

ON IDEALS OF ENDOMORPHISM RING OF PROJECTIVE MODULE

Soon-Sook Bae

0. Introduction

The object of the paper is to study the relationship between submodules of projective module and ideals of endomorphism ring of projective module. In a projective module ${}_R M$ if ${}_R M$ has a small submodule, then the endomorphism ring $\text{End}({}_R M)$ has a small left ideal. If ${}_R M$ has the largest submodule, then $\text{End}({}_R M)$ is a local ring.

Throughout this paper, every ring is an associative ring with identity and every module is a left module. For an element a in a ring R , ${}^1(a)$ means the left ideal generated by a , in fact, ${}^1(a) = Ra + Za$. The ring of R -endomorphisms of a left R -module ${}_R M$, denoted by $\text{End}({}_R M)$, will be written on the right side of M as right operators on M , that is, ${}_R M_{\text{End}({}_R M)}$ will be considered in this paper. For mappings $f: M \rightarrow N$, $g: N \rightarrow L$, the composition mapping $f \cdot M \rightarrow L$ will be written by fg in order. $\text{Im}f$ is denoted by the image of f .

1. Results

For a submodule L of a module ${}_R M$, consider the set I^L of all endomorphisms whose images are contained in L , then the zero 0 is in I^L , which says that I^L is not empty. For each $f, g \in I^L$, $\text{Im}(f+g) \leq \text{Im}f + \text{Im}g \leq L + L \leq L$ and for any h in $\text{End}({}_R M)$, $\text{Im}(hg) \leq \text{Im}g \leq L$ so we have a left ideal I^L .

Properties 1.

(1) For any left ideal I of $\text{End}({}_R M)$ let $L = \sum_{f \in I} \text{Im}f$, then $I \leq I^L$

- (2) If $L_1 \leq L_2 \leq M$, then $I^{L_1} \leq I^{L_2}$ in $\text{End}({}_R M)$
 (3) $I^M = \text{End}({}_R M)$ and $I^0 = 0$
 (4) For submodules $L_a (a \in A)$ of ${}_R M$, $I \cap a L_a = a \cap I^L a$
 and $\sum_{a \in A} I^{L_a} \leq I^{\sum_{a \in A} L_a}$

Definition 2. A submodule L of an R -module ${}_R M$ is said to be fully invariant if every endomorphism on ${}_R M$ sends L into L .

Not all submodule of a module need not be fully invariant for example, $0 \oplus \{0, 2\}$ is not fully invariant of $Z_4 \oplus 0$ And $\{0, 2\}$ is a fully invariant submodule z_4 .

Proposition 3. If L is a fully invariant submodule of ${}_R M$, then I^L is a both sided ideal of $\text{End}({}_R M)$

Proof. It suffices to prove that I^L is right sided ideal. Let $f \in I^L$ and $g \in \text{End}({}_R M)$ be arbitrary given, then $\text{Im}(fg) = (\text{Im}f)g \leq Lg \leq L$ Since L is fully invariant, which tells that $fg \in I^L$.

Remark 4. Every left ideal I of $\text{End}({}_R M)$ has a fully invariant submodule $\cap \{\text{Ker } f | f \in I\}$. Since for each $x \in \cap \{\text{Ker } f | f \in I\}$, $xf = 0$ for every $f \in I$, and $xhf = 0$ for all $h \in \text{End}({}_R M)$ because I is a left sided ideal. Hence xh is contained in $\cap \{\text{Ker } f | f \in I\}$.

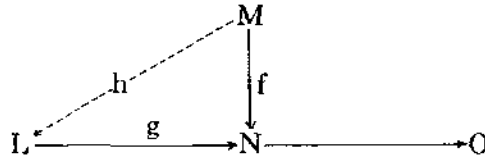
Let ι_j be the multiplication by j on Z_4 , then ι_j becomes an endomorphism of Z_4 . For a submodule $\{0, 2\}$ of Z_4 which is fully invariant, we obtain a both sided ideal $I^{0,2} = \{\iota_0, \iota_2\}$.

For fully invariant submodules $L_a (a \in A)$ of ${}_R M$, their sum $\sum_{a \in A} L_a$ and intersection $\cap \{L_a | a \in A\}$ are also fully invariant.

It may happen to exist distinct submodules L', L'' of ${}_R M$ such that $I^{L'} = I^{L''}$ (for example, in the set of real numbers as a Z -module, the set Q of rational numbers and the set Z of integers are such submodules, i. e., $I^Q = I^Z = 0$), then we are going to take L as their intersection $L' \cap L''$.

Generally, if $I^L = I^a (a \in A)$, then L will be regarded as the intersection $\cap \{L_a | a \in A\}$. From now on, in I^L , L means the least submodule of ${}_R M$ which induces a left ideal I^L .

A left R -module M is said to be projective if for any exact sequence and for any homomorphism $f: M \rightarrow N$ there is an R -homomorphism $h: M \rightarrow L$ such that diagram commutes.



A submodule L of a left module M is said to be small (or superfluous) if for every submodule $K \leq M$, $L + K = M$ implies $K = M$.

Lemma 5. Every epimorphism of $\text{End}({}_R M)$ is left invertible if ${}_R M$ is projective.

Proof This is easily followed by the proposition 5, p83 in [1].

Theorem 6. If a submodule M is small, then the left ideal I^L is small in $\text{End}(M)$.

Proof. We need only consider all left ideals of $\text{End}(M)$. Suppose I is a left ideal of $\text{End}(M)$ such that $I^L + I = \text{End}(M)$.

Then the identity 1 of $\text{End}(M)$ can be written as a sum of $f \in I$ $i \in I^L$, that is $1 = f + i$. Thus $M = \text{Im} 1 = \text{Im}(f + i) \leq \text{Im} f + \text{Im} i \leq L + \text{Im} i$. By hypothesis, L is small which implies $\text{Im} i = M$. Thus i is an epimorphism which is in I . By Lemma 5, i is left invertible, whence $I = \text{End}(M)$.

Let M be a left module. Then the radical of M , ([2])

$$\begin{aligned}
 \text{Rad} M &= \bigcap \{K \leq M \mid K \text{ is maximal in } M\} \\
 &= \sum \{L \leq M \mid L \text{ is small in } M\}
 \end{aligned}$$

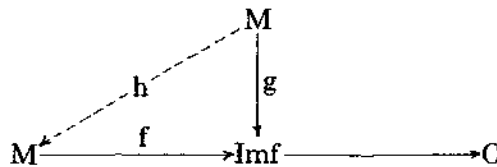
Theorem 7. If a projective module M has the largest submodule L , then $\text{End}(M)$ is a local ring, and M has a small submodule.

Proof From the fact that L is largest in M , every homomorphic image of non-epimorphism is contained in L . Let J be any ideal of $\text{End}(M)$ such that $J \neq \text{End}(M)$, then for each $f \in J$, $\text{Im} f \leq L$ so that $f \in I^L$. Hence $J \leq I^L$. This implies I^L is the largest left ideal of $\text{End}(M)$. By Proposition 4 in [1] on p57, and Corollary on p58, the radical

of $\text{End}(M)$ is I^L which is a both sided ideal, since the largest ideal is a maximal ideal in a ring. Now it remains to show that M has a small submodule. Since $\text{rad}M=L=\text{sum}$ of small submodules of M and since a sum of submodules in an empty set is zero, thus there is at least one small submodule.

Theorem 8. In a projective module M , if L is a homomorphic image of endomorphism, then the left ideal I^L is principal.

Proof. Let $L=Imf$ for f in $\text{End}(M)$. If $g \in I^L$, then $Img \leq Imf=L$. Considering a diagram



there is an R -homomorphism $h : M \rightarrow M$ such that $g=hf$

This means $I^L=I(f)$.

Corollary 9. In a projective module M , if L is a fully invariant submodule which is an image of an endomorphism.

Then a both sided ideal I^L is principal in $\text{End}(M)$.

Proof. By Proposition 3, I^L is a both sided ideal. Hence $I^L=I(f)$ in $\text{End}(M)$.

References

1. Joachim Lambek, "Lectures on Rings and Modules", Chelsea Publ. New York (1976).
2. F. Anderson and K.R. Fuller, "Rings and Categories of Modules" New York Springer-Verlag(1973)
3. Roger Ware, "Endomorphism ring of projective modules" Trans. of AMS Vol 155, Nr.1 March 1971. 233-255.
4. I. Kaplansky, "Projective modules" Ann. of Math,(2) 68(1958) 372-377 Mr 20 # 6453
5. I. Kaplansky, "Fields and Rings" Univ. of Chicago Press, Chicago (1969)

6. O. Zariski and P. Samuel, "Commutative Algebra" Vol I, Univ. Series in Higher Math, Van Nostrand N.J.(1958) Mr 19, 833.

Department of Mathematics
Kyungnam University
Masan 630-701, Korea