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THE COMPUTATION METHOD OF THE MILNOR NUMBER OF HYPERSURFACE SINGULARITIES DEFINED BY AN IRREDUCIBLE WEIERSTRASS POLYNOMIAL $z^n+a(x,y)z+b(x,y)=0$ in C^3 AND ITS APPLICATION

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Introduction

Let $V=\{(x,y,z) : f=z^n-npz+(n-1)q=0$ for $n\geq 3\}$ be a complex analytic subvariety of a polydisc in $\boldsymbol{C}^3$ where $p=p(x,y)$ and $q=q(x,y)$ are holomorphic near $(x,y)=(0,0)$ and f is an irreducible Weierstrass polynomial in z of multiplicity n. Suppose that V has an isolated singular point at the origin. Recall that the z-discriminant of f is $D(f)=c(p^n-q^{n-1})$ for some number c. Suppose that $D(f)$ is square-free. Then we prove that by Theorem 2.1 $\mu(p^n-q^{n-1})=\mu(f)-(n-1)+n(n-2)I(p,q)+1$ where $\mu(f)$, $\mu(p^n-q^{n-1})$ are the corresponding Milnor numbers of f, p^n-q^{n-1}, respectively and $I(p,q)$ is the intersection number of p and q at the origin. By one of applications suppose that $W_t=\{(x,y,z) : g_t=z^n-np_t^{n-1}z+(n-1)q_t^{n-1}=0\}$ is a smooth family of complex analytic varieties near $t=0$ each of which has an isolated singularity at the origin, satisfying that the z-discriminant of g_t, that is, $D(g_t)$ is square-free. If $\mu(g_t)$ are constant near $t=0$, then we prove that the family of plane curves, $D(g_t)$ are equisingular and also $D(f_t)$ are equisingular near $t=0$ where $f_t=z^n-np_tz+(n-1)q_t=0$.

1. Preliminaries

Let O_n be the ring of germs of holomorphic functions near the origin in $\boldsymbol{C}^n$. Let $f:(\boldsymbol{C}^n,0)\to(\boldsymbol{C},0)$ be a germ at the origin of holomorphic function with an isolated singular point. The Milnor number of f is defined by the dimension of $O_n/\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right)$ as a finite dimensional $\boldsymbol{C}$-vector space and it is denoted by $\mu(f)$. Let $e_n(J)$ be defined by the

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dimension of O_n/J as a finite dimensional $\boldsymbol{C}$-vector space where $J=(f_1, \dots, f_n)$ is an ideal in O_n generated by $f_1, \dots, f_n$.

THEOREM 1.1. *Let* $V=\{(x, y, z) : f=z^n+a_1z^{n-1}+\dots+a_iz^{n-i}+\dots+a_n=0\}$ *be a complex analytic subvariety of a polydisc near the origin in* $\boldsymbol{C}^3$ *where the* $a_i=a_i(x, y)$ *are holomorphic near* $(x, y)=(0, 0)$ *and* f *is a Weierstrass polynomial in* z *of multiplicity* n. *Suppose that the origin in* $\boldsymbol{C}^3$ *is an isolated singular point of* V *and that the* z*-discriminant of* f, *denoted by* $D(f)$, *is square-free. Then we have*

$$\mu(D)=\mu(f)-(n-1)+2k(f)+3\phi(f)+1$$

where $\phi(f)=e_3(f, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial z^2})$ *and* $2k(f)=e_4(J)$ *such that* J *is an ideal in* O_4 *generated by* $f(x, y, z)$, $f'_z(x, y, z_1)$, $\frac{f'_z(x, y, z_2)-f'_z(x, y, z_1)}{z_2-z_1}$ *and* $\frac{1}{(z_2-z_1)^3}\{f(x, y, z_2)-f(x, y, z_1)-\frac{z_2-z_1}{2}[f'_z(x, y, z_1)+f'_z(x, y, z_2)]\}$ *with respect to coordinates* x, y, z_1, z_2.

Proof. See [[1], Proposition 1.1, p. 263].

2. The computation method of the Milnor number of hypersurface singularities defined by $z^n-npz+(n-1)q=0$ with some condition

THEOREM 2.1. *Let* $V=\{(x, y, z) : f=z^n-npz+(n-1)q=0$ *for* $n\geq 3\}$ *be a complex subvariety of a polydisc in* $\boldsymbol{C}^3$ *where* $p=p(x, y)$ *and* $q=q(x, y)$ *are holomorphic near* $(x, y)=(0, 0)$ *and* f *is a Weierstrass polynomial in* z *of multiplicity* n. *Suppose that the origin in* $\boldsymbol{C}^3$ *is an isolated singular point of* V. *Then the* z*-discriminant of* f *is* $D(f)=c(p^n-q^{n-1})$ *for some number* c. *Suppose that* $D(f)$ *is square-free. Then we have the following*:

$$\mu(p^n-q^{n-1})=\mu(f)-(n-1)+n(n-2)I(p, q)+1$$

where $I(p, q)=dim\ O_2/(p, q)$ *as a* $\boldsymbol{C}$*-vector space.*

Proof. By Theorem 1.1, it is enough to compute $2k(f)$ and $3\phi(f)$ for this f. The ideal $(f, f_z, f_{zz})=(Z^{n-2}, p, q)$. Thus $\phi(f)=\dim O/(f, f_z, f_{zz})=\dim O/(z^{n-2}, p, q)=(n-2)I(p, q)$. Now by Theorem 1.1, let us calculate four generators of the ideal J for $2k(f)$, in order to prove that $2k(f)=(n-2)(n-3)I(p, q)$ for $n\geq 3$:

(1) $f(x, y, z_1)=z_1{}^n-npz_1+(n-1)q$

(2) $f'_z(x,y,z_1)=nz_1^{n-1}-np=n(z_1^{n-1}-p)$

(3) Let $a(z_1,z_2)=\dfrac{f'_z(x,y,z_1)-f'_z(x,y,z_2)}{z_1-z_2}$

Then for $z_1\neq z_2$, $(z_1-z_2)a(z_1,z_2)=[(nz_1^{n-1}-np)-(nz_2^{n-1}-np)]$ $=n(z_1^{n-1}-z_2^{n-1})$. So $a(z_1,z_2)=n(z_1^{n-2}+z_1^{n-3}z_2+\ldots+z_2^{n-2})$.

(4) Let $b(z_1,z_2)=\dfrac{1}{(z_2-z_1)^3}[f(x,y,z_2)-f(x,y,z_1)-\dfrac{z_2-z_1}{2}(f'_z(x,y,z_1)+f'_z(x,y,z_2))]$. Then for $z_1\neq z_2$, $(z_2-z_1)^3b(z_1,z_2)=(z_2^n-npz_2+(n-1)q)-(z_1^n-npz_1+(n-1)q)-\dfrac{z_2-z_1}{2}(nz_1^{n-1}-np+nz_2^{n-1}-np)=(z_2-z_1)(z_2^{n-1}+z_2^{n-2}z_1+\ldots+z_2z_1^{n-2}+z_1^{n-1}-np)-\dfrac{1}{2}(z_2-z_1)(nz_1^{n-1}+nz_2^{n-1}-2np)=\dfrac{1}{2}(z_2-z_1)[(2-n)z_2^{n-1}+2z_2^{n-2}z_1+\ldots+2z_2z_1^{n-2}+(2-n)z_1^{n-1}]$.
Thus $2(z_2-z_1)^2b(z_1,z_2)=(2-n)z_2^{n-1}+2z_2^{n-2}z_1+\ldots+2z_2z_1^{n-2}+(2-n)z_1^{n-1}$. Now

$(z_1+z_2)a(z_1,z_2)-2n(z_2-z_1)^2b(z_1,z_2)=n(n-1)(z_1^{n-1}+z_2^{n-1})\in J.$
Also, since $(z_1-z_2)a(z_1,z_2)$ gives that $z_1^{n-1}-z_2^{n-1}\in J$, $z_1^{n-1}, z_2^{n-1}\in J$. Since $z_1^{n-1}\in J$, p and q are in J from two equations (1) and (2). Therefore $J=(a(z_1,z_2),b(z_1,z_2),p,q)$ is in O_4. Now if we prove that $a(z_1,z_2)$ and $b(z_1,z_2)$ are relatively prime in a ring of convergent power series of z_1,z_2, then we know that $\dim O_2/(a(z_1,z_2),b(z_1,z_2))=(n-2)(n-3)$ since $a(z_1,z_2)$ and $b(z_1,z_2)$ are homogeneous polynomials in z_1,z_2 of degree $n-2$ and $n-3$, respectively. Therefore $2k(f)=\dim O_4/J=(n-2)(n-3)I(p,q)$ where $I(p,q)$ is the intersection number $\dim O_2/(p,q)$ as a C-vector space. Then $3\phi(f)+2k(f)=[3(n-2)+(n-2)(n-3)]I(p,q)=n(n-2)I(p,q)$. Let us prove that $a(z_1,z_2)$ and $b(z_1,z_2)$ have no common factor in a ring of convergent power series of z_1,z_2. Now $a(z_1,z_2)=n(z_2-\omega_1z_1)\ldots(z_2-\omega_{n-2}z_1)$, where $\omega_k=e^{2\pi k/n-1}$ for $k=1,\ldots,n-2$. Replacing z_2 by ω_kz_1, then $2(z_2-z_1)^2b(z_1,z_2)=(2-n)z_1^{n-1}-nz_1^{n-1}=(2-2n)z_1^{n-1}$. Since $n>1$, $a(z_1,z_2)$ and $b(z_1,z_2)$ have no common factor in O_2.

3. Some application

Definition 3.1. Let $V=\{(y,z):f(y,z)=0\}$ and $W=\{(y,z):g(y,z)=0\}$ be germs of analytic varieties of a polydisc in C^2 where f,g are holomorphic and square-free near the origin and the origin is an isolated singular point of V and W, both. V and W are said to be topologically equivalent or equisingular if there exists a germ at the origin of homeomorphism $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0)=0$ where U_1 and U_2 are open subsets containing the origin in $\boldsymbol{C}^2$. In this case, we call $f(y, z)$ and $g(y, z)$ topologically equivalent or equisingular.

LEMMA 3.2 (A generalization of Milnor's Theorem). *Let $F(x, y)$ be $F_1 \cdot F_2 \cdots F_k$ where F_i is a plane curve with only singularities at the origin in $\boldsymbol{C}^2$ and F_i may be reducible. Then $\mu(F) = \sum_{i=1}^{k} \mu(F_i) + 2\sum_{i<j} I(F_i, F_j) - k + 1$ where each $I(F_i, F_j)$ is the intersection number of F_i and F_j at the origin for $i \neq j$.*

THEOREM 3.3. *Let $V_t = \{(x, y, z) : g_t = z^n - np_t^{n-1}z + (n-1)q_t^{n-1} = 0\}$ be a smooth family of complex analytic varieties of a polydisc in $\boldsymbol{C}^3$ where $p_t = p(x, y, t)$ and $q_t = q(x, y, t)$ are holomorphic near $(x, y) = (0, 0)$ and smooth near $t=0$, and g_t is a Weierstrass polynomial in z of multiplicity n at the origin in $\boldsymbol{C}^3$. Suppose that the origin in $\boldsymbol{C}^3$ is an isolated singular point of U_t and the z-discriminant of g_t, $D(g_t)$, is square-free for each t. Suppose that the Milnor number $\mu(g_t)$ are constant for all t near 0. Then $\mu(D(g_t))$ are constant for such all t. Moreover, if $f_t = z^n - np_t z + (n-1)q_t = 0$ and $\mu(g_t)$ are constant for such t with the same assumption as above, then $\mu(D(f_t))$ are constant, too.*

Proof. By Theorem 2.1, $\mu(D(g_t)) = \mu(g_t) - (n-1) + n(n-2)I(p_t^{n-1}, q_t^{n-1}) + 1$. But $\mu(D(g_t)) = \mu(p_t^{n(n-1)} - q_t^{(n-1)^2}) = \mu(p_t^n - \omega_1 q_t^{n-1}) + \ldots + \mu(p_t^n - \omega_{n-1} q_t^{n-1}) + 2 \cdot {}_{n-1}C_2 I(p_t^n, q_t^{n-1}) - (n-1) + 1$ where ω_i is a root of an equation $a^{n-1} = 1$, by Lemma 3.2. So $\mu(p_t^n - \omega_1 q_t^{n-1}) + \ldots + \mu(p_t^n - \omega_{n-1} q_t^{n-1}) = \mu(g_t)$ for all t near 0. Since $\mu(g_t)$ are constant and μ is nonnegative and upper semi-continuous, for each fixed i $\mu(p_t^n - \omega_i q_t^{n-1})$ is constant and so $p_t^n + \omega_i q_t^{n-1}$ is equisingular near $t=0$ by [3]. Consider the family $W_t = \{(x, y, z) : f_t = z^n - np_t z + (n-1)q_t = 0\}$. Then the z-discriminant for f_t, $D(f_t)$ is equisingular near $t=0$ because ω_j may be equal to one for some j. Since $\mu(p_t^n - q_t^{n-1}) = \mu(f_t) - (n-1) + n(n-2)I(p_t, q_t) + 1$ for some $\omega_j = 1$ and $\mu(p_t^n - q_t^{n-1})$ is constant then $\mu(f_t)$ is constant and also $I(p_t, q_t)$ is constant near $t=0$. Therefore, then fact that $I(p_t, q_t)$ is constant near $t=0$ implies that $\mu(D(g_t))$ is constant for such t.

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