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SEPARATION AXIOMS ON THE INITIAL CONVERGENCE SPACES

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1. Introduction

Kent([3]) introduced convergence structure on a nonempty set. Carstens, Kent, and Richardson ([1], [5], [8], and [10]) investigated some properties of convergence spaces and applied the concepts as product of convergence spaces. In discussing product of convergence spaces, they restricted to finite product. The purpose of this paper is that most of the results extend to arbitrary product using the initial convergence structure.

For a nonempty set S, $F(S)$ denotes the set of all filters and $P(S)$ the set of all subsets of S For each $s \in S$, $\dot{s}$ is the principal ultrafilter containing $\{s\}$

A convergence structure on S is defined to be a function q from $F(S)$ into $P(S)$, satisfying the following conditions:

(1) for each $s \in S$, $s \in q(\dot{s})$;

(2) if ψ and ϕ are in $F(S)$ and $\psi \supset \phi$, then $q(\psi) \supset q(\phi)$;

(3) if $s \in q(\psi)$, then $s \in q(\psi \cap \dot{s})$.

The pair (S,q) is called a convergence space. If $s \in q(\psi)$, then we say that ψ q-converges to s. The set function $\Gamma_q : P(S) \to P(S)$ is defined on all subsets $A \subset S$ by:

$$\Gamma_q(A) = \{s \in S \mid \textit{there is an ultrafilter } \psi \textit{ q-converging to } s \textit{ with } A \in \psi\}.$$

Let $\Gamma_q^0(A)=A$ If α is an ordinal number and $\alpha-1$ exists, then $\Gamma_q^\alpha(A)$ is defined by $\Gamma_q(\Gamma_q^{\alpha-1}(A))$, if α is a limit ordinal, $\Gamma_q^\alpha(A)$ is defined by the union of $\Gamma_q^\beta(A)$ for $\beta<\alpha$. then Γ_q is a closure operator in the topological sense, except idempotency.

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A function f, mapping a convergence (S,q) onto a convergence space (T,p), is said to be *continuous*, if $f(\psi)$ p-converges to $f(s)$ whenever ψ q-converges to s.

Let X be a nonempty set, (X_λ,q_λ) be a convergence space for each $\lambda\in\Lambda$, f_λ be a function X onto (X_λ,q_λ). The *initial convergence structure* q on X induced by the family $\{f_\lambda\,|\,\lambda\in\Lambda\}$ is defined to be a function from $F(X)$ into $P(X)$, satisfying the following condition:

for each element $x\in X$, $\psi\in F(X)$, $x\in q(\psi)$ if and only if $f_\lambda(\psi)$ q_λ-converges to $f_\lambda(x)$ for each $\lambda\in\Lambda$

2. Separation axioms

In this section, we shall investigate some separation axioms for initial convergence structure. Hereafter, (X,q) means initial convergence space induced by the family $\{f_\lambda\,|\,\lambda\in\Lambda\}$, where f_λ is a function X onto convergence space (X_λ,q_λ) for each $\lambda\in\Lambda$, and $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ means initial convergence space induced by the family $\{P_\lambda\,|\,\lambda\in\Lambda\}$,where P_λ is canonical projection of $\prod_{\lambda\in\Lambda}X_\lambda$ onto X_λ for each $\lambda\in\Lambda$, i.e., $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is product convergence space of$\{(X_\lambda,q_\lambda)\}$.

A convergence space (S,q) is said to be T_0 if either $\dot{s}$ fails to q-converging to t or $\dot{t}$ fails to q-converging to s for each distinct s and t in S.

Theorem 2.1. *Let (X_λ,q_λ) be a convergence space for each $\lambda\in\Lambda$. If there exists at least one $\mu\in\Lambda$ such that f_μ is injective and X_μ is T_0, then (X,q) is also T_0 convergence space.*

Proof. Let $\dot{x}$ q-converge to y for each distinct x,y in X. Then $f_\lambda(\dot{x})=\dot{f_\lambda}(x)$ q_λ-converges to $f_\lambda(y)$ for each $\lambda\in\Lambda$. Since (X_μ,q_μ) is T_0, $\dot{f_\mu}(y)$ fails to q_μ-converging to $f_\mu(x)$. Thus $\dot{y}$ fails to q-converging to x.

Corollary 2.2. *(X_λ,q_λ) is T_0 convergence space for each $\lambda\in\Lambda$ if and only if $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is T_0 convergence space.*

Proof. For each $\lambda\in\Lambda$, $P_\lambda(x)=P_\lambda(y)$ implies $x=y$.

Conversely, let $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ be T_0 convergence space. For each $\lambda_0\in\Lambda$, $\dot{x}_{\lambda_0}$ q_{λ_0}-converges to y_{λ_0} for each distinct $x_{\lambda_0},y_{\lambda_0}$ in X_{λ_0}. Let

x,y be points of $\prod_{\lambda\in\Lambda} X_\lambda$ with $P_{\lambda_0}(x)=x_{\lambda_0}$, $P_{\lambda_0}(y)=y_{\lambda_0}$ and $P_\lambda(x)=P_\lambda(y)$ for any $\lambda\neq\lambda_0$. Then $\dot{x}$ q'-converges to y Since $(\prod_{\lambda\in\Lambda} X_\lambda, q')$ is T_0 convergence space, $\dot{y}$ fails to q'-converging to x But $P_\lambda(\dot{y})=\dot{P_\lambda}(y)=\dot{y}_\lambda=\dot{x}_\lambda=\dot{P_\lambda}(x)$ q_λ-converges to $x_\lambda=P_\lambda(x)$ for each $\lambda\neq\lambda_0$ Therefore $P_{\lambda_0}(\dot{y})=\dot{y}_{\lambda_0}$ fails to q_{λ_0}-*converge* $P_{\lambda_0}(x)=x_{\lambda_0}$ Thus $(X_{\lambda_0}, q_{\lambda_0})$ is T_0 convergence space.

A convergence space (S,q) is said to be T_1 if $\dot{s}$ fails to q-converging to t for each distinct s and t in S.

Theorem 2.3. *Let (X_λ, q_λ) be a convergence space for each $\lambda\in\Lambda$ If there exists at least one $\mu\in\Lambda$ such that f_μ is injective and X_μ is T_1, then (X,q) is T_1.*

Proof. Let $\dot{x}$ q-converge to y. Then $f_\lambda(\dot{x})=\dot{f_\lambda}(x)$ q_λ-converges to $f_\lambda(y)$ for each $\lambda\in\Lambda$. Since (X_μ, q_μ) is T_1 convergence space and f_μ is injective, $f_\mu(x)=f_\mu(y)$ *and* $x=y$. Thus (X,q) is T_1

Corollary 2.4. *(X_λ, q_λ) is a T_1 convergence space for each $\lambda\in\Lambda$ if and only if $(\prod_{\lambda\in\Lambda} X_\lambda,\ q')$ is T_1 convergence space.*

A convergence space (S,q) is said to be *Hausdorff*, if each filter q-converges to at most one point in S.

Theorem 2.5. *Let (X_λ, q_λ) be a convergence space for each $\lambda\in\Lambda$ If there exists at least one $\mu\in\Lambda$ such that f_μ is injective and X_μ is Hausdorff, then (X,q) is Hausdorff*

Proof. Let ψ q-converge to x,y. Then for each $\lambda\in\Lambda$, $f_\lambda(\psi)$ q_λ-converge to $f_\lambda(x)$, $f_\lambda(y)$. Since (X_μ, q_μ) is Hausdorff and f_μ is injective, $x=y$. Thus (X,q) is also Hausdorff.

Corollary 2.6. *(X_λ, q_λ) is Hausdorff convergence space for each $\lambda\in\Lambda$ if and only if $(\prod_{\lambda\in\Lambda} X_\lambda, q')$ is also Hausdorff.*

Let q, p be two convergence structures on a set S. (S,q) is said to be *p-regular* if $\Gamma_p(\psi)$ q-converges to s whenever ψ q-converges to s, and p-regular Hausdorff convergence space is called p-T_3 *space*.

Lemma 2.7 *([6]). Let $f:(S,q)\to(T,p)$ be a continuous function, $A\subset S$. Then $f(\Gamma_q(A))\subset\Gamma_p(f(A))$.*

If ψ is any filter on S, $\Gamma_q(\psi)$ is the filter generated by $\{\Gamma_q(F)|F\in\psi\}$. (S,q) is said to be *regular* if $\Gamma_q(\psi)$ q-converges to s whenever ψ q-converges to s, and regular Hausdorff convergence space is called T_3.

Theorem 2.8. *Let (X_λ,q_λ) be a convergence space, p_λ be another convergence structure on X_λ for each $\lambda\in\Lambda$. If (X_λ,q_λ) is p_λ-regular convergence space for each $\lambda\in\Lambda$, then (X,q) is p-regular, where p is the initial convergence structure induced by the family $\{f_\lambda|f_\lambda:X\to(X_\lambda,p_\lambda),\lambda\in\Lambda\}$.*

Proof Let ψ q-converge to x, then $f_\lambda(\psi)$ q_λ-converges to $f_\lambda(x)$ for each $\lambda\in\Lambda$. Since (X_λ,q_λ) is p_λ-regrular, $\Gamma_{p\lambda}(f_\lambda(\psi))$ q_λ-converges to $f_\lambda(x)$ By Lemma 2.7, $f_\lambda(\Gamma_p(\psi))$ q_λ-converges to $f_\lambda(x)$. Thus $\Gamma_p(\psi)$ q-converges to x

Corollary 2.9. *If (X_λ,q_λ) is p_λ-regular convergence space for each $\lambda\in\Lambda$, then $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is p'-regular, where p' is product convergence structure of p_λ on $\prod_{\lambda\in\Lambda}X_\lambda$*

In p-regular convergence space (S,q), put $p=q$, then the following corollaries are easily verified.

Corollary 2.10. *If (X_λ,q_λ) is regular convergence space for each $\lambda\in\Lambda$, then (X,q) is also regular*

Corollary 2.11. *If (X_λ,q_λ) is regular convergence space for each $\lambda\in\Lambda$, then $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is regular.*

Corollary 2.12. *If (X_λ,q_λ) is T_3 convergence space for each $\lambda\in\Lambda$, then $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is T_3.*

A regular convergence space (S,q) is said to be *symmetric* if ψ q-converges to t whenever ψ q-converges to s and $\dot{s}$ q-converges to .

Theorem 2.13. *If (X_λ,q_λ) is a symmetric convergence space for each*

$\lambda \in \Lambda$, *then* (X,q) *is symmetric.*

Proof Let ψ q-converge to x and $\dot{x}$ q-converge to y. Then for each $\lambda \in \Lambda$, $f_\lambda(\psi)$ q_λ-converges to $f_\lambda(x)$ and $f_\lambda^{\cdot}(x)$ q_λ-converges to $f_\lambda(y)$. Since (X_λ, q_λ) is symmetric space, $f_\lambda(\psi)$ q_λ-converges to $f_\lambda(y)$. Thus ψ q-converges to y.

Corollary 2.14. *If* (X_λ, q_λ) *is symmetric convergence space for each* $\lambda \in \Lambda$, *then* $(\prod_{\lambda \in \Lambda} X_\lambda, q')$ *is symmetric.*

3. Regular modification and regular series

Richardson and Kent ([10]) defined the *regular modification* $(r(S), r(q))$ of a convergence space (S,q),

$(r(S),r(q))$ is the finest regular convergence on the same underlying set which is coarser than (S,q);

$(r(S),r(q))$ is the supremum of all regular convergence spaces coarser than (S,q)

A convergence space (S,q) is said to be *R-Hausdorff* if $(r(S),r(q))$ is Hausdorff.

Lemma 3.1. *Let* q_λ *and* p_λ *be convergence structures on* X_λ *such that* $q_\lambda \geqq p_\lambda$ *for each* $\lambda \in \Lambda$, q *be the initial convergence structure on* X *induced by the family* $\{f_\lambda \mid f_\lambda : X \to (X_\lambda, q_\lambda), \lambda \in \Lambda\}$ *and* p *be the initial convergence structure on* X *induced by the family* $\{f_\lambda \mid f_\lambda : X \to (X_\lambda, p_\lambda), \lambda \in \Lambda\}$. *Then* $(X,q) \geqq (X,p)$.

Theorem 3.2. *Let* q^* *be the initial convergence structure on* X *induced by the family* $\{f_\lambda \mid f_\lambda : X \to (r(X_\lambda), r(q_\lambda)), \lambda \in \Lambda\}$. *Then* $(r(X), r(q)) \geqq (X, q^*)$

Proof. Since, for each $\lambda \in \Lambda$, $(r(X_\lambda), r(q_\lambda))$ is the finest regular convergence space on the same underlying set coarser than (X_λ, q_λ), by Corollary 2.10, (X,q^*) is a regular convergence space. Since $q_\lambda \geqq r(q_\lambda)$ for each $\lambda \in \Lambda$, by Lemma 3.1, $q \geqq q^*$ But $r(q)$ is the finest regular convergence structure on X which is coarser than q, thus $r(q) \geqq q^*$.

Let q^* be convergence structure defined in Theorem 3.2. Then by Theorem 2.5, 3.2, and Corollary 2.6, the following corollaries are easilly verified.

Corollary 3.3. $(r(\prod_{\lambda\in\Lambda} X_\lambda),\ r(q'))\geqq \prod_{\lambda\in\Lambda}(r(X_\lambda),\ r(q_\lambda))$.

Corollary 3.4. *Let (X_λ, q_λ) be R-Hausdorff convergence space for each $\lambda\in\Lambda$ Then*

a) If there exists at least one $\mu\in\Lambda$ such that f_μ is injective, then (X, q^*) is Hausdorff.

b) (X,q) is R-Hausdorff

Corollary 3.5. *Let $X=\prod_{\lambda\in\Lambda} X_\lambda$. Then the following statements are equivalent.*

a) $(X_\lambda,\ q_\lambda)$ is R-Hausdorff for each $\lambda\in\Lambda$.

b) (X,q^*) is Hausdorff

Furthermore, if (X_λ,q_λ) is R-Hausdorff for each $\lambda\in\Lambda$, then $(r(X),r(q'))$ is Hausdorff.

For any convergence space (S,q), let $(\rho(S), \rho(q))$ be the convergence space defined on the same underlying set as follows:

ψ $\rho(q)$-converges to s if and only if each ultrafilter ϕ finer than ψ q-converges to s.

Then by definition of p-regular and [10], the following theorems hold.

Theorem 3.6. *Let q and p be two convergence structures on S with the property;*

if ultrafilter ψ q-converges to s, then $\Gamma_p(\psi)$ q-converges to s. Then

a) $(\rho(S),\rho(q))$ is $\rho(p)$-regular.

b) If (S,q) is p-regular, then $(\rho(S),\rho(q))$ is $\rho(p)$-regular.

Theorem 3.7. *If (S_λ,q_λ) is p_λ-regular on the same underlying set S, $q^*=sup\{q_\lambda \mid \lambda\in\Lambda\}$, and $p^*=sup\{p_\lambda\mid\lambda\in\Lambda\}$ then (S, q^*) is p^*-regular.*

Let (S,q) be a convergence space. The set of all natural numbers (including 0) will be defined by N. An ordinal family $\{(r_\alpha(S),r_\alpha(q))\}$ is defined recursively on the same underlying set S.

$r_0(S)=S;$

ψ $r_1(q)$-converges to s if and only if there exist $n\in N$ and filter ϕ q-converging to s such that $\psi\supset\Gamma_q^n(\phi)$;

ψ $r_2(q)$-converges to s if and only if there exist $n \in N$ and filter ϕ q-converging to s such that $\psi \supset \Gamma^n_{r_1(q)}(\phi)$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,

ψ $r_\alpha(q)$-converges to s if and only if there exist $n \in N$, filter ϕ q-converging to s, and $\beta < \alpha$ such that $\psi \supset \Gamma^n_{r_\beta(q)}(\phi)$

The family $\{(r_\alpha(S), r_\alpha(q))\}$ will be called the *R-series* (or *regularity series*) of (S, q). Richardson and Kent ([10]) introduced definition of R-series.

Theorem 3.8. *Let* (X, q^*_α) *be the initial convergence space induced by the family* $\{f_\lambda | f_\lambda : X \to (r_\alpha(X_\lambda), r_\alpha(q_\lambda)), \lambda \in \Lambda\}$ *for any ordinal number* α. *Then* $(r_\alpha(X), r_\alpha(q)) \geqq (X, q^*_\alpha)$.

Proof. Transfinite induction.

Let ψ $r_1(q)$-converge to x. Then there exist $n \in N$ and filter ϕ q-converging to x such that $\psi \supset \Gamma^n_q(\phi)$ For each $\lambda \in \Lambda$, $f_\lambda(\phi)$ q_λ-converges to $f_\lambda(x)$ and

$$f_\lambda(\psi) \supset f_\lambda(\Gamma^n_q(\phi)) \supset \Gamma^n_q(f_\lambda(\phi))$$

Therefore, $f_\lambda(\psi)$ $r_1(q_\lambda)$-converges to $f_\lambda(x)$ for each $\lambda \in \Lambda$

Thus ψ q^*_α-converges to x

Assume that $(r_\beta(X), r_\beta(q)) \geqq (X, q^*_\beta)$ for all $\beta < \alpha$

Let ψ $r_\alpha(q)$- converge to x Then there exist $n \in N$, filter ϕ q-converging to x, and $\gamma < \alpha$ such that $\psi \supset \Gamma^n_{r_\gamma(q)}(\phi)$ Therefore, for each $\lambda \in \Lambda$, $f_\lambda(\phi)$ q_λ-converges to $f_\lambda(x)$ and

$$f_\lambda(\psi) \supset f_\lambda(\Gamma^n_{r_\gamma(q)}(\phi)) \supset \Gamma^n_{r_\gamma(q)}(f_\lambda(\phi))$$

Thus $f_\lambda(\psi)$ $r_\alpha(q_\lambda)$-converges to $f_\lambda(x)$ for each $\lambda \in \Lambda$

Corollary 3.9. *Let* $(\prod_{\lambda \in \Lambda} X_\lambda, q')$ *be the product convergence space of family* $\{(X_\lambda, q_\lambda) | \lambda \in \Lambda\}$ *of convergence spaces. Then, for any ordinal number* α,

$$(r_\alpha(\prod_{\lambda \in \Lambda} X_\lambda), r_\alpha(q')) \geqq \prod_{\lambda \in \Lambda}(r_\alpha(X_\lambda), r_\alpha(q_\lambda))$$

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