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ONNEGATIVE MINIMUM BIASED ESTIMATION IN VARIANCE COMPONENT MODELS*

Jong Hoo Lee

ABSTRACT

In a general variance component model, nonnegative quadratic estimators of the components of variance are considered which are invariant with respect to mean value translaion and have minimum bias (analogously to estimation theory of mean value parameters). Here the minimum is taken over an appropriate cone of positive semidefinite matrices, after having made a reduction by invariance.

Among these estimators, which always exist the one of minimum norm is characterized. This characterization is achieved by systems of necessary and sufficient condition, and by a cone restricted pseudoinverse. In models where the decomposing covariance matrices span a commutative quadratic subspace, a representation of the considered estimator is derived that requires merely to solve an ordinary convex quadratic optimization problem. As an example, we present the two way nested classification random model.

An unbiased estimator is derived for the mean squared error of any unbiased or biased estimator that is expressible as a linear combination of independent sums of squares. Further, it is shown that, for the classical balanced variance component models, this estimator is the best invariant unbiased estimator, for the variance of the ANOVA estimator and for the mean squared error of the nonnegative minimum biased estimator. As an example, the balanced two way nested classification model with ramdom effects is considered.

1. Introduction

Many Statistician studied estimating nonnegative variance com-

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ponents, by unbiased quadratic estimators.

The proposed nonnegative estimators either lack some desirable optimality properties or they exist only for special models, respectively are applicable only under particular assumption.

The MINQUE (minimum norm quadratic unbiased estimator), introduced by C.R.Rao(1970, 1972, 1973), page 303–305, is usually defined on the whole space of appropriate symmetric matrices. In order to get nonnegative estimators C.R.Rao (1972). Section 7, suggested resticting the class of possible estimators to the corresponding cone of positive semidefinite matrices, mentioning the resulting problem of finding a "nonnegative MINQUE", if it exists, as likely a difficult one. This problem is further considered, e.g., by LaMotte (1973) and Pukelsheim(1977, 1979, 1981).

However, such estimators exist only in very special cases. For example, in the analysis of variance (ANOVA) models, besides the overall variance σ_e^2, none of the other variane components permit the existence of a positive semidefinite matrix that is an unbiased estimator, as pointed out by LaMotte(1973 b).

In this paper we consider minimum bias estimators (as introduced by Chipman (1964) for estimating mean value parameters), which are invariant under the group of mean value translations. Here the minimum is taken over the appropriate cone of positive semidefinite matrices after having made a reduction by invariance. These estimators always exist, and of course thery guarantee nonnegative estimates. Moreover, they are unbiased if nonnegative unbiased quadratic estimation is possible. To get a unique estimator, we choose the on with minimal norm.

We characterize the minimum norm minimum bias invariant positive semidefinite estimator by introducing a cone–restricted pseudoinverses, and we obtain a useful necessary and sufficient condition for a matrix to be the desired estimator are derived.

In models where the decomposing covariance matrices span a commutative quadratic subspace the computation of this estimator is reduced to an ordinary convex quadratic optimization problem. As an example, the balanced two–way nested classification model with random effects is considered where the estimators for the three variance components are stated explicitly.

2. Formulation of the problem and definition

HMS the Hilbert space of all real symmetric $n \times n$ matrices, where the inner product is given by $(A, B)_{HMS} = trAB$, defining the norm $\|A\|^2_{HMS} = trA^2$, for $A, B \in HMS$. Further let PSD denote the closed convex cone of positive semidefinite matrices in HMS,

$$PSD = \{A \mid A \in HMS,\ x'Ax \geqq 0 \ for\ all\ x \in R^n\}.$$

We consider the linear variance component model

$$z \sim (X\beta,\ \sum_{i=1}^{m} \alpha_i U_i), \tag{2.1}$$

that consists fo an n-dimensional random variable z with mean value

$$Ez = X\beta$$

and variance-covariance matrix

$$cov(z) = \sum_{i=1}^{m} \alpha_i U_i$$

where the $n \times k$ design matrix X and the m symmetric positive semidefinite $n \times n$ matrices U_i are known, $U_i \in PSD,\ i=1,\cdots\cdots,\ m$, while the parameter β varies in R^k and the parameter $\alpha = (\alpha_1, \cdots, \alpha_m)$ varies in R^m_+, the nonnegative orthant of R^m.

The problem considered here is to find quadratic estimates for the variance components $\alpha_1, \cdots, \alpha_m$, that are nonegative and invariant with respect to the group G of mean value translations,

$$G = \{z \to z + X\beta,\ \beta \in R^k\}.$$

A maximal invariant linear statistic y with respect to G is given then by (Seely 1971)

$$y = Proj_{R(X)^\perp} z = (I - XX^+)z,$$

where I is the $n \times n$ *identity*, X^+ is the Moore-Penrose generralized inverse, and $R(X)$ denotes the range of X.

We get the reduced linear variance component model

$$y \sim (0,\ \sum_{i=1}^{m} \alpha_i V_i), \quad \alpha_i \in R^m_+,\ V_i \in PSD. \tag{2.2}$$

where $Vi=(I-XX^+)U_i(I-XX^+)$, $i=1,\cdots,m$. Often a model of the kind (2.2) is given also by the experimental arrangement, for instance by grouping and measuring of differences.

Now let $A \in HMS$, then a quadratic invariant estimate for a linear form $p'\alpha$, $p \in R^m$, is given by $y'Ay$, with the bias

$$Ey'Ay-p'\alpha=\sum_{i=1}^{m}\alpha_i(trAV_i-p_i).$$

Here $y'\tilde{A}y$ is an unbiased estimate of $p'a$ if

$$tr\tilde{A}V_i=P_i, \text{ for all } i=1,\cdots,m. \tag{2.3}$$

It is additionally of "minimum norm" if $\tilde{A}$ solves the problem

$$minimize\{trB^2|B\in HMS,\ trBV_i=p_i,\ i=1,\cdots,m\}. \tag{2.4}$$

Then $\tilde{A}$ is called the MINQUE(minimum norm quadratic unbiased estimator).

However, the MIQUE doesn't always exist.

Then the condition of unbiasedness (2.3) may be weakened to that of finding a best approxmate solution of (2.3), i.e., minimizing the discrepancy $\sum_i(trAV_i-p_i)^2$ over HMS.

Definition 2.1 For estimating the linear form $p'a$ the matrix $\tilde{A} \in$HMS is the MINQUE that gives minimum bias with respect to HMS, if $\tilde{A}$ solves the following problem.

$$\begin{aligned}&\text{minimize } trA^2 \textit{ subject to}\,; A\in HMS, \text{ and}\\ &\min_{A\in HMS}\sum_1^m(trAV_i-p_i)^2\end{aligned} \tag{2.5}$$

Let us introduce the linear operator

$$g:\ HMS\to R^m,\ A\to gA=\begin{pmatrix}trAV_1\\ \vdots\\ trAV_m\end{pmatrix} \tag{2.6}$$

Then (2.5) is equivalent to

$$\text{minimize }\{trA^2|A\in HMS,\ min_{A\in HMS}\|gA-p\|_{R^m}\} \tag{2.7}$$

respectively, find a best approxximate solution $\tilde{A}$, of minimum norm of the linear equation $gA=p, A\in HMS$. By definition of a pseudoinverse operator, e.g., Holmes (1972), page 220,

$$\tilde{A}=g^{+}p, \qquad (2.8)$$

g^{+} the pseudoinverse of g, which for matrices is identical to the Moore –Penrose generalized inverse, cf., Mitra(1975). The estimator $\tilde{A}$ always exists and $\tilde{A}$ is equal to the MINQUE if $p\in R(g)$.

We now show how to compute g^{+} The adjoint g^{*} of g is given by

$$g^{*}: R^{m}\rightarrow HMS,\ a\rightarrow g^{*}a=\sum_{i=1}^{m} a_i V_i,\ a=(a_1,\cdots,a_m). \qquad (2.9)$$

Then

$$gg^{*}=(trV_iV_j)_{i,j}=1,\cdots,\ m \qquad (2.10)$$

and

$$g^{*}g:\ HMS\rightarrow HMS,\ A\rightarrow\sum_{i=1}^{m}(trAV_i)V_i \qquad (2.11)$$

Using now the following properties of pseudoinverses in Hilbert spaces (e.g. Holmes(1972), page 222),

$$g^{+}=g^{*}(gg^{*})^{+}, \qquad (2.12)$$

$$g^{+}=(g^{*}g)^{+}g^{*}, \qquad (2.13)$$

we get a computational representation of g^{+}, resp. of $\tilde{A}$ or of the MINQUE $\tilde{A}$ if $p\in R(g)$.

Lemma 2.1. The estimator $\tilde{A}$ satisfies the "normal equation"

$$\sum_{i=1}^{m}(tr\tilde{A}V_i)V_i=\sum_{i=1}^{m}[p]_iV_i, \qquad (2.14)$$

and permits the computational representation

$$\tilde{A}=g^{+}p=\sum_{i=1}^{m}[(gg^{*})^{+}p]_iV_i. \qquad (2.15)$$

where $[w]_i$ denotes the ith component of vector w.

Now, an estimate $y'Ay$ with $A\in HMS$ can be negative, while estimating nonnegative variance components. Therefore, C.R.Rao(1972), Section 7, guggested finding a MINQUE over the cone PSD. i.e., find a solution $\tilde{A}^0$ of the problem

$$minimize\{trA^2 \mid A \in PSD,\ gA=p\}, \tag{2.16}$$

a problem futher considered for instance by LaMotte(1973 b) and pukelsheim (1981). However, the kind of models where such an estimator $\tilde{A}^0$ exists for all variance components is very limited. For instance, as pointed out by LaMotte(1973b), in ANOVA models the only component that might be estimable in this way is the overall variance σ_e^2 ; cf., also the 2-way nested layout considered in Section 4.

Thus we are led to exchange the unbiasedness condition in (2.16), $gA=p$ by the claim to minimize the discrepancy $\|gA-p\|_{R^m}$ over PSD.

Definition 2.2 $\hat{A} \in PSD$ is the nonnegative MINQ minimum bias estimator of the linear form $p'\alpha$ if $\hat{A}$ solves the following problem,

$$\begin{aligned} &minimize\ trA^2\ subject\ to : A \in PSD, \text{ and} \\ &\|gA-p\| = min_{B \in PSD}\|gB-p\|. \end{aligned} \tag{2.17}$$

Lemma 2.1. $\hat{A}$ always exists and is uniquely determind.

Proof. The cone PSD is closed and convx, g is a continuous linear mapping with closed range, $R(g|_{PSD})$ is closed and convex, and so there exists a best $R(g|_{PSD})$-approximation to p, say p_a, that is unique. Now $PSD \cap \{A|gA=p_a$ is nonempty, closed and convex, so has a unique element of minimum norm and this is just $\hat{A}$.

Definition 2.3. The operator $g|^+_{PSD} : R^m \to PSD$ is the *PSD*-restricted pseudoinverse of g if for every $q \in R^m$ the best apporximatoe solution $A(q)$ of minimum norm of the linear equation

$$gA=q \ subject\ to\ A \in PSD$$

is given by $A(q)=g|^+_{PSD}q$.

So the solution $\hat{A}$ of (2.17) is given by $\hat{A}=g|_{PSD^+}p$, and by Lemma 2.1 $g|_{PSD^+}$ exists and is a single valued function.

If nonnegative unbiased estimability is given then $\tilde{A}$ automatically becomes the "nonnegative MINQUE" $\tilde{A}^0$.

3. Main results

First we consider the class of nonnegative minimum bias estimators. For the following convex program, describing the nonnegative minimum bias estimators for a linear form $q'\alpha$,

$$minimize\{\|gA-q\|^2 \mid A\in PSD\} \quad with \quad q\in R^m \tag{3.1}$$

By Lemma 2.1 there always exists a solution of (3.1).

Lemma 3.1. Let f be a Frechet(Gateaux) differentiable convex functional on a real normed space X. Let P be a convex cone in X. A necessary and sufficient condition that $x_0\in P$ minimize f over P is that

$$\delta f(x_0;\, x)\geqq 0 \ for\ all\ x\in P \tag{3.2}$$

$$\delta f(x_0 \cdot x_0)=0, \tag{3.3}$$

Proof Necessary ; If x_0 minimizes f, then for any $x\in P$ we must have

$$\frac{d}{d\alpha} f(x_0+\alpha(x-x_0))|_{\alpha=0}\geqq 0$$

Hence

$$\delta f(x_0\,;\, x-x_0)\geqq 0 \tag{3.4}$$

Setting $x=x_0/2$ yields

$$\delta f(x_0;x_0)\leqq 0 \tag{3.5}$$

while setting $x=2x_0$ yields

$$\delta f(x_0;x_0)\geqq 0 \tag{3.6}$$

Together, equations (3.4), (3.5) and (3.6) imply (3.2) and (3.3).
Sufficiency ; For x_0, $x\in P$ and $0<\alpha<1$ we have

$$f(x_0+\alpha(x-x_0))<f(x_0)+\alpha[f(x)-f(x_0)]$$

or

$$f(x)-f(x_0) \geqq \frac{1}{\alpha}[f(x_0+\alpha(x-x_0))-f(x_0)]$$

As $\alpha \to 0$, the right side of this equsation tends toward $\delta f(x_0; x\text{-}x_0)$; hence we have

$$f(z)-f(x_0) \geqq \delta f(x_0; x\text{-}x_0). \tag{3.7}$$

If (3.2) and (3.3) hold, then $\delta f(x_0; x\text{-}x_0) \geqq 0$ for *all* $x \in P$ and, hence, from (3.7)

$$f(x)-f(x_0) \geqq 0 \text{ for all } x \in P.$$

Definition 3.1. Since HMS with the inner product $trAB$ for A, $B \in HMS$ is a Hilbert space, the "positive" dual cone of the "positive" cone PSD is given by

$$PSD^* = \{B \in HMS \mid trAB \geqq 0 \text{ for all } A \in PSD\}.$$

cf., Luenberger (1969), page 215, Berman (1973), page 5. Now PSD is self−dual, and its interior consists of PD, the set of positive definite matrices in HMS, cf., Berman(1973), page 55, i.e.,

$$\text{PSD}^* = \text{PSD}, \text{ intPSD} = \text{PD}$$

and

PD is nonempty.

Theorem 3.1. The matrix $A_0 \in PSD$ is solution of (3.1) if and only if

$$g^*gA_0 - g^*q \in PSD, \tag{3.8}$$

and

$$trA_0\ (g^*gA_0 - g^*q) = 0. \tag{3.9}$$

Proof. we consider the minimization solution of convex functional $f(A) = \|gA-q\|^2$. By Lemma 3.1 necessary and sufficient conditions that $A_0 \in PSD$, minimize f on PSD is that

$$\delta f(A_0; A-A_0) = 2tr(A-A_0)g^*(gA_0-q), \tag{3.10}$$

Therefore

$$\delta f(A_0; A)=2trAg^*(gA_0-q)\geqq 0 \ \textit{for all}\ A\in PSD \tag{3.11}$$

then by the self-duality of PSD,

$$g^*(gA_0-q)\in PSD.$$

$$\delta f(A_0; A_0)=2trA_0g^*(gA_0-q)=0. \tag{3.12}$$

Thus is proved

Now we give necessary and sufficient conditions for a matrix A to be an estimator Â, *i.e.*, to be the minimum norm solution of the program (3.1) for a $q\in \boldsymbol{R}^m$, denoted by $g|^+_{PSD}q$, cf., Definition 2.2 and 2.3.

Theorem 3.2 Let $q\in \boldsymbol{R}^m$, then $\widetilde{A}=g|^+_{PSD}q$, is the element of minimum norm in solution of (3.1), if the following condition hold for some $b^0\in \boldsymbol{R}^m$.

$$\widetilde{A}=g^*b^0\in PSD \tag{3.13}$$

$$tr\widetilde{A}(\widetilde{A}+g^*b^0)=0. \tag{3.14}$$

Proof. By definition 2.3 $\widetilde{A}$ is a solutin of (3.1), *i.e.*, $\widetilde{A}$ minimizes $\|gA-q\|$ over PSD. Denote by $S(q)$ the set of all solutions of (3.1) and let $q_a=g\widetilde{A}$, then q_a is the unique best $R(g|_{PSD})$ approximation to q and

$$S(q)=\{A\,|\,A\in PSD,\ gA=q_a\},\ \widetilde{A}\in S(q). \tag{3.15}$$

we have to show that $\widetilde{A}$ is the element of minimum norm in $S(q)$. For $A\in HMS$, $b^0\in \boldsymbol{R}^m$ we define

$$\mathrm{M(A)}=\frac{1}{2}\,trA^2+(gA-q_a)^{-1}b^0. \tag{3.16}$$

Differentiating $M(A)$ with respect to A gives $grad_A M(A)=A+g^*b^0$. By lemma 3.1 we get minimum $M(A)$. i.e., Gateaux differential is

$$\delta M(\widetilde{A};A-\widetilde{A})=tr(A-\widetilde{A})\ (\widetilde{A}+g^*b^0)$$

so that satisfies following

$$\delta M(\tilde{A};A)=trA(\tilde{A}+g^{*}b^{0})\geqq 0 \text{ for all } p\in PSD.$$

$$\delta M(\tilde{A};\tilde{A})=tr\tilde{A}(\tilde{A}+g^{*}b^{0})\geqq 0$$

then $M(A)$ is minimal at $\tilde{A}$. Thus we get for all $A_1\in S(q)$.

$$\frac{1}{2}tr\tilde{A}^2=\frac{1}{2}tr\tilde{A}+(g\tilde{A}-q_a)'b^0=M(\tilde{A})=min_{A\in HMS}M(A)$$

$$=min_{A\in PSD}M(A)\leqq min_{A\in S(q)}M(A)\leqq\frac{1}{2}trA^2{}_1\cdot$$

Theorem 3.3 Let $\hat{A}=g|_{PSD^+}q,\ \ q\in R^m$, then there hold (3.8), (3.9), (3.13) and (3.14).

Proof. Let $\hat{A}=g|_{PSD^+}q$, then $\hat{A}$ is solution of (3.1), thus Theorem 3.1 gives conditions (3.8), (3.9). Let $g\hat{A}=q_a$ then q_a is unique. $N(q)$ may be define as (3.15).

$$N(q)=\{A\,|\,A\in PSD,\ gA=q_a\},\ \hat{A}\in N(q). \tag{3.17}$$

We consider the convex functional (3.16).

$$M(A)=\frac{1}{2}trA^2+(gA-q_a)^1b$$

since $tr\hat{A}^2\leqq trA^2$ for all $A\in N(q)$,

$$\frac{1}{2}tr\hat{A}^2=\frac{1}{2}tr\hat{A}^2+(g\hat{A}-q_a)^1b=M(\hat{A})=min_{A\in N(q)}M(A)$$

such that by Lemma 3.1 minimum condition of M(A) over N(q) is

$$\delta M(\hat{A};A)=trA(\hat{A}+g^{*}b^{0})\geqq 0 \text{ for all } A\in PSD \tag{3.18}$$

$$\delta M(\hat{A},\hat{A})=tr\hat{A}(\hat{A}+g^{*}b^{0})=0. \tag{3.19}$$

Since *PSD* is self−duality, thus we get by (3.18), $\hat{A}+g^{*}b^{0}\in PSD$ some $b^0\in R^m$ therefore $\hat{A}$ satisfies two conditions (3.13) and (3.14)

4. Application

In many situations discussed, for example, by Gump (1951) and Searie (1971, p416) an estimator for a linear form $p'\sigma$, where p is a given $m\times 1$ vector and $\sigma=(\sigma_1,\cdots,\sigma_m)'$ is a vector of unknown variance components, is given by linear combination of independent mean squares $M_1,\cdots,M_m$; that is, the estimator is expressble in the form

$$\bar{p}'\bar{\sigma}=\sum_{i=1}^{m} q_i M_i = q' M \tag{4.1}$$

where $q=(q_1,\cdots,q_m)'$ is a given vector and $M=(M_1,\cdots,M_m)'$.
The expected mean values are $\tau_i=E(M_i)>0$ $(i=1,\cdots,m)$
Futher, $\tau=(\tau_1,\cdots,\tau_m)' = L'\sigma$, where L is a nonsingular $m\times m$ matrix. We assume that for some positive integer f_i, f_iM_i/τ_i has a central x^2 distribution with f_i df $(i=1,\cdots,m)$ so that $cor(M)=2diag(\tau_1^2/f_1,\cdots,\tau_m^2/f_m)$. Note that $f_1M_1,\cdots,f_mM_m$ are the sume of squares.
For example, let us consider the balanced two-way nested

$$z_{ijk}=\mu+a_i+b_{ij}+e_{ijk}$$
$$i=1,\cdots,r>1,\ j=1,\cdots,s>1,\ k=1,\cdots,t>1,\ n=rst \tag{4.2}$$

where $a_1,\cdots,a_r,b_{11},\cdots,b_{rs}, e_{111},\cdots,e_{rst}$ are independent(1-dimensional random variables with $Ea_i=Eb_{ij}=Ee_{ijk}=0$ and $Ea_i^2=\sigma_a^2$, $Eb_{ij}^2=\sigma_b^2$, $Ee_{ijk}^2=\sigma_e^2$, and $\mu\in R$ is the mean value parameter. Denote $1_k=(1,\cdots,1)'\in R^k$, $J_k=1_k' 1_k$, $\bar{J}_k=(1/k)J_k$, I_k the $k\times k$ identity matrix, $K_k=I_k-\bar{J}_k$, $k\in N$, and $\otimes$ the Kronecker product of two matrices. $Proj_{R(1_n)^\perp}=I_n-(1/n)1_n1_n'=K_n$, cf., Graybill (1976) page 634,

$$z_{ijk}-\bar{z}_{\cdots}=(\bar{z}_{i\cdot\cdot}-\bar{z}_{\cdots})+(\bar{z}_{ij\cdot}-\bar{z}_{i\cdot\cdot})+(1_{ijk}-\bar{z}_{ij\cdot}),$$

respectively the orthogonal decomposition

$$K_nz=p_1z+p_2z+p_3z \tag{4.3}$$

where

$$p_1=K_r\otimes\bar{J}_s\otimes\bar{J}_t,\ P_2=I_r\otimes K_s\otimes\bar{J}_t,\ p_3=I_r\otimes K_s\otimes K_t$$

are pairwise orthogonal projection matrices; note that $I_k, \bar{J}_k, K_k$ are projectors and $\bar{J}_kK=K\bar{J}_k=0$. Further $U_1=I_r\otimes J_s\otimes J_t$, $U_2=I_r\otimes I_s\otimes J_t$, $U_3=I_r\otimes I_s\otimes I_t$.

Using the above decomposition of K_n we easily get

$$\begin{cases} V_1=K_nU_1K_n=K_r\otimes J_s\otimes J_t=stP_1 \\ V_2=K_nU_2K_n=K_r\otimes \bar{J}_s\otimes J_t+I_r\otimes K_n\otimes J_t=tP_1+tP_2. \\ V_3=K_nU_3K_n=K_n=P_1\otimes P_2\otimes P_3 \end{cases} \quad (4.4)$$

so that

$$\Phi=\begin{pmatrix} st & 0 & 0 \\ t & t & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \Phi^1\Phi=\begin{pmatrix} s^2t^2+t^2+1 & t^2+1 & 1 \\ & t^2+1 & t^2+1 & 1 \\ & 1 & 1 & 1 \end{pmatrix}$$

$$\Phi'p=\begin{pmatrix} p_1st+p_2t+p_3 \\ p_2t+p_3 \\ p_3 \end{pmatrix}, \qquad \Phi^{-1}=\frac{1}{st^2}\begin{pmatrix} t & 0 & 0 \\ -t & st & 0 \\ 0 & -st & st^2 \end{pmatrix} \quad (4.5)$$

Letting MS_A, MS_B and MS_E represent the factor A (among group), B within A, and mean squared error (within-group), the estrimates are given by

$$\begin{cases} E(MS_A)=E(\|P_1z\|^2/(r-1))=st\sigma_a^2+t\sigma_b^2+\sigma_e^2 \\ E(MS_B)=E\|P_2z\|^2/r(s-1))=t\sigma_b^2+\sigma_e^2 \\ E(MS_B)=E\|P_2z\|^2/rs(t-1))=t\sigma_e^2 \end{cases} \quad (4.6)$$

The MINQUE's estimators are given by

$$\begin{cases} \hat{\sigma}_a^2=(MS_A-MS_B)/st \\ \hat{\sigma}_b^2=(MS_B-MS_E)/t \\ \hat{\sigma}_e^2=MS_E \end{cases} \quad (4.7)$$

We introduce the following Lemma.

Lemma 4.1 Let $M=(M_1,\cdots,M_m)'$ be an m-dimensional random vector with $E(M)=\tau=(\tau_1,\cdots,\tau_m)'$ and $\hat{C}$ be an unbiased estimator for cov(M). Then

1. An unbiased estimaotr for $\tau\tau'$ is given by $MM'-\hat{C}$.
2. When the linear combination q^1M is regarded as an estimator of $b'\tau$, then its mean squared error $MSE(q^1M)=q'\,M\,cov(M)q+(q'\tau-b'\tau)^2$ is unbiasedly estimated by $M\hat{S}E(q'\,M)=q'\,\hat{C}q+(q-b)'\,(MM'-\hat{C})(q-b)$.
3. If $\mathrm{cov}(M)=diag(\gamma_1M_1^2/(1+\gamma_1),\cdots,\gamma_mM_m^2/(1+\gamma_m))$ is an unbiased estimator of cov(M) (See. Hartung, 1986).

Observing that $\mathrm{var}(\bar{p}'\bar{\sigma})=q'\mathrm{cov}(M)q$, we find that the mean square error(MSE) of estimator (4.1) is

$$MSE(\bar{p}'\bar{\sigma})=q'\,cov(M)q+(q'\tau-p'\sigma)^2. \tag{4.8}$$

As shown in the Lemma 4.1, an unbiased estimator of $MSE(\bar{p}'\bar{\sigma})$ is

$$M\hat{S}E(\bar{p}'\bar{\sigma})=q'\hat{C}q+(q-L^{-1}p)'\,(MM'-\hat{C})\,(q-L^{-1}p). \tag{4.9}$$

Using the Theorem 5.1 of Hartung (1981), we get the following two case results.

⟨*Case 1*⟩ $\hat{d}_a=\Phi^{-1}p_a=q_a$.

(1). If $p_a=(1,0,0)'$, then the following facts are obtained.

$$\hat{d}_a=q_a=(1/st,\ -1/st,\ 0)'$$

$$M=\begin{pmatrix} MS_A \\ MS_B \\ MS_E \end{pmatrix},\qquad E(M)=\tau=\begin{pmatrix} st\sigma_a^2+t\sigma_b^2+\sigma_e^2 \\ t\sigma_b^2+\sigma_e^2 \\ \sigma_e^2 \end{pmatrix} \tag{4.10}$$

$f_1=r-1,\ f_2=r(s-1),\ f_3=rs(t-1),\ q_a'\tau-p_a'\sigma=0$

$$cov(M)=2diag[(st\sigma_a^2+t\sigma_b^2+\sigma_e^2)^2/(r-1)],\quad (t\sigma_b^2+\sigma_e^2)^2/r(s-1),\ \ \sigma_e^4/rs(t-1)] \tag{4.11}$$

$$q_a^1M=(MS_A-MS_B)/st=\hat{\sigma}_a^2 \tag{4.12}$$

so that

$$var(\hat{\sigma}_a^2)=2[(st\sigma_a^2+t\sigma_b^2+\sigma_e^2)^2/(r-1)+(t\sigma_b^2+\sigma_e^2)^2/r(s-1)]/s^2t^2 \tag{4.13}$$

and

$$\hat{C}=2diag[MS_A^2/(r+1),\ MS_B^2/(r(s-1)+2),\ MS_E^2/(rs(t-1)+2)] \tag{4.14}$$

is an unbiased estimator of cov(M). In the special case in wiich $p'\sigma$ is an unbiased estimator formula (4.9) simplifies to $M\hat{S}E(\bar{p}\ \bar{\sigma})=v\hat{a}r(\bar{p}\ \bar{\sigma})=q\ \hat{C}q$, therefore

$$v\hat{a}r(\hat{\sigma}_a^2)=2[MS_A^2/(r+1)+MS_B^2/(r(s-1)+2)]/s^2t^2. \tag{4.15}$$

(2). If $P_b=(0,1,0)'$, then

$$\hat{d}_b=q_b=\Phi^{-1}p_b=(0,1/t,\ -1/t)',$$

$$MSE(\overline{p'\sigma})=q_b'cov(M)q_b$$

$$var(\hat{\sigma}_b^2)=2[(t\sigma_b^2+\sigma_e^2)^2/r(s-1)+\sigma_e^4/rs(t-1)]/t^2 \tag{4.16}$$

$$v\hat{a}r(\hat{\sigma}_b^2)=2[MS_B^2/(r(s-1)+2)+MS_E^2/(rs(t-1)+2)]/t^2 \tag{4.17}$$

(3). If $p_e=(0,0,1)'$, then

$$\hat{d}_e=q_e=(0,0,1)'$$

$$var(\hat{\sigma}_e^2)=2\sigma_e^4/rs(t-1). \tag{4.18}$$

$$v\hat{a}r(\hat{\sigma}_e^2)=2MS_E^2/(rs(t-1)+2), \tag{4.19}$$

⟨*Case 2*⟩ $\tilde{d}\Phi^1(\Phi\tilde{d}-p)=0$

(1). *If* $p_a=(1,0,0)'$, then

$$\tilde{d}_a=q_a=(st/(s^2t^2+t^2+1),0,0)'$$

$$L^{-1}p_a(1/st,\ -1/st,\ o)^1 \qquad \text{where } L=\Phi$$

$$\tilde{\sigma}_a^2=stMS_A/(s^2t^2+t^2+1).$$

$$MSE(\tilde{\sigma}_a^2)=\frac{s^2t^2(st\sigma_a^2+t\sigma_b^2+\sigma_e^2)}{(r-1)(s^2t^2+t^2+1)^2}+\left[\frac{t^2(s\sigma_b^2-\sigma_a^2)^2\,st\sigma_b^2-\sigma_a^2}{s^2t^2+t^2+1}\right] \tag{4.20}$$

$$M\tilde{S}E(\tilde{\sigma}_a^2)=\frac{2s^2t^2MS_A^2}{(r+1)(s^2t^2+t^2+1)^2}+\frac{(r-1)\ (t^2+1)MS_A^2}{(r+1)s^2t^2(s^2t^2+t^2+1)^2} -\frac{2(t^2+1)MS_A\cdot MS_B}{s^2t^2(s^2t^2+t^2+1)}+\frac{r(s-1)MS_A^2}{s^2t^2(r(s-1)+2)} \tag{4.21}$$

(2). If $p_b=(0,1,0)'$, then

$$\tilde{d}_b=q_b=(0,t/(t^2+1),0)',\ \overline{p_b'\sigma}=tMS_B/(t^2+1)=\tilde{\sigma}_b^2$$

$$MSE(\tilde{\sigma}_b^2)=\frac{(t\sigma_b^2+\sigma_e^2)^2}{t^2(t^2+1)r(s-1)}+\frac{(t\sigma_e^2+\sigma_b^2)^2}{(t^2+1)^2} \tag{4.22}$$

$$M\hat{S}E(\bar{\sigma}_b^2)=\frac{2t^2MS_B^2}{(t^2+1)\ (r(s-1)+2)}+\frac{rs(t-1)MS_E^2}{(rs(t-1)+2)t^2}+\frac{r(s-1)MS_B^2}{t^2(t^2+1)^2(rs(t-1)+2)}-\frac{2MS_E\cdot MS_B}{t^2(t^2+1)} \quad (4.23)$$

(3). If $p_e=(0,0,1)'$, then

$$\tilde{d}_e=q_e=(0,0,1)'$$

$$var(\tilde{\sigma}_e^2)=2\sigma_e^4=2\sigma_e^4/rs(t-1),\ v\hat{a}r(\tilde{\sigma}_e^2)=2MS_E^2/(rs(t-1)+2) \quad (4.24)$$

Remark. In models satisfying some assumptions, the ANOVA estimator of $p'\sigma$ is $\widehat{p'\sigma}=p'L^{-1}M$ and the nonnegative minimum biased estimator of $p'\sigma$ is $\widetilde{p'\sigma}=\tilde{d}'M$, where $\tilde{d}\in R^m_+$ minimizes uniquely $(Ld-p)'\ (Ld-p)$ over R^m_+. Thus · both estimators are of the form (4. 1).

The unbiased estimator $M\hat{S}E(\bar{p}'\bar{\sigma})$ is the best unbiased estimator of $MSE(\bar{p}'\bar{\sigma})$, where $p'\sigma$ is any unbiased or biased estimator of $p'\sigma$ of the general form $\bar{p}'\bar{\sigma}=q'M$.

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Korea Mariritime University
Pusan 606-791, Korea