

RANGE THEOREMS FOR ACCRETIVE OPERATORS IN BANACH SPACES

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1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, An operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $y_i \in Ax_i, i = 1, 2$ and $r > 0$. An accretive operator $A \subset E \times E$ is said to be m -accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$, where I is the identity operator.

The following theorem was given by Kartsatos in [3] :

Theorem. A. *Let X, X^* be uniformly convex. Let T be an m -accretive operator with $D(T)$ containing zero. Moreover, let*

$$\|T(0)\| < r < \liminf_{\substack{\|x\| \rightarrow \infty \\ x \in D(T)}} \|Tx\|$$

where r is a positive constant. Then $B_r(0) = \{x \in E \mid \|x\| < r\} \subset R(T)$.

This result of Kartsatos was an extension of Lange's result in case that E is a Hilbert space (cf. [3]). In proof of this result, existence theorems for ordinary differential equations with accretive mappings in Banach spaces were used.

In this paper, we extend Theorem A to more general case by using the fixed point property for nonexpansive self-mappings. Our proof is simpler than that of [3] on account of using Banach limits.

2. Preliminaries

Let E be a real Banach space and let E^* be its dual. Let $U = \{x \in E \mid \|x\| = 1\}$ be the unit sphere of E . The norm of E is said to

be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if this limit is attained uniformly for (x, y) in $U \times U$. It is well known that the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*) . With each $x \in E$, we associated the set

$$J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for any $x \in E$. The multi-valued operator $J: E \rightarrow E^*$ is called the duality mapping of E . It is well known that if E is smooth, then J is single valued.

Let K be a subset of E , then we denote by $\delta(K)$ the diameter of K . A point $x \in K$ is a diametral point of K provided

$$\sup\{\|x-y\| : y \in K\} = \delta(K)$$

A closed convex subset C of a Banach space E is said to have normal structure for each closed bounded convex subset K of C , which contains at least two points, there exists an element of K which is not a diametral point of K . Banach space E is said to have normal structure if any bounded, convex and closed subset of the space has normal structure. All uniformly convex Banach spaces have normal structure. It is also known that all uniformly smooth Banach spaces have normal structure. (cf. [2, p.45]).

Let C be a closed convex subset of E . Then C is said to have the fixed point property for nonexpansive self-mappings if for every nonexpansive mapping $T: C \rightarrow C$, there is a point $p \in C$ such that $T(p) = p$. Kirk [4] proved that if E is a reflexive Banach space and C is a bounded closed convex subset of E having normal structure, then C has the fixed point property for nonexpansive self-mappings.

The closed convex hull of subset B of E will be denoted by $\text{clco}(B)$

Recall that an operator $A \in E \times E$ is accretive if and only if for each $x_1 \in D(A)$ and $y_1 \in Ax_1$, $i=1,2$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. If A is accretive, we can define, for each positive r , the resolvent of A , $J_r : R(I+rA) \rightarrow D(A)$ by $J_r = (I+rA)^{-1}$ and the Yosida approximation of A , $A_r : R(I+rA) \rightarrow R(A)$ by $A_r = (1/r)(I - J_r)$. We know that $A_r x \in A J_r x$ for every $x \in R(I+rA)$ and that $\|A_r x\| \leq |Ax|$ for every $x \in D(A) \cap R(I+rA)$, where $|Ax| = \inf\{\|y\| : y \in Ax\}$ (cf. [1]). We also know that if $D(A) \subset R(I+rA)$ for each $r > 0$, then $A^{-1}0 = F(J_r)$, where $F(J_r)$ is the set of fixed points of J_r .

Finally, we mention that a Banach limit LIM is a bounded linear functional on l^∞ of norm 1 such that

$$\liminf_{n \rightarrow \infty} t_n \leq \text{LIM } t_n \leq \limsup_{n \rightarrow \infty} t_n$$

and

$$\text{LIM } t_n = \text{LIM } t_{n+i}$$

for all $\{t_n\}$ in l^∞ . Let $\{x_n\}$ be a bounded sequence of E . Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \text{LIM } \|x_n - z\|$$

for each $z \in E$.

3. Main results

We now prove a theorem which extends Theorem A to more general case.

Theorem. *Let E be a reflexive Banach space with normal structure, Let $A \in E \times E$ be an m -accretive operator. Assume that for some $x_0 \in D(A)$ and $r > 0$*

$$|Ax_0| < r \leq \liminf_{\substack{\|x\| \rightarrow \infty \\ x \in D(A)}} |Ax|$$

Then $B_r(0) = \{x \in E : \|x\| < r\} \subset R(A)$

The proof of theorem follows from the following Lemma.

Lemma. *Under the assumption of Theorem,*

$$B_\mu(0) \subset R(A), \text{ where } \mu = (r - |Ax_0|)/2.$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. In fact, we may replace A by A' given by $A'x = A(x + x_0)$ for each $x \in D(A') = D(A) - x_0$. Let $p \in B_\mu(0)$, where $\mu = (r - |A0|)/2$. Then since A is m -accretive, there exist $(x_n, y_n) \in A$ such that

$$x_n + ny_n = np \text{ or } (1/n)x_n + y_n = p, \quad n = 1, 2, \dots$$

We show that $\{x_n\}$ is bounded. For any $y_0 \in A0$, since A is accretive, there exist $j_n \in J(x_n - 0) = J(x_n)$ such that

$$\begin{aligned} \|p\| \|j_n\| &\geq (p, j_n) = (y_n - y_0, j_n) + (y_0, j_n) + \frac{1}{n} (x_n, j_n) \\ &\geq (y_0, j_n) + \frac{1}{n} \|x_n\|^2. \end{aligned}$$

so, it follows that

$$\frac{1}{n} \|x_n\|^2 \leq \|p\| \|j_n\| - (y_0, j_n) \leq (\|p\| + \|y_0\|) \|j_n\|$$

and hence

$$\frac{1}{n} \|x_n\| \leq \|p\| + \|y_0\|$$

Assume that $\lim_{k \rightarrow \infty} \|x_{n_k}\| = \infty$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Then

$$r \leq \liminf_{\substack{\|x\| \rightarrow \infty \\ x \in D(A)}} |Ax| \leq \liminf_{k \rightarrow \infty} |Ax_{n_k}| \leq \liminf_{k \rightarrow \infty} \|y_{n_k}\|$$

Thus, for any $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ such that

$$\|y_{n_k}\| \geq r - \varepsilon, \text{ for every } k > N(\varepsilon).$$

Consequently, for any $k > N(\varepsilon)$,

$$r - \|p\| - \varepsilon \leq \|y_{n_k}\| - \|p\| \leq \|y_{n_k} - p\| = \frac{1}{n_k} \|x_{n_k}\|$$

$$\leq \|p\| + |A0|$$

Since $\epsilon > 0$ is arbitrary, we have

$$|A0| \geq r - 2\|p\| > r - 2\mu = |A0|.$$

This is a contradiction. So, $\{x_n\}$ is bounded, and hence $y_n \rightarrow p$.
Now let $A_0x = Ax - p$, for each $x \in D(A)$. Then A_0 is m -accretive and

$$R(I + A_0) \cap \text{clco}(D(A_0)) = \text{clco}(D(A)).$$

We also have that $(x_n, y_n - p) \in A_0$. Let $C = \text{clco}(D(A))$. For a Banach limit, define $\phi : C \rightarrow R$ by

$$\phi(z) = \text{LIM} \|x_n - z\|$$

for each $z \in C$ and $r = \inf\{\phi(z) : z \in C\}$. Then since the function ϕ on C is continuous convex, $\phi(z) \rightarrow \infty$ as $\|z\| \rightarrow \infty$, and E is reflexive, there exists $v \in C$ with $\phi(v) = r$. Let $K = \{v \in C : \phi(v) = r\}$. Then it follows that K is nonempty, bounded, closed and convex. Let $v \in K$ and $J_1^{A_0} = (I + A_0)^{-1}$. Then $J_1^{A_0}$ is a nonexpansive mapping from $R(I + A_0)$ to $D(A_0) = D(A)$. We also have

$$\begin{aligned} \text{LIM} \|x_n - J_1^{A_0} v\| &\leq \text{LIM} \|x_n - J_1^{A_0} x_n\| + \text{LIM} \|x_n - v\| \\ &\leq \text{LIM} |A_0 x_n| + \text{LIM} \|x_n - v\| \\ &\leq \text{LIM} \|y_n - p\| + \text{LIM} \|x_n - v\| \\ &= \text{LIM} \|x_n - v\|. \end{aligned}$$

where $|A_0 x| = \inf\{\|y\| : y \in A_0 x\}$. Therefore K is invariant under $J_1^{A_0}$. Since K has the fixed point property for nonexpansive mappings, there exists $v \in K$ with $J_1^{A_0} v = v$. This implies $0 \in A_0 v$.

Thus we obtain $p \in Av$, which proves $B_r(0) \subset R(A)$.

proof of Theorem. Let $y \in B_r(0)$ and let

$$M = \{t \in [0, 1] \cdot ty \in R(A)\}.$$

Then by Lemma, $M \neq \emptyset$. We complete the proof of by showing $1 \in M$. Let $t_0 = \sup M$. Choose $t_n \in M$ such that $t_n < t_0$ and $t_n \rightarrow t_0$, and then take

$x_n \in D(A)$ with $t_n y \in Ax_n$. Define $A_n \subset E \times E$ by

$$A_n x = A(x + x_n) - t_n y$$

for any $x \in D(A_n) = D(A) - x_n$. Then $0 \in A_n 0$ and

$$\lim_{\substack{\|x\| \rightarrow \infty \\ x \in D(A_n)}} \inf |A_n x| \geq r - t_n \|y\| > 0.$$

In fact, we have

$$\begin{aligned} |A_n x| &= \inf \{ \|z\| : z \in A_n x \} = \inf \{ \|w - t_n y\| : w \in A(x + x_n) \} \\ &\geq \inf \{ \|w\| : w \in A(x + x_n) \} - t_n \|y\| \\ &= |A(x + x_n)| - t_n \|y\| \end{aligned}$$

for any $x \in D(A_n)$ and hence

$$\begin{aligned} \lim_{\substack{\|x\| \rightarrow \infty \\ x \in D(A_n)}} \inf |A_n x| &\geq \lim_{\substack{\|x\| \rightarrow \infty \\ x \in D(A_n)}} \inf |A(x + x_n)| - t_n \|y\| \\ &= \lim_{\substack{\|x\| \rightarrow \infty \\ x \in D(A)}} \inf |Ax| - t_n \|y\| \geq r - t_n \|y\|. \end{aligned}$$

By Lemma, $B_{\mu_n}(0) \subset R(A_n)$, where $\mu_n = (r - t_n \|y\|) / 2$.

On the other hand, if $t \geq t_0$ is close enough to t_0 , it is possible to select n so that

$$(t - t_n) \|y\| \leq (r - t_n \|y\|) / 2.$$

So, from $B_{\mu_n}(0) \subset R(A_n)$, there exists $z_n \in D(A_n)$ such that $(t - t_n)y \in A_n z_n$, that is $ty \in A_n z_n + t_n y = A(z_n + x_n)$. Hence $t \in M$. This implies $t_0 = 1 \in M$, which proves that $B_r(0) \subset R(A)$.

As a direct consequence of Theorem, we have the following interesting corollaries.

Corollary 1. *Under the assumption of Theorem, if*

$$\lim_{\substack{\|x\| \rightarrow \infty \\ x \in D(A)}} \inf |Ax| = \infty,$$

then $R(A) = E$.

Proof. $B_r(0) \subset R(A)$ for every $r > 0$

Corollary 2. Assume that E is as in Theorem. Let A be an m -accretive operator. Assume that for some $x_0 \in D(A)$, $y_0 \in Ax_0$ and $r, s > 0$,

$$r \leq \inf_{\substack{\|x-x_0\| \geq s \\ x \in D(A)}} |Ax - y_0|,$$

where $|Ax - y_0| = \inf\{\|y - y_0\| : y \in Ax\}$. Then $B_r(y_0) \subset A(B_s(x_0) \cap D(A))$.

Proof. We consider the operator $A'x = A(x + x_0) - y_0$, for any $x \in D(A') = D(A) - x_0$. Then A' is m -accretive on $D(A')$ and $0 \in A'0$. Moreover,

$$\begin{aligned} \liminf_{\substack{\|r\| \rightarrow \infty \\ x \in D(A')}} |A'x| &\geq \inf_{\substack{\|u\| \geq s \\ x \in D(A')}} |A'u| = \inf_{\substack{\|x-x_0\| \geq s \\ x \in D(A)}} |Ax - y_0| \\ &\geq r > |A'0|. \end{aligned}$$

Theorem implies now that $B_r(0) \subset R(A')$, which is equivalent to $B_r(y_0) \subset R(A)$. Let $y \in B_r(y_0)$ and $u \in D(A)$ such that $y \in Au$ and assume that $\|u - x_0\| \geq s$. Then

$$r \leq \inf_{\substack{\|x-x_0\| \geq s \\ x \in D(A)}} |Ax - y_0| \leq \|y - y_0\|,$$

a contradiction to $y \in B_r(y_0)$. Thus $u \in B_s(x_0)$ and the proof is complete.

References

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