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ON IDEALS OF ENDOMORPHISM RING OF PROJECTIVE MODULE

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0. Introduction

The object of the paper is to study the relationship between submodules of projective module and ideals of endomorphism ring of projective module. In a projective module ${}_RM$ if ${}_RM$ has a small submodule, then the endomorphism ring $\mathrm{End}({}_RM)$ has a small left ideal. If ${}_RM$ has the largest submodule, then $\mathrm{End}({}_RM)$ is a local ring.

Throughout this paper, every ring is an associative ring with identity and every module is a left module. For an element a in a ring R, ${}^l(a)$ means the left ideal generated by a, in fact, ${}^l(a)=Ra+Za$. The ring of R-endomorphisms of a left R-module ${}_RM$, denoted by End $({}_RM)$, will be written on the right side of M as right operators on M, that is, ${}_RM_{\mathrm{END}({}_RM)}$ will be considered in this paper. For mappings $f: M\to N$, $g: N\to L$, the composition mapping $f\cdot M\to L$ will be written by fg in order. Imf is denoted by the image of f.

1. Results

For a submodule L of a module ${}_RM$, consider the set I^L of all endomorphisms whose images are contained in L, then the zero 0 is in I^L, which says that I^L is not empty. For each f, $g\in I^L$, $Im(f+g)\leq Imf+Img\leq L+L\leq L$ and for any h in $\mathrm{End}({}_RM)$, $Im(hg)\leq Img\leq L$ so we have a left ideal I^L.

Properties 1.

(1) For any left ideal I of $\mathrm{End}({}_RM)$ let $L=\sum_{f\in I}\mathrm{Im}f$, then $I\leqq I^L$

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(2) If $L_1 \leq L_2 \leq M$, then $I^{L_1} \leq I^{L_2}$ in $\mathrm{End}(_RM)$
(3) $I^M = End(_RM)$ and $I^0 = 0$
(4) For submodules $L_a (a \in A)$ of $_RM$, $I \cap a\ L_a = a \cap I^L a$
and $\sum_{a \in A} I^{La} \leq I^{\sum_{a \in A} La}$

Definition 2. A submodule L of an R-module $_RM$ is said to be fully invariant if every endomorphism on $_RM$ sends L into L.

Not all submodule of a module need not be fully invariant for example, $0 \oplus \{0, 2\}$ is not fully invariant of $Z_4 \oplus 0$ And$\{0, 2\}$ is a fully invariant submodule z_4.

Proposition 3. If L is a·fully invariant submodule of $_RM$, then I^L is a both sided ideal of $\mathrm{End}(_RM)$

Proof. If suffices to prove that I^L is right sided ideal. Let $f \in I^L$ and $g \in \mathrm{End}(_RM)$ be arbitary given, then $Im(fg) = (Imf)g \leq Lg \leq L$ Since L is fully invariant, which tells that $fg \in I^L$.

Remark 4. Every left ideal I of $\mathrm{End}(_RM)$ has a fully invariant submodule $\cap \{\mathrm{Ker}\, f | f \in \mathrm{I}\}$. Since for each $x \in \cap \{\mathrm{Ker} f | f \in \mathrm{I}\}$, $xf = 0$ for every $f \in I$, and $xhf = 0$ for all $h \in \mathrm{End}(_RM)$ because I is a left sided ideal. Hence xh is contained in $\cap \{\mathrm{Ker} f | f \in \mathrm{I}\}$.

Let ι_j be the multiplication by j on Z_4, then ι_j becomes an endomorphism of Z_4. For a submodule $\{0, 2\}$ of Z_4 which is fully invariant, we obtain a both sided ideal $I^{\{0,2\}} = \{\iota_0, \iota_2\}$.

For fully invariant submodules $L_a (a \in A)$ of $_RM$, their sum $\sum_{a \in A} L_a$ and intersection $\cap \{L_a | \, | a \in A\}$ are also fully invariant.

It may happen to exist distinct submodules L', L'' of $_RM$ such that $I^{L''} = I^{L'}$ (for example, in the set of real numbers as a Z-module, the set Q of rational numbers and the set Z of integers are such submodules, ι, $e.$, $I^Q = I^Z = 0$), then we are going to take L as their intersection $L' \cap L''$.

Generally, if $I^{La} = I^L (a \in A)$, then L will be regarded as the intersection $\cap \{La | \, | a \in A\}$. From now on, in I^L, L means the least submodule of $_RM$ which induces a left ideal I^L.

A left R-module M is said to be projective if for any exact sequence and for any homomorphism $f: M \to N$ there is an R-homomorphism $h: M \to L$ such that diagram commutes.

$$\begin{tikzcd} & M \arrow[d, "f"] \arrow[dl, dashed, "h"'] & \\ L \arrow[r, "g"] & N \arrow[r] & O \end{tikzcd}$$

A submodule L of a left module M is said to be small (or superfluous) if for every submodule $K \leq M$, $L+K=M$ implies $K=M$.

Lemma 5. Every epimorphism of End$({}_RM)$ is left invertible if ${}_RM$ is projective.

Proof This is easily followed by the proposition 5, p83 in [1].

Theorem 6. If a submodule M is small, then the left ideal I^L is small in End(M).

Proof. We need only consider all left ideals of End(M).
Suppose I is a left ideal of End(M) such that $I^L+I=End(M)$.

Then the identity 1 of End(M) can be written as a sum of $f \in I$ $\iota \in I^L$, that is $1=f+\iota$ Thus $M=Im1=Im(f+\iota) \leq Imf+Im\iota \leq L+Im\iota$. By hypothesis, L is small which implies $Im\,\iota=M$. Thus ι is an epimorphism which is in I. By Lemma 5, i is left invertible, whence $I=End(M)$.

Let M be a left module. Then the radical of M, ([2])

$$RadM=\cap\{K \leq M \mid K \text{ is maximal in } M\}$$
$$=\sum\{L \leq M \mid L \text{ is small in } M\}$$

Theorem 7. If a projective module M has the largest submodule L, then End(M) is a local ring, and M has a small submodule.

Proof From the fact that L is largest in M, every homomorphic image of non-epimorphism is contained in L, Let J be any ideal of End(M) such that $J \neq End(M)$, then for each $f \in J$, $Imf \leq L$ so that $f \in I^L$. Hence $J \leq I^L$. This implies I^L is the largest left ideal of End (M). By Proposition 4 in [1] on p57, and Corollary on p58, the radical of End(M) is I^L which i a both sided ideal, since the largest ideal is a maximal ideal in a ring. Now it remains to show that M has a small submodule. Since $radM=L=sum$ of small submodules of M and since a sum of submodules in an empty set is zero, thus there is at least one small submodule.

Theorem 8. In a projective module M, if L is a homomorphic image of endomorphism, then the left ideal I^L is principal.

Proof. Let $L=Imf$ for f in End(M). If $g \in I^L$, then $Img \leq Imf=L$. Considering a diagram

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
M \arrow[r, "f"] & Imf \arrow[r] & O
\end{tikzcd}$$

there is an R-homomorphism $h: M \rightarrow M$ such that $g=hf$
This means $I^L={}^1(f)$.

Corollary 9. In a projective module M, if L is a fully invariant submodule which is an image of an endomorphism.
Then a both sided ideal I^L is principal in End(M).

Proof. By Proposition 3, I^L is a both sided ideal. Hence $I^L=(f)$ in End(M).

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