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ON AN IDENTITY RELATED TO THE HOMOMORPHISM CLOSURE PRORERTY

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0. Introduction

Let Hom(H,G) be the set of all homomorphisms of a group H into another group G. For φ and ψ in Hom(H,G), define a mapping $\varphi+\psi: H\rightarrow G$ by $(\varphi+\psi)(x)=\varphi(x)+\psi(x)$ for all x in H. It is shown that $\varphi+\psi$ is again a homomorphism if G is abelian. In fact, Hom(H,G) is a group under this operation for any group H if and only if G is abelian ([3],[4],[5]). Furthermore, Hom(H,G) is abelian for every group H in that case. In this paper, we will generalize the condition on a universal algebra for Hom(B,A) to be an algebra of the same type as A and B. As in the group case, this condition is simply given by an identity, called the *medial law* It is assumed that the readers have basic concepts of universal algebra, such as *types, subalgebras, homomorphisms, congruences, direct products.* For terminology and definitions one can refer to [2] or [7].

1. Preliminary Definitions and Theorems.

A *universal algebra* (or *algebra* in short) is a pair (A,Ω) where A is a set, called the *underlying set* of the algebra, and Ω is a set of mappings $f: A^n\rightarrow A$, n depending on f. Each of these mappings is called an *operation* of the algebra. If $f\in\Omega$ is a mapping of A^n into A, n is called the *arity* of f and denoted by $a(f)$.

From now on, we will use only the underlying set A for the algebra

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(A,Ω) dropping out the operations if doing so does not make and confusion.

Let (A,Ω) and (B,Ω) be algebras of the same type and let Hom (B,A) be the set of all homomorhisms of B into A. For every operation $f \in \Omega$ with arity n and for homomorphisms $\varphi_1,\varphi_2,\cdots,\varphi_n$ in Hom (B,A) define a mapping $f(\varphi_1,\varphi_2,\cdots,\varphi_n) : B\rightarrow A$ by

$$f(\varphi_1,\varphi_2,\cdots,\varphi_n)(b)=f(\varphi_1(b),\varphi_2(b)\cdots,\varphi_n(b))$$

for all $b\in B$ If $f(\varphi_1,\varphi_2\cdots,\varphi_n)$ is again a homomosrphism for each operation f, for all $\varphi_1,\cdots,\varphi_n$ in Hom (B,A) and for every algebra B, we say the algebra A has the *homomorphism closure property* (HCP for convenience). Thus, in this case, Hom(B,A) is also an algebra of the same type as A and B.

Let us investigate this property a little farther with groups. If H and G are groups, for φ and ψ in Hom(H,G), define $\varphi+\psi:H\rightarrow G$ by $(\varphi+\psi)(x)=\varphi(x)$ for all $x\in H$ It can be easily seen that $\varphi+\psi$ is also a homomorpism provided G is abelian. That is, every abelian group has the HCP. Under this operation, the zero mapping sending every element of H onto the identity element of G is the identity element of Hom(H,G). For φ in Hom(H,G), define $-\varphi(x)=-(\varphi(x))$, and $-\varphi$ becomes the inverse of φ under this operation. The associativity follows trivially. Obviously, $\varphi+\psi=\varphi+\psi$ Thus if G is abelian, Hom (H,G) is also an abelian group. Conversely, suppose G has the HCP. Let $\pi_1,\pi_2:G\times G\rightarrow G$ be the projections given by $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for all $x,y\in G$. By the HCP of G, $\pi_1+\pi_2$ is a homomorphism. Thus for any $x,y\in G$,

$$\begin{aligned} x+y&=\pi_1(x,y)+\pi_2(x,y)\\ &=(\pi_1+\pi_2)(x,y)\\ &=(\pi_1+\pi_2)((o,y)+(x,o))\\ &=(\pi_1+\pi_2)(y,o)+(\pi_1+\pi_2)(o,x)\\ &=\pi_1(o,y)+\pi_2(o,y)+\pi_1(x,o)+\pi_2(x,o)\\ &=y+x, \end{aligned}$$

which says G is abelian. Thus, for groups, the HCP is equivalent to being abelian.

2. The HCP and the medial law.

For an n ary operation f and an m-ary operation g of an algebra A, we say f and g commute each other if

$$f(g(a_{11},a_{12},\cdots,a_{1m}),g(a_{21},a_{22},\cdots,a_{2m}),\cdots,g(a_{n1},a_{n2},\cdots,a_{nm}))$$
$$=g(f(a_{11},a_{21}\cdots,a_{n1}),f(a_{12},a_{22},\cdots,a_{n2}),\cdots,f(a_{1m},a_{2m},\cdots,a_{nm}))$$

for all $a_{ij}\in A$, $i=1,2,\cdots,n, j=1,2,\cdots,m$. An algebra is called *medial* if every pair of operations (not necessarily distinct) commute each other.

A guoupoid is a set closed under a binary operation defined on it. For a groupoid, the medial law can be written as $(x+y)+(x+w)=(x+z)+(y+w)$.

Lemma 2.1. (Murdoch [9]). Every medial groupoid with an identity element is a commutative monoid.

Proof If o is the identity element,

$$x+y=(o+x)+(y+o)=(o+y)+(x+o)=y+x,$$

and the groupoids is commutative, and

$$x+(y+z)=(x+o)+(y+z)=(x+y)+(o+z)=(x+y)+z,$$

which makes the groupoid associative.□

Suppose A is a medial groupoid and B any groupoid. For any φ, $\psi\in$ Hom(B,A) and $x,y\in B$,

$$\begin{aligned}(\varphi+\psi)(x+y)&=\varphi(x+y)+(x+y)\\&=(\varphi(x)+\varphi(y))+(\varphi(x)+\psi(y))\\&=(\varphi(x)+\psi(x)+(\varphi(y)+\psi(y))\\&=(\varphi+\psi)(x)+(\varphi+\psi)(y).\end{aligned}$$

Thus, $\varphi+\psi$ is also a homomorphism and so A has the HCP. In fact, for groupoids, the medial law, not the commutative law, implies the HCP. We state a theorem for groupoids.

Theorem 2.2. (Evans [4]). The following four conditions are equivalent any groupoid A.

i) A is medial;

ii) A has HCP;

iii) If φ and ψ are in $\mathrm{Hom}(A^2,A)$, then so is $\varphi+\psi$;

iv) The mapping $(x, y)\to x+y$ is a homomorphism of A^2 into A.

Generalizing this theorem to general algebraic systems, we have the following theorem.
Parts of the theorem appeared in [6] and [8].

Theorem 2.3. The following four conditions are equivalent for any algebra A.

i) A is medial;

ii) A has HCP;

iii) If f is an n-ary operaton and $\varphi_1,\varphi_2,\cdots,\varphi_n$ are homomorphisms of A^n into A, then $f(\varphi_1,\varphi_2\cdots,\varphi_n)$ is also a homomorphism of A^n into A;

iv) If f is an n-ary operation on A, then $f\in \mathrm{Hom}(A^n,A)$.

Proof. i) $\Rightarrow$ ii). Let f be an n-ary operation, B be any similar algebra and $\varphi_1,\varphi_2,\cdots,\varphi_n \in \mathrm{Hom}(B,A)$. For any m-ary operation g and $b_i\in B, i=1,2,\cdots,m$,

$$\begin{aligned}
&f(\varphi_1,\varphi_2,\cdots,\varphi_n)(g(b_1,b_2,\cdots,b_m))\\
&=f(\varphi_1(g(b_1,b_2\cdots,b_m)),\cdots\cdots,\varphi_n(g(b_1,b_2,\cdots,b_m)))\\
&=f(g(\varphi_1(b_1),\varphi_1(b_2),\cdots,\varphi_1(b_m)),\cdots\cdots,g(\varphi_n(b_1)\varphi_n(b_2),\cdots,\varphi_n(b_m)))\\
&=g(f(\varphi_1(b_1),\varphi_2(b_1),\cdots,\varphi_n(b_1))\cdots\cdots,f(\varphi_1(b_m),\varphi_2(b_m),\cdots,\varphi_n(b_m)))\\
&=g(f(\varphi_1,\varphi_2,\cdots,\varphi_n)(b_1),\cdots\cdots,f(\varphi_1,\varphi_2,\cdots,\varphi_n)(b_m)),
\end{aligned}$$

proving $f(\varphi_1,\varphi_2,\cdots,\varphi_n)$ is a homomorphism. ii) $\Rightarrow$ iii). Obvious. iii) $\Rightarrow$ iv). Let f be an n-ary operation and π_i be the i-th projection of A^n onto A, then clearly π_i is a homomorphism. Then $f(\pi_1,\pi_2\cdots,\pi_n)$ is a homomorphism of A^n into A and

$$f(\pi_1,\pi_2,\cdots,\pi_n)(x_1,x_2,\cdots,x_n)=f(x_1,x_2\cdots,x_n)$$

for all $x_1,x_2,\cdots,x_n\in A$. Thus $(x_1,x_2,\cdots,x_n)\to f(x_1,x_2,\cdots,x_n)$ is a homomorphism of A^n into A. iv) $\Rightarrow$ i). The condition iv) simply says that every operation is a homomorphism of a direct product of A into A. But, a homomorphism is nothing more than a mapping commuting with all operations. Thus, A is medial.□

As we noted, homomorphisms of a group into an abelian group form an abelian group.
Likewise, we say little more for medial algeras.

Theorem 2.4. If A is a medial algebra and B is any algebra of the same type, then the set Hom(B,A) is also a medial algebra of the same type.

Proof Hom(B,A) is closed under all operations by the preceding theorem. Suppose f and g be n-ary and m-ary operations, respectively. Let $\varphi_{ij} \in$ Hom(B,A) for $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$. Then,

$$\begin{aligned}
&f(g(\varphi_{11},\varphi_{12},\cdots,\varphi_{1m}),\cdots\cdots,g(\varphi_{n1},\varphi_{n2},\cdots,\varphi_{nm}))(b)\\
&=f(g(\varphi_{11}(b),\varphi_{12}(b),\cdots,\varphi_{1m}(b)\cdots\cdots,g(\varphi_{n1}(b),\varphi_{n2}(b),\cdots,\varphi_{nm}(b)))\\
&=g(f(\varphi_{11}(b),\varphi_{21}(b),\cdots,\varphi_{n1}(b)\cdots\cdots,f(\varphi_{1m}(b),\varphi_{2m}(b),\cdots,\varphi_{nm}(b)))\\
&=g(f(\varphi_{11},\varphi_{21},\cdots,\varphi_{n1}),\cdots\cdots,f(\varphi_{1m},\varphi_{2m},\cdots,\varphi_{nm}))(b)
\end{aligned}$$

for all $b \in B$ Thus,

$$\begin{aligned}
&f(g(\varphi_{11},\varphi_{12},\cdots,\varphi_{1m}),\cdots\cdots,g(\varphi_{n1},\varphi_{n2},\cdots,\varphi_{nm}))\\
&=g(f(\varphi_{11},\varphi_{21},\cdots,\varphi_{n1}),\cdots\cdots,f(\varphi_{1m},\varphi_{2m},\cdots,\varphi_{nm}))
\end{aligned}$$

and so f and g commute as operations on Hom(B,A). That is, Hom (B,A) is medial.□

Example 2.5. A vector space V over a field F can be treated as a universal algebra with a binary operation and a unary oreration for each element in F The commutativity of the field, the commutativity of the addition of the vector space, and the distributibity make these operations commute one another. That is, vector spaces are medial. Thus, the family of all homomorphisms, which are linear transformations, of a vector space into another is again a vector space.

An algebra A is said to have the *endomorphiom closure property* (ECP, in short) if $f(\varphi_1,\varphi_2,\cdots,\varphi_n)$ is an endomorphism for every operation f and endomorphisms $\varphi_1,\varphi_2,\cdots,\varphi_n$ of A

Corollary 2.6. Every medial algebra has the ECP and the family of all endomorphisms of the algebra is also a medial algebra of the same type.

3.The ECP and varieties.

A word(or *term*)in symbols $X=\{x_1,x_2,x_3,\cdots\}$is an expression built up from X using operations inductively as follow: every element in X is a word, and if f is an operation and $u_1,u_2,\cdots,u_{a(f)}$ are words then $f(u_1,u_2,\cdots,u_{a(f)})$ is also a word. It is easy to see that every word u defines a derived mapping of a cartesian product of A into A in the natural way of substitutions. A mapping of an algebra derived from a word is called a *term function*. In this reason, a word is sometimes called a *polynomial*. An *identity*(or *law*)is an equation $u=v$ for some words u and v. An algebra $(A,\Omega)$is said to satisfy the identity $u=v$ if u and v derive the same term function, that is, $u(a_1,a_2,\cdots,a_n)=v(a_1,a_2,\cdots,a_n)$ for all $a_1,a_2,\cdots,a_n\in A$. We say an algebra satisfies a set of identities if it satisfies every identity in the set. A variety V is a class of similar algebras such that

i)if $A \in V$ and B is a subalgebra of A, then $B \in V$;

ii)if $A \in V$ and B is a homomorphic image of A, then $B \in V$;

iii)if $A_i\in V$ for $i\in I$, then $\times\{A_i| i\in I\}\in V$.

An algebra A in a variety V is called *V-free* on a generating set X provided that every mapping of X into any algebra B in V can be extended to a homomorphism of A. This generating set is called a *free generating set* of A.

Lemma 3.1 ([1]). Every variety V has a V-free algebras on every set of generators, and V-free algebras are uniquely determined up to isomorphism by the cardinalities of their free generating sets.

Lemma 3.2 ([1]). Let V be a variety and A be a V-free algebra. If $u(a_1,a_2,\cdots,a_n)=v(a_1,a_2,\cdots,a_n)$ is a relation among elements $a_1,a_2,\cdots,$ a_n of a free generating set of A.
Then, $u(x_1,x_2,\cdots,x_n)=v(x_1,x_2,\cdots,x_n)$ is an identity satisfied by all algebras in V.

By corollary 2.6, every medial algebra has the ECP. However, the ECP does not implies the medial law in general ([3]). Thus, an algebra may have the ECP while it satisfies 'weaker' identities than the medial law. But for varieties, if all algebras in a variety have the ECP, then every algebra in the variety must be medial, as the following theorem asserts.

Theorem 3.3. Let V be a variety of algebras. Then every algebra in V has the ECP if and only if every algebra in V is medial.

Proof. The necessity is clear by corollary 2.6. For the converse, suppose every algebra in V has the ECP and let f be an n-ary and g be an m-ary operation. Take the V-free algebra F on the free generating set $\{a_{ij} \mid i=1,2,\cdots,n, j=1,2,\cdots,m\}$ For each $i=1,2,\cdots,n$, let φ_i be the endomorphism of F such that $\varphi_i(a_{1j})=a_{ij}$ for $j=1,2,\cdots,m$, which exists due to Lemma 3.1.Then, by the definition of $f(\varphi_1,\varphi_2,\cdots,\varphi_n)$,

$$\begin{aligned}
&f(\varphi_1,\varphi_2,\cdots,\varphi_n)(g(a_{11},a_{12},\cdots,a_{1m}))\\
&=f(\varphi_1(g(a_{11},a_{12},\cdots,a_{1m})),\cdots\cdots\varphi_n(g(a_{11},a_{12},\cdots,a_{1m})))\\
&=f(g(\varphi_1(a_{11}),\varphi_1(a_{12}),\cdots,\varphi_1(a_{1m}),\cdots\cdots,g(\varphi_n(a_{11}),\varphi_n(a_{12}),\cdots,\varphi_n(a_{1m})))\\
&=f(g(a_{11},a_{12},\cdots,a_{1m}),\cdots\cdots,g(a_{n1},a_{n2},\cdots,a_{nm})).
\end{aligned}$$

On the other hand, since $f(\varphi_1,\varphi,\cdots,\varphi_n)$ is an endomorphism.

$$\begin{aligned}
&f(\varphi_1,\varphi_2,\cdots,\varphi_n)(g(a_{11},a_{12},\cdots,a_{1m}))\\
&=g(f(\varphi_1,\varphi_2,\cdots,\varphi_n)(a_{11}),\cdots\cdots,f(\varphi_1,\varphi_2,\cdots,\varphi_n)(a_{1m}))\\
&=g(f(\varphi_1(a_{11}),\varphi_2(a_{11}),\cdots,\varphi_n(a_{11})),\cdots\cdots,f(\varphi_1(a_{1m}),\varphi_2(a_{1m}),\cdots,\varphi_n(a_{1m})))\\
&=g(f(a_{11},a_{21},\cdots,a_{n1}),\cdots\cdots,f(a_{1m},a_{2m},\cdots,a_{nm})).
\end{aligned}$$

Thus,

$$\begin{aligned}
&f(g(a_{11},a_{12},\cdots,a_{1m}),\cdots\cdots,g(a_{n1},a_{n2},\cdots,a_{nm}))\\
&=g(f(a_{11},a_{21},\cdots,a_{n1}),\cdots\cdots,f(a_{1m},a_{2m},\cdots,a_{nm}))
\end{aligned}$$

By Lemma 3.2, every algebra in V satisfies the identity,

$$\begin{aligned}
&f(g(x_{11},x_{12},\cdots,x_{1m}),\cdots\cdots,g(x_{n1},x_{n2},\cdots,x_{nm}))\\
&=g(f(x_{11},x_{21},\cdots,x_{n1}),\cdots\cdots,f(x_{1m},x_{2m},\cdots,x_{nm}))
\end{aligned}$$

That is, f and g commute each other. Hence every algebra in V is medial.□

Corollary 3.4. The medial law is the weakest law for a variety to have all ite members medial.

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