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SOME RESULTS ON FLSTER THEORY OF BCK-ALGEBRAS

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1. Introduction and preliminaries

K. Iseki [2] has introduced the notion of a BCK-algebra which is an algebraic formulation of a propositional calculus. In his various papers, Iseki studied the structure of these algebras(see[3], [4], [5]). Also E. Y. Deeba [1] introduced the notion of filters, and studied their basic properties. In this paper, we obtain an equivalent condition and property of F-ascending chain condition, and also some properties of irreducible filters in a BCK-algebra.

In [1], Deeba has defined a filter as follows :

Definition. A non-empty set F of a BCK-algebra X is called a filter of X if

(1) $x \in F$ and $x \leq y$ imply that $y \in F$,
(2) $x \in F$ and $y \in F$ imply that $glb\{x, y\} \in F$.

Let x be a bounded implicative BCK-algebra and F a filter of X, Define a relation $\sim$ on X as follows : $x \sim y$ if and only if $1*(x*y) \in F$, $1*(y*x) \in F$ Then$\sim$is an equivalence relation on X, and so X can be partitioned in to equivalence classes. The class containing $x \in X$ will be denoted by Fx It is clear that $x \sim y$ and only if $F_x = F_y$. Denote the set of all such equivalence classes by X/F. Define a binary operation o in X/F as follow :$F_x \circ F_y = F_{x*y}$ · Further we write $F_x \leq F_y$ if and only if $x*y \sim 0$. Then$(X/F ; \circ, F_0)$ is a BCK-algebra.

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2. Main results

Lemma 2.1. Let $f : X \to X'$ be a homomorphism of BCK–algebras. If F' is a filter of X', then $f^{-1}(F')$ is a filter of X

Proof Let $x \in f^{-1}(F')$ and assume that $x \leqq y$ Then $f(x) \in F'$ and $f(x) \leq f(y)$ Since F' is a filter of X', we have $f(y) \in F'$. It follows that $f^{-1}(f(y)) \in f^{-1}(F')$, and hence $y \in f^{-1}(F')$ Next, let $x, y \in f^{-1}(F')$. Then we have $f(x), f(y) \in F'$. Since F' is a filter and f is isotone, we have $f(glb\{x, y\}) = glb\{f(x), f(y)\} \in F'$. It follows that $glb\{x, y\} \in f^{-1}(F')$. This completes the proof.

Theorem 2.2 Let $f : X \to X'$ be an epimorphism of BCK–algebras. If F$'$ and G$'$ are distinct filters of X', then $f^{-1}(F')$ and $f^{-1}(G')$ are also distinct filters of X.

Proof. It follows from Lemma 2.1 that $f^{-1}(F')$ and $f^{-1}(G')$ are filters of X. Note that $f(f^{-1}(F')) = F'$ if f is surjective suppose that $f^{-1}(F') = f^{-1}(G')$. Then we obtain $F' = f(f^{-1}(F')) = f(f^{-1}(G')) = G'$, a contradiction. Hence $f^{-1}(F')$ and $f^{-1}(G')$ are distinct filters of X.

Definition 2.3. Let X be a BCK–algebra. We shall say that X satisfies the F–maximal condition if every non–empty set of filters of X has a maximal element.

Definition 2.4. A BCK–algebra is said to satisfy the F–ascending chain conition if each ascending chain of filters $F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots$; terminates after a finite number of steps.

Teorem 2.5. Let X be a BCK–algebra. Then the followings are equivalent :

(1) X satisfies the F–maximal condition.
(2) X satisfies the F–ascending chain condition.

Proof. Suppose that X satisfies the F–maximal condition, and let $F_1 \subset F_2 \subset \cdots$ be an ascending sequence of filters of X. Then the set $\{F_i : i = 1, 2, 3, \cdots\}$ of filters has a maximal element F_n. Hence we have $F_i = F_n$ for all $i \geq n$, that is, X satisfies the F–ascending chain condition. Conversely, assume that X satisfies the F-ascending chain condition. Let ∇ be a non-empty set of filters of X and suppose that ∇ has no maximal element. Take F_1 in ∇. Since F_1 is not maximal, there exists a filter F_2 in ∇ such that $F_1 \subset F_2$. Repeating this argument we obtain an infinite ascending sequence $F_1 \subset F_2 \subset \cdots$ of filters, a contradiction. Therefore X satisfies the F-maximal condition.

Theorem 2.6. Let X be a bounded implicative BCK-algebra. If X satisfies the F-ascending chain condition, then every quotient algebra of X by filter satisfies the same F-ascending chain condition.

Proof. Let F be a filter of X and $F_1' \subset F_2' \subset F_3' \subset \cdots$ an ascending sequence of filters of X/F. Since $p : X \to X/F$ is the canonical epimorphism, $\{p^{-1}(F_i')\}$ is an ascending sequence of filters of X. Since X satisfies the F-ascending chain condition, there is a natural number m such that $p^{-1}(F_m') = p^{-1}(F_i')$ for all $i \geq m$. The fact that p is the canonical epimorphism implies $F_i' = F_m'$ for all $i \geq m$. Therefore X/F satisfies the F-ascending chain condition.

Definition 2.7. A filter F of a BCK-algebra X is said to be irreducible if $F = G \cap H$ implies $F = G$ or $F = H$ for filters G, H

Proposition 2.8. If F is a filter in a BCK-algebra X, and x is not contained in F, then there is an irreducible filter G such that $F \subset G$ and $x \notin G$

Proof. See [8, p. 6].

Proposition 2.9. Let F be a filter in a BCK-algebra X. If, for any x, y of $X-F$, there is an element z of $X-F$ satisfying $x \leqq z$ and $y \leq z$, then F is irreducible.

Proof. See [8, p. 6].

Theorem 2.10. Let X be a BCK-algebra satisfying the F-ascending chanin condition. Then every filter in X can be written as the intersection of a finite number of irreducible filters.

Proof. Let $\bigtriangledown$ be the set of all filters of X, which cannot be written as the intersection of a finite number of irreducible filters. If the theorem is false then $\bigtriangledown$ is not empty. Since X satisfies the F–ascending chanin conditon, $\bigtriangledown$ has a maximal element, say F. Then F is not irreducible. Thus we have $F=G\cap H$ for filters G and H in X such that $F\subsetneqq G$ and $F\subsetneqq H$. However then G and H are not in $\bigtriangledown$. Hence G and H can both be written as the intersection of a finite number of irreducible filters, and the same is true for F, which is a contradiciton. This completes the proof.

Lemma 2.11 [1]. If F_α , $\alpha\in$ I, is a totally ordered family of filters of a BCK–algebra X ordered by inclusion, then both $\cup F_\alpha$ and $\cap F_\alpha$ are filters of X.

Theorem 2.12. Let A be an ideal of a BCK–algebra X. If F is a filter containing A, then F contains a filter which contains A and has no smaller filter containing A.

Proof. Let $\bigtriangledown_A$ be the set of all filters which contain A and are contained in F. The $F \in\bigtriangledown_A$, and so $\bigtriangledown_A$ is not empty. We write $F'\leq F''$ if $F''\subset F'$ for all F', $F''\in\bigtriangledown_A$. This gives a partial order on $\bigtriangledown_A$. We claim that $\bigtriangledown_A$ is an inductive system. For this purpose, let $\bigtriangledown_{A}'$ be a non–empty totally ordered subset of $\bigtriangledown_A$. By Lemma 2.11 the intersection of all filters of $\bigtriangledown_{A}'$ is a filter, say G. This certainly contains A and is contained in F. Consequently $G\in\bigtriangledown_A$. Since $G\subset F'$ for every $F'\in\bigtriangledown_{A}'$, we have $F'\leq G$ for every F' in $\bigtriangledown_A$. Thus G is an upper bound for $\bigtriangledown_{A}'$. Hence $\bigtriangledown_A$ is an inductive system. Then by Zorn's Lemma $\bigtriangledown_A$ has a maximal element, say F^*, and hence $A\subset F^*\subset F$. Suppose that F^{**} is a filter with $A\subset F^{**}\subset F^*$.
Then $F^{**}\in\bigtriangledown_A$ and $F^*\leq F^{**}$, and so $F^*=F^{**}$ by the maximality of F^*. This completes the proof.

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