $Tf_x=Nf_{x0}$, $Tf_y=Nf_{y0}$, The procedure of elvaluating Eq. (8) is 1) discrete Fourier transform $v(mx_0, ny_0, 0)$, which is a the sampled and windowed version of continuous function $v(x, y, 0)$ at distance $d=0$, to get $\hat{V}(pf_{x0}, qf_{y0}, 0)$, then 2) multiply by the windowed and sampled Green's unction $G(pf_{x0}, qf_{y0}, d)\ W(pf_{x0}, qf_{y0}, 0)$ and 3) calculate inverse DFT to get estimated $Pd(m_{x0}, m_{y0}, d)$

Because we need to use the notation for periodic function later, let us define the discrete Fourier series (DFS) of a periodic function $\tilde{f}(mx_o, ny_o)$ with period of T_x, T_y and sampling interval xo, yo in the x and y directions, respectively. For simplicity, we will consider the one dimensional case, since the two dimensional case easily can be extended from the result. When $f(x)$ is a inverse Fourier transform of $F(f_x)$, the inverse DFS of $\tilde{F}(pk_{xo})$ is 6.

$$\tilde{f}(m_{xo})=1/T_x\sum_{p=0}^{N-1}\tilde{F}(PK_{xo})\,\mathrm{Exp}[j(2\pi N)mp] \tag{10}$$

where $k_{xo}=2\pi f_{xo}=2\pi/T_x$, $T_{fx}=1/x_o$

$$\tilde{f}(mx_0)=\sum_{i=-\infty}^{\infty} f(mx_0+iT_x), \tag{11}$$

and

$$\tilde{f}(PK_{xo})=\sum_{l=-\infty}^{\infty} F(pk_{x0}-1\,T_{fx}). \tag{12}$$

Here, the periodic functions $\tilde{f}(mx_o)$ and $\tilde{F}(pk_{xo})$ are aliased versions of non-periodic functions $f(mx_o)$ and $F(pk_{xo})$, respectively. As we know, the DFT assumes that one finite sequence of discrete data is a one period of an infinite sequence of period T_x. In this way the Fourier transform is evaluated.

Williams and Maynard[1] stated the evaluation of Eq.(7) as,

$$p(x, y, d)=FT^{-1}[V(f_x, f_y, 0)\ G(f_x, f_y, d)\ Ш(f_x/f_{xo}, f_y/f_{yo})\ \Pi(f_x/T_{kx}, f_y/T_{ky})\ ] \tag{13}$$

where Ш is a sampling function and Π an is window function, following Bracewell's notation[7]. Applying convolution theorem (really, the converse of the convolution theorem) to Eq.(13) yields,

$$p(x, y, d)=FT^{-1}[f_x, f_y, 0]**FT^{-1}[G(f_x, f_y, d)]**FT^{-1}[(Ш(F_x/f_{xo}, f_y/f_{yo})]**FT^{-1}[\Pi(f_x/T_{kx}, f_y/T_{ky})]$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_0\, y_0\, v(x, y, 0)\,\delta(x-mx_0)\,\delta(y-ny_0)$$

$$**g(x, y, d)\ **(1/(T_xT_y))\ Ш(x/T_x, y/T_y)$$

$$**\mathrm{sinc}(x/x_0, y/y_0).$$

If we write $v_N(x, y)$ as,

$$v_N(x, y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_0\, y_0\, v(x, y, 0)\,\delta(x-mx_0)\,\delta(y-ny_0), \tag{15}$$

then Eq.(14) becomes,

$$p(x, y, d)=A\ v_N(x, y)**\ Ш(x/T_x, y/T_y)**g(x, y, d)\ **\mathrm{sinc}(x/x_o, y/y_o)$$

$$=A\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} v_N(x+iT_x, y+jT_y)\ **g(x, y, d)**\ \mathrm{sinc}(x/x_0, y/y_0) \tag{16}$$

where A is some constant. Wiliams and Maynard explained that the first convolution in last equation of Eq.(16) leads to replicated sources with terms $v_N(x+iT_x, y+jT_y)$, because the double summation sums up all the replicated terms. They thought, since g(x,y,d) extends over all real space, that convolution suffers from wrapping around. Thus the aliasing effect in Fourier domain is the same as wrapping around in spatial domain.

We believe that there are misunderstandings in interpreting Eq.(7) as Eq.(16). Even though the sampling function and the window function are written in separate forms, once the sampling and window function are multiplied with any infinite analog function to be discrete Fourier transformed, it is inappropriate to separate those terms as in Eq.(14) and to combine them with any other arbitrary terms. The sampling function Ш and the window function Π should stick with $G(f_x, f_y, d)$. Thus it is am isleading to interpret Eq.(16) as meaning that there are replicated sources.

There still exists a large bias error in reconstructing object fields from source pressure gradient distributions and it happens under the same conditions as given by Williams and Maynard[1]. This can be explained as follows. First rewrite Eq.(8) in the inverse DFTed form,

$$p_d(mx_0\ ny_0, d) = v(mx_0, ny_0, 0)**[\hat{g}(mx_0, ny_0, d) **w(mx_0, ny_0)] \quad (17)$$

Obviously $g(mx_0, ny_0, d) \neq g(mx_0, ny_0, d)$ for $0 \leq mx_0 \leq Tx$, $0 \leq ny_0 \leq T_y$. The aliased version of function g (mx_0, ny_0, d) is equal to $g(mx_0, ny, d)$ only when the continuous function g(x, y) is space limited within T_x and T_y, which is not true in the case of Green's function as given in Eq. (3). Also notice that the $v(mx_0, ny_0, 0)$ recoverd from the original sequence exactrly whether it is undersampled or not. Now. $v(mx_0, ny_0, 0)$ is not "wrapped－around" as a result of forward and inverse DFT operation. because the original spatial function is tapered to zero outside of the window. i. e. sptially limited. It is $g(mx_0, ny_0, d)$ not $v(mx_0, ny_0, 0)$ which can cause bias errors when we evaluationg E. q. (7) using the DFT to estimate p(x, y, d). The Green's function given in Eq. (6) in frequency domain is virtually band－limted function because the evanescent waves component decays out fast. As a result, the Fourier transform pair which is sampling therorem, the given in Eq. (3) is not space-limited. Thus, according to the sampling in frequency domain is always undersampled.

In the following section, we will see how this aliased version of Green's function affects the reconstruction result. Again, consider one dimensional case for simplicity. The inverse DFTed Green's function $\tilde{g}(mx_0)$ is an aliased version of $g(mx_0)$ given in Eq.(3). According to Eq.(11), the aliased Green's function can be written as,

$$\hat{g}(mx_0, d) = \tilde{g}(mx_0, d) = \sum_{i=-\infty}^{\infty} g(mx_0 + iT_x) \quad (18)$$

$$= \frac{-1}{2\pi} \sum_{i=-\infty}^{\infty} \frac{\mathrm{Exp}\{jk[(mx_0 + iT_x)^2 + d^2]^{1/2}\}}{[(mx_0 + iT_x)^2 + d\ ]^{1/2}}$$

When d=0,

$$\tilde{g}(mx_0, d) = -\frac{1}{2\pi} \sum_{i=-\infty}^{\infty} \frac{\mathrm{Exp}\ [jk(mx_0 + iT_x)]}{mx_0 + iT_x}$$

$$= \mathrm{Exp}[jkmx_0] \sum_{i=-\infty}^{\infty} \frac{\mathrm{Exp}[jkT_x i]}{mx_0 + T_x i} \quad (19)$$

Since $kT_x = 2\pi T_x / \lambda$, whenever the value of T_x / λ is an integer the exponent is unity for all i and the diverges. However, this divergence happens only when d=0.

The large bias error discussed in Williams and Maynard's paper[1] arises, not because of the replicated source problem, but because of the coincidence of a sampling position with the singularity of $G(f_x, f_y, d)$ in the Fourier domain. As we see from Eq.(6), the singularity happens when,

$$k^2 = k_x^2 + k_y^2 = (pk_{x0})^2 + (qk_{y0})^2, \quad (20)$$

which defines a circle centered at the origin with radius of k in the Fourier domain. Since the sampling spacings kx_o, ky_o in Fourier adomain are related to aperture size as $k_{xo}=2\pi/T_x$, $k_{yo}=2\pi/T_y$, Eq.(20) can be written as,

$$(2\pi/\lambda)^2 = (p\ 2\pi/T_x)^2 + (q\ 2\pi/T_y)^2 \quad (21)$$

Thus, whenever T_x/λ and T_y/λ become integers, the equality of Eq.(21) is satisfied and the singularity happens. Thus the inverse DFT will fail under these conditions. Note that this situation can arise at any value of d. Fig.2 shows the relation between the value of k and the position of singularities. The reason why the bias errors are dominant for small integers is explained graphically in Fig.2 Let us say here $k_{yo}=0$ for simplicity. Fig.3−a shows the case of $T_x<\lambda$. In this case, the singularity occurs between p = 0 and p = 1 in the Fourier domain, that is,

$$k=2\pi/\lambda<2\pi/T_x= K_{xo} \quad (22)$$

Because the singularity happens in between the sampling points, and the Green's function decays fast after the singularity (as d increases by one wavelength, the amplitude decays about 3.5×10^{-6})[8], the resulting function sampled at discrete integer points shows a quite different shape from the original function. Because we sample the data only at integer points, in case of the condition given in Eq.(22), the sampled sequence gives some value at origin and very small values at the other points. This looks like a impulse function, thus the inverse DFT of the sequence results in an almost a constant value as seen in Fig.(3) of Williams and Maynard'd paper. When $T_x=\lambda$, then a singularity arises at p=1, which is the point of $k=k_{ko}$ shown in Fig.3-b, and the DET will fail because there is a infinite value at p=1. This phenomenon happens continua-

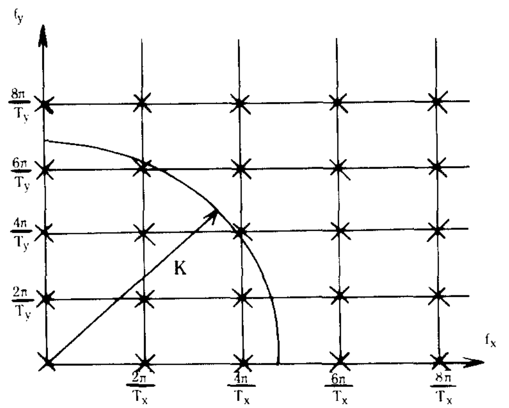

Fig. 2 Lattice points of Green's function singularity position (marked as x) in 2 dimensional Fourier space with radiation circle. Only first quadrant is shown.

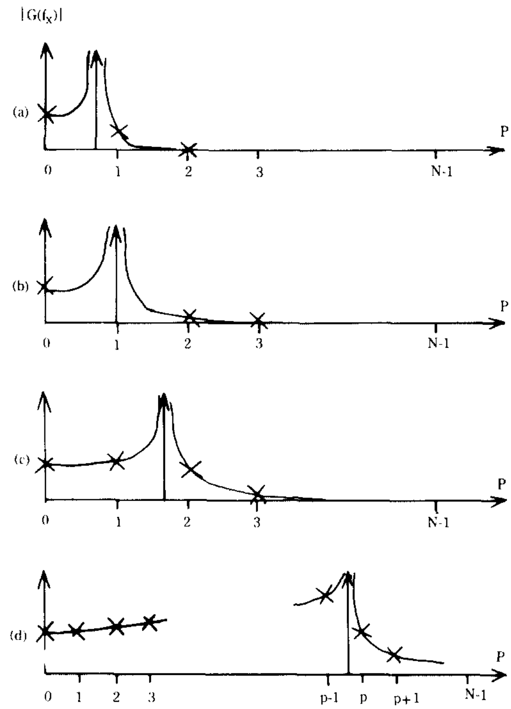

Fig. 3 The relationship between the sampling points and the position of singularity when (a) $0<T_x<\lambda$, (b) $T_x=\lambda$, (c) $\lambda<T_x<2\lambda$ and (d) T_x is much larger than λ.

lly whenver the condition of Eq.(21) is met, i.e. whenever both T_x/λ and T_y/λ are integers. Fig.3-c is for the case when $\lambda<T_x<2\lambda$, and also shows quite different sampled data than the original function. When the value of T_x/λ is getting close to an integer (equivalently, k value is close to a sampling poing), the sampled value near the singularity becomes larger. As a result a large bias errors occurs. As λ becomes smaller, in other words as the value of k is getting larger, the spacing of adjacent singularities become closer, see in Fig.l, and the bias errors are lessened, as shown graphically in Fig.3-d.

3. Conclusion

The conditions of large bias error are the same as those reported earlier by Williams and Maynard's. We showed that the coincidence of singularities of $G(pk_{xo}, qk_{yo}, d)$ and the sampling points causes large bias errors. Also, the divergence of the sum in Eq.(19) is neither related to the location of singularities nor the distance, d. As shown in Eq.(17), there is no replicated source problem in evaluating Rayleigh's integral numerically by the reason of the periodic assumption of input sequence in DFT[9] or by the periodic sampling of the function in the Fourier domain[1]. Because the source functions are not wrapped over each other after forward and inverse FFT, the wrap around error is not due to an overlap of the individual sources but because of the undersampling of the Green's function in the frequency domain. Actually, the replicated and overlapped one is inverse Fourier transformed Green's function rather than the source function. As a matter of fact, as the distanced, increases, the frequency content (frequency of the oscillation in frequency domain) of pass band in Eq.(6) is also increased. Thus, the more the distance increases, the more we undersample the Green's function in frequency domain and have wrap around error in spatial domain. By this reason, there exists a maximum propagation distance for field reconstrucation. Note that the pass band-width in Eq.(6) is not a function of d but a function of k. We will discuss about this subject elsewhere. The sampling rate in frequency domain is proportional to the window width in space. Thus, the incresase of sampling rate is frequency domain is a more accurate interpretation for the "zero padding" required to reduce errors in spatial domain than that of moving replicated surces away from each other[10].

Other factors to be considered are circular convolution and windowing effect in both spatial and Fourier domain. This will be discussed in a future publication.

Reference

1) E. G. Williams and J. D. Maynard, "Numerical evaluation of the Rayleigh integral for planar radiators using the FFT," J. Acoust. Soc. Am. 72(6), pp. 2020-2030, Dec. 1982.
2) C. J. Bouwkamp, "Diffraction theory, "Rept. Progr. Physics, vol. 17, pp. 35-39, 1954.
3) R. T. Beyer and S. V. Letcher, "Physical Ultrasonics," Academic Press, New York, 1969. Ch. 1.
4) G. C. Sherman, "Application of the convolution theorem to Rayleigh's integral formulas," J. Opt. Soc. Am. vol. 57 (4), pp. 546-547, Apr. 1967.
5) J. W. Goodman, "Introduction to Fourier Optics," McGraw-Hill, New York, 1968. Ch. 3.
6) A. Papoulis, "Signal Analysis," McGraw-Hill, New York, 1977. Ch. 3.
7) R. Bracewell, "The Fourier Transform and Its Application." 2nd ed. McGraw-Hill, New York, 1978.
8) J. R. Shewell, "Inverse diffraction and a new

reciprocity theorem," J. Opt. Soc. Am. 58(12), pp. 1596-1603, Dec. 1968.

9) E. A. Sziklas and A. E. Siegman, "Mode Calculation in Stable Resonators with Flowing Saturable Gain. 2 : Fast Fourier Transform Method," Applied Opatics, vol 14, No. 8, pp. 1874-1889. Aug. 1975.

10) J. P. Powers, "Computer Simulation of Linear Acoustic Diffraction," Acoust. Holography, edited by L. W. Kessler, Plenum, New York, vol. 7. pp. 193-205, 1976.