Pusan Kyŏngnam Mathematical Journal
Vol 5, 1~24, 1989

ON CERTAIN SUBCLASSES OF ANALYTIC P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. Aouf*

Abstract

Let $S_p^*(\alpha,\beta,\mu)$ denote the class of functions $f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}$ $(a_{p+n}\geq o,\ p\in N)$ analytic and p-valent in the unit disc $U=\{z:|z|<1\}$ and satisfy the condition

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-p}{\mu\dfrac{zf'(z)}{f(z)}+p-(1+\mu)\alpha}\right|<\beta,\ z\in U,$$

where $o\leq\alpha<p$, $o<\beta\leq 1$ and $o\leq\mu\leq 1$. Further $f(z)$ is said to belong to the class $C_p^*(\alpha,\beta,\mu)$ if $zf'(z)/p\in S_p^*(\alpha,\beta,\mu)$.

In this paper we obtain for these classes sharp results concerning coefficient estimates, disortion theorems, closure theorems, Hadamard products and some distortion theorems for the fractional calculus.

I. Introduction.

Let S_p denote the class of functions

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}\qquad(p\in N)$$

analytic and p-valent in the unit disc $U=\{z: z<1\}$
We say that $f(z)$ belongs to the class $S_p(\alpha,\beta,\mu)$ if $f(z)\in S_p$ satisfies the condition

Received August 20, 1988
Present address : Dept. of Maths., Faculty of Science, Univ. of Qatar, P.O. Box 2713, Doha-Qatar.

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\mu\dfrac{zf'(z)}{f(z)} + p - (1+\mu)\alpha} \right| < \beta \qquad (z \in U)$$

for $o \leq \alpha < p$, $o < \beta \leq 1$ and $o \leq \mu \leq 1$. Further $f(z)$ is said to belong to the class $C_p(\alpha,\beta,\mu)$ if $zf'(z)/p \in S_p(\alpha,\beta,\mu)$

In particular, $S_p(0,1,1)$ and $S_p(\alpha,1,1)$ are respectively the classes of p−valent starlike functions and p−valent starlike functions of order α, $o \leqq \alpha < p$. Also $S_1(0,1,0)$, $S_1(0,\beta,1)$ and $S_1(0,\beta,\mu)$ are respectively the classes of functions studied by Singh [15], Padmanabhan [11] and Lakshminarasimhan [5].

Let T_p denote the subclass of S_p consisting of functions analytic and p–valent which can expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \quad (a_{p+n} \geq o,\ p \in N).$$

We denote by $S_p^*(\alpha,\beta,\mu)$ and $C_p^*(\alpha,\beta,\mu)$ the classer obtained by taking intersections of the classes $S_p(\alpha,\beta,\mu)$ and $C_p(\alpha,\beta,\mu)$ with T_p, respectively.

It is easy to see that

$$S_p^*(\alpha_2,\beta,\mu) \subset S_p^*(\alpha_1,\beta,\mu) \ \ if\ \alpha_1 < \alpha_2,$$
$$S_p^*(\alpha,\beta_1,\mu) \subset S_p^*(\alpha,\beta_2,\mu) \ \ if\ \beta_1 < \beta_2$$

and

$$C_p^*(\alpha_2,\beta,\mu) \subset C_p^*(\alpha_1,\beta,\mu) \ \ if\ \alpha_1 < \alpha_2,$$
$$C_p^*(\alpha,\beta_1,\mu) \subset C_p^*(\alpha,\beta_2,\mu) \ \ if\ \beta_1 < \beta_2$$

In 1976, Gupta and Jain [3] studied the class $S_1^*(\alpha,\beta,1)$. Moreover Silverman [12], Silverman and Silvia [13], [14], Ahuja and Jain [1], Owa [7] and Owa and Aouf [9], [10] have studied certain subclasses of univalent functions with negative coefficients. For other classes of analytic p–valent functions with negative, Goel and Sohi [2], Srivastava and Owa [16] and Owa [8] showed some results.

2. Coefficient estimates.

Theorem 1. A function

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} (a_{p+n} \geq 0,\ p \in N)$$

is in the class $S_p^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}a_{p+n}\leqq(1+\mu)\beta(p-\alpha).$$

This result is sharp.

Proof. Let $|z|=1$. Then we get

$$|zf'(z)-pf(z)|-\beta\,|uzf'(z)+[p-(1+\mu)\alpha]f(z)|$$

$$=\left|\sum_{n=1}^{\infty}-n\ a_{p+n}\ z^{p+n}\right|-\beta\left|(1+\mu)\ (p-\alpha)z^p\right.$$

$$\left.-\sum_{n=1}^{\infty}[\mu n+(1+\mu)\ (p-\alpha)]a_{p+n}z^{p+n}\right|$$

$$\leq\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}a_{p+n}-(1+\mu)\beta(p-\alpha)\leq 0$$

Hence, by the maximum modulus theorem, f(z) is in the class $S_p^*(\alpha,\beta,\mu)$.

On the other hand, assume that

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{u\frac{zf'(z)}{f(z)}+p-(1+\mu)\alpha}\right|<\beta \qquad (z\in U)$$

Since $|Re(z)|\leq|z|$ for any z, we have

$$Re\left\{\frac{\sum_{n=1}^{\infty}n\ a_{p+n}\ z^{p+n}}{(1+\mu)\,(p-\alpha)\,z^p-\sum_{n=1}^{\infty}[\mu n+(1+\mu)\,(p-\alpha)\,]a_{p+n}\,z^{p+n}}\right\}\leq\beta \quad (2.1)$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.1) and letting $z\to 1^-$ through real values, we get

$$\sum_{n=1}^{\infty}n\ a_{p+n}\leq\beta\{(1+\mu)\ (p-\alpha)-\sum_{n=1}^{\infty}[\mu n+(1+\mu)\ (p-\alpha)]\ a_{p+n}\}$$

which implies that

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n}\leq(1+\mu)\beta(p-\alpha).$$

The function

$$f(z)=z^p-\frac{(1+\mu)\beta(p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}z^{p+n} \qquad (n\geq 1)$$

is an extremal function.

Corollary 1. Let a function

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

be in the class $S_p^*(\alpha,\beta,\mu)$. Then we have

$$a_{p+n}\leq\frac{(1+\mu)\ \beta\ (p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}$$

for any $n\geq 1$. The equality holds for the function

$$f(z)=z^p-\frac{(1+\mu)\ \beta\ (p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}z^{p+n}.$$

Theorem 2. A function

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

is in the class $C_p^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=1}^{\infty}(p+n)\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n}\leqq(1+\mu)\beta(p-\alpha)p.$$

This result is sharp.

Proof. The function $f(z)$ is in the class $C_p^*(\alpha,\beta,\mu)$ if and only if $zf'(z)/p\in S_p^*(\alpha,\beta,\mu)$. Now, since

$$\frac{zf'(z)}{p}=z^p-\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}\ z^{p+n},$$

by replacing a_{p+n} by $(\frac{p+n}{p})$ a_{p+n} in Theorem 1, we have the theorem.

Corollary 2. Let a function

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

be in the class $C_p^*(\alpha,\beta,\mu)$. Then we have

$$a_{p+n}\leq\frac{(1+\mu)\beta(p-\alpha)p}{(p+n)\ \{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}}$$

for any $n\geqq 1$. The equality holds for the function

$$f(z)=z^p-\frac{(1+\mu)\beta(p-\alpha)p}{(p+n)\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}}z^{p+n}$$

3. Distortion theorems.

Theorem 3. Let a function

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

be in the class $S_p^*(\alpha,\beta,\mu)$. Then we have

$$\left|f(z)\right|\geq|z|^p-\frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^{p+1}$$

and

$$\left|f(z)\right|\leq|z|^p+\frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^{p+1}$$

for $z\in U$. Further

$$\left|f'(z)\right|\geq\ p|z|^{p-1}-\frac{(1+\mu)\beta(p-\alpha)\ (p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^p$$

and

$$\left|f'(z)\right|\leq\ p|z|^{p-1}+\frac{(1+\mu)\ \beta\ (p-\alpha)\ (p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^p$$

for $z \in U$. These estimations are sharp.

Proof. By using Theorem 1, we obtain

$$\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}\sum_{n=1}^{\infty} a_{p+n}$$
$$\leq \sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n} \leq (1+\mu)\beta(p-\alpha)$$

which implies that

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}$$

Consequently we have

$$\left|f(z)\right| \geq |z|^p - \sum_{n=1}^{\infty} a_{p+n}\,|z|^{p+n}$$
$$\geq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$
$$\geq |z|^p - \frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p+1}$$

and

$$\left|f(z)\right| \leq |z|^p + \sum_{n=1}^{\infty} a_{p+n}|z|^{p+n}$$
$$\leq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$
$$\leq |z|^p + \frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p+1}$$

for $z \in U$.

In order to show the second half of the theorem, by using

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq \frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)\ )]},$$

we obtain

$$\left|f'(z)\right| \geq\ p|z|^{p-1} - \sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^{p+n-1}$$

$$\geq p|z|^{p-1}-|z|^{p}\sum_{n=1}^{\infty}(p+n)a_{p+n}$$

$$\geq p|z|^{p-1}-\frac{(1+\mu)\beta(p-\alpha)\ (p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^{p}$$

and

$$\left|f'(z)\right| \leq p|z|^{p-1}+\sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^{p-n-1}$$

$$\leq p|z|^{p}+|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}$$

$$\leq p|z|^{p-1}+\frac{(1+\mu)\beta(p-\alpha)\ (p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}|z|^{p}$$

for $z \in U$ The bounds are sharp and are attained for the function

$$f(z)=z^{p}-\frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}z^{p-1}$$

Corollary 3. Under the hypotheses of Theorem 3, $f(z)$ is included in te disc with center at the origin and radius

$$\frac{(1+\beta\mu)+2(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}$$

Further $f'(z)$ is included in the disc with center at the origin and radius

$$\frac{p(1+\beta\mu)+(1+\mu)\beta(p-\alpha)\ (2p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}$$

Theorem 4. Let a function

$$f(z)=z^{p}-\sum_{n=1}^{\infty}a_{p+n}z^{p-n}(a_{p+n}\geq o,\ p\in N)$$

be in the class $C_{p}^{*}(\alpha,\beta,\mu)$ Then we have

$$\left|f(z)\right| \geq |z|^{p}-\frac{(1+\mu)\beta(p-\alpha)p}{(p+1)\{1+\beta[\mu+(1+\mu)\ (p-\alpha]\}}|z|^{p+1}$$

and

$$\left|f(z)\right| \leq |z|^{p} + \frac{(1+\mu)\beta(p-\alpha)p}{(p+1)\{1+\beta[\mu+(1+\mu)\ (p-\alpha]\}}\,|z|^{p+1}$$

for $z \in U$. Further

$$\left|f'(z)\right| \geq p|z|^{p-1} - \frac{(1+\mu)\beta(p-\alpha)p}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p}$$

and

$$\left|f'(z)\right| \leq p|z|^{p-1} + \frac{(1+\mu)\beta(p-\alpha)p}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p}$$

for $z \in U$ If $p \in N-\{1\}$, then we have

$$\left|f''(z)\right| \geq p(p-1)|z|^{p-2} - \frac{(1+\mu)\beta(p-\alpha)p(p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p-1}$$

$$\left|f''(z)\right| \leq p(p-1)|z|^{p-2} - \frac{(1+\mu)\beta(p-\alpha)p(p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p-1}$$

for $z \in U$. The estimates for $f(z)$ and $f'(z)$ are sharp and are attained for the fnction

$$f(z) = z^{p} - \frac{(1+\mu)\beta(p-\alpha)p}{(p+1)\ \{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}\,z^{p+1}$$

Proof. The proofs for $|f(z)|$ and $|f'(z)|$ are obtained by using the same technique as in the proof of Theorem 3 with the aid of Theorem 2. Further, for $p \in N-\{1\}$ and $z \in U$, we have

$$\begin{aligned}\left|f''(z)\right| &\geq p(p-1)|z|^{p-2} - \sum_{n=1}^{\infty}(p+n)(p+n-1)a_{p+n}|z|^{p+n-2} \\ &\geq p(p-1)|z|^{p-2} - |z|^{p-1}\sum_{n=1}^{\infty}(p+n)^{2}a_{p+n} \\ &\geq p(p-1)|z|^{p-2} - \frac{(1+\mu)\beta(p-\alpha)p(p+1)}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}\,|z|^{p-1}\end{aligned}$$

and

$$\left|f''(z)\right| \leq p(p-1)|z|^{p-2}+\sum_{n=1}^{\infty}(p+n)\ (p+n-1)a_{p+n}|z|^{p+n-2}$$
$$\leq p(p-1)|z|^{p-2}+|z|^{p-1}\sum_{n=1}^{\infty}(p+n)^2\ a_{p+n}$$
$$\leq p(p-1)|z|^{p-2}+\frac{(1+\mu)\beta(p-\alpha)p(p+1)}{1+\beta[\mu+(1+\mu)(p-\alpha)]}|z|^{p-1}$$

by using Theorem 2.

Corollary 4. Under the hypotheses of Theorem 4, $f(z)$ is included in the disc with center at the origin and radius

$$\frac{(p+1)\ (1+\beta\mu)+(1+\mu)\ \beta\ (p-\alpha)\ (2p+1)}{(p+1)\ \{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}$$

and $f'(z)$ is included in the disc with center at the origin and radius

$$\frac{p(1+\beta\mu)+2(1+\mu)\beta(p-\alpha)p}{1+\beta[\mu+(1+\mu)\ (p-\alpha)]}$$

Further $f''(z)$ is included in the disc with center at the origin and radius

$$\frac{p(p-1)(1+\beta\mu)+2(1+\mu)\beta(p-q)p^2}{1+\beta[\mu+(1+\mu)(p-\alpha)]}$$

4. Closure theorems.

Theorem 5. Let

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\,z^{p+n}(a_{p+n}\geq o,\ p\in N)$$

be in the class $S_p^*(\alpha,\beta,\mu)$.

Proof. Since $f(z)$ and $g(z)$ are in $S_p^*(\alpha,\beta,\mu)$, we have from Theorem 1

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}a_{p+n}\leq(1+\mu)\beta(p-\alpha) \tag{4.1}$$

and

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}b_{p+n}\leq(1+\mu)\beta(p-\alpha). \tag{4.2}$$

From (4.1) and (4.2) we get

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}(a_{p+n}+b_{p+n})\leq(1+\mu)\beta(p-\alpha),$$

which implies that $h(z)\in S_p^*(\alpha,\beta,\mu)$.

The analogus of Theorem 5 for the class $C_p^*(\alpha,\beta,\mu)$ is :

Theorem 6. Let

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}\quad(a_{p+n}\geq o,\ p\in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\,z^{p+n}\quad(b_{p+n}\geq o,\ p\in N)$$

be in the class $C_p^*(\alpha,\beta,\mu)$. Then the function

$$h(z)=z^p-\tfrac{1}{2}\sum_{n=1}^{\infty}(a_{p+n}+b_{p+n})z^{p+n}$$

is also in the class $C_p^*(\alpha,\beta,\mu)$.

Theorem 7. Let

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}\quad(a_{p+n}\geq o,\ p\in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\,z^{p+n}\quad(b_{p+n}\geq o,\ p\in N)$$

be in the classes $S_p^*(\alpha,\beta,\mu)$ and $C_p^*(\alpha,\beta,\mu)$, respectively. Then the function

$$k(z)=z^p-\left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}(a_{p+n}+b_{p+n})z^{p+n}$$

is in the class $S_p^*(\alpha,\beta,\mu)$.

Proof. Since $f(z)\in S_p^*(\alpha,\beta,\mu)$ and $g(z)\in C_p^*(\alpha,\beta,\mu)$, by using Theorem 1 and Theorem 2, we get

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n}\leq(1+\mu)\beta\ (p-\alpha)$$

and

$$\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)(p-\alpha)]\}b_{p+n}\leq\frac{(1+\mu)\beta(p-\alpha)p}{(p+1)}$$

Therefore we have

$$\left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}(a_{p+n}+b_{p+n})\leq(1+\mu)\beta(p-\alpha)$$

which implies that $k(z)\in S_p^*(\alpha,\beta,\mu)$.

Theorem 8. Let

$$f_p(z)=z^p \qquad (p\in N)$$

and

$$f_{p+n}(z)=z^p-\frac{(1+\mu)\beta(p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}z^{p+n}(z\in N)$$

for n=1,2,3,⋯⋯. Then $f(z)$ belongs to the class $S_p^*(\alpha,\beta,\mu)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_{p+n}f_{p+n}(z),$$

where $\lambda_{p+n}\geq o$ and $\sum_{n=1}^{\infty}\lambda_{p+n}=1$.

Proof. Assume that

$$f(z)=\sum_{n=1}^{\infty}\lambda_{p+n}\ f_{p+n}(z)$$

$$=z^{p}-\sum_{n=1}^{\infty}\frac{(1+\mu)\beta(p-\alpha)g}{n+\beta[\mu n+(1+\mu\ (p-\alpha)]}\ \lambda_{p+n}\ z^{p+n}.$$

Then we obtain

$$\sum_{n=1}^{\infty}\{\lambda_{p+n}\frac{n+\beta[\mu n+(1+\mu)(p-\alpha)]}{(1+\mu)\beta(p-\alpha)}\cdot\frac{(1+\mu)\beta(p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}\}$$

$$=\sum_{n=1}^{\infty}\lambda_{p+n}=1-\lambda_{p}\leq 1$$

This shows that $f(z)\in S_p^*(\alpha,\beta,\mu)$ by Theorem 1.

On the other hand, let

$$f(z)=z^{p}-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}(a_{p+n}\geqq o,\ p\in N)$$

be in the class $S_p^*(\alpha,\beta,\mu)$. Then, by using Corollary 1, we get

$$a_{p+n}\leq\frac{(1+\mu)\beta(p-\alpha)}{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}$$

for any $n\geq 1$. On putting

$$\lambda_{p+n}=\frac{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]}{(1+\mu)\beta(p-\alpha)]}a_{p+n}\qquad(n=1,2,3,\cdots\cdots)$$

and

$$\lambda_{p}=1-\sum_{n=1}^{\infty}\lambda_{p+n},$$

we have the expression

$$f(z)=\sum_{n=1}^{\infty}\lambda_{p+n}\ f_{p+n}(z)$$

This completes the proof of the theroem.

Theorem 9. Let

$$f_p(z)=z^p \qquad (p \in N)$$

and

$$f_{p+n}(z)=z^p-\frac{(1+\mu)\beta(p-\alpha)p}{(p+n)\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}}z^{p+n} \qquad (p \in N)$$

for $n=1,2,\cdots\cdots$. Then $f(z)$ belongs to the class $C_p^*(\alpha,\beta,\mu)$ if and only if can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_{p+n}\ f_{p+n}(z),$$

where $\lambda_{p+n}\geqq o$ and $\sum_{n=1}^{\infty}{}_{p+n}=1$.

The proof of Theorem 9 is given in much the same way as Theorem 8.

5. Hadamard products.

Let $f*g(z)$ denote the Hadamard product of two fundctions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}\ (a_{p+n}\geq o,\ p \in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\ z^{p-n}\ (b_{p+n}\geq o,\ p \in N)$$

that is

$$f*g(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ b_{p+n}\ z^{p+n}$$

Theorem 10. Let the functions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}(a_{p+n}\geq o,\ p \in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\ z^{p+n}(a_{p+n}\geq o,\ p \in N)$$

belong to the classes $S_p^*(\alpha,\beta,\mu)$ and $S_p^*(\alpha_2, \beta_2, \mu)$, respectively. Then the Hadamard product $f*g(z)$ belongs to the class $C_p^*(\alpha_0,\beta_0,\mu)$, where $C_p^*(\alpha_0, \beta_0, \mu) = (C_p^*(\alpha_1, \beta_1, \mu) \wedge C_p^*(\alpha_2, \beta_2, \mu)$.

Proof. The proof is similar to the proof of [*4*, Theorem *1*].

Since $f(z) \in S_p^*(\alpha_1,\beta_1,\mu)$ implies $f(z) \in S_p^*(0,1,1)$, we have

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq p.$$

Therefore

$$(p+n)a_{p+n} \leq p, \text{ for all } n=1,2,\cdots, p \in N \tag{5.1}$$

Similarly we have

$$(p+n)b_{p+n} \leq p, \text{ for all } n=1,2,\cdots\cdots, p \in N \tag{5.2}$$

In order to establish the required result we need to show that $f*g(z)$ belongs to both $C_p^*(\alpha_1,\beta_1,\mu)$ and $C_p^*(\alpha_2,\beta_2,\mu)$.

By using (5.2) we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(p+n)\{n+\beta_1[\mu n+(1+\mu)\ (p-\alpha_1)]\}a_{p+n}\ b_{p+n} \\
&\qquad \leq \sum_{n=1}^{\infty} p\{n+\beta_1[\mu n+(1+\mu)\ (p-\alpha_1)]\}a_{p+n} \\
&\qquad \leq (1+\mu)\beta_1(p-\alpha_1)p, \text{ since } f(z) \in S_p^*(\alpha_1,\beta_1,\mu).
\end{aligned}$$

This shows that $f*g(z) \in C_p^*(\alpha_1,\beta_1,\mu)$

Similarly, by using(*5.1*), we can show that $f*g(z) \in C_p(\alpha_2, \beta_2, \mu)$.

Hence $f*g(z)$ belongs to $C_p^*(\alpha_1,\beta_1,\mu) \wedge C^*_p(\alpha_2,\beta_2,\mu)$.

Corollary 5. Let the functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\ z^{p+n} (a_{p+n} \geq 0,\ p \in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\ z^{p+n}(b_{p+n}\geq 0,\ p\in P)$$

be in the same class $S_p^*(\alpha,\beta,\mu)$. Then the Hadamard product $f^*g(z)$ belongs to the class $C_p^*(\alpha,\beta,\mu)$.

Theorem 11. Let the functions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}(a_{p+n}\geq 0,\ p\in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\ z^{p+n}(b_{p+n}\geq 0,\ p\in N)$$

be in the classes $C_p^*(\alpha_1,\beta_1,\mu)$ and $C_p^*(\alpha_2,\beta_2,\mu)$, respectively. Then the Hadamard product $f*g(z)$ is in the class $C_p^*(\alpha,\beta,\mu)$, where $\alpha=$ Min$(\alpha_1,\ \alpha_2)$ and $\beta=$Max $(\beta_1,\ \beta_2)$.

Proof The proof is similar to the proof of [7, Theorem 1].

Since $f(z)\in C_p^*(\alpha_1,\beta_1,\mu)$ and $g(z)\in C_p^*(\alpha_2,\beta_2,\mu)$, by using Theorem 2,

$$\sum_{n=1}^{\infty}(p+n)\{n+\beta_1[\mu n+(1+\mu)\ (p-\alpha_1)]\}a_{p+n}$$
$$\leq(1+\mu)\ \beta_2(p-\alpha_2)p.$$

and

$$\sum_{n=1}^{\infty}(p+n)\{n+\beta_2[\mu n+(1+\mu)\ (p-\alpha_2)]\}b_{p+n}$$
$$\leq(1+\mu)\ \beta_2(p-\alpha_2)p.$$

Hence we have

$$\sum_{n=1}^{\infty}a_{p+n}\leq\frac{(1+\mu)\beta_1(p-\alpha_1)p}{(p+1)\{1+\beta_1[\mu+(1+\mu)\ (p-\alpha_1)]\}}<1$$

and

$$\sum_{n=1}^{\infty}b_{p+n}\leq\frac{(1+\mu)\beta_2(p-\alpha_2)p}{(p+1)\{1+\beta_2[\mu+(1+\mu)\ (p-\alpha_2)]\}}<1$$

Therefore,

$$\sum_{n=1}^{\infty}(p+n)\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n}\ b_{p+n}$$

$$\leq Max\ \{\sum_{n=1}^{\infty}(p+n)\ \{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}a_{p+n},$$

$$\sum_{n=1}^{\infty}(p+n)\{n+\beta[\mu n+(1+\mu)\ (p-\alpha)]\}b_{p+n}\}$$

$$\leq(1+\mu)\beta(p-\alpha)p,$$

where $\alpha=Min(\alpha_1,\alpha_2)$ and $\beta=Max(\beta_1,\beta_2)$ Consequently, the Hadamard product $f*g(z)$ belongs to the class $C_p^*(\alpha,\beta,\mu)$ by Theorem 2.

Corollary 6. Let the functions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}(a_{p+n}\geq 0,\ p\in N)$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}\ z^{p+n}(a_{p+n}\geq 0,\ p\in N)$$

be in the same class $C_p^*(\alpha,\beta,\mu)$. then the Hadamard product $f*g(z)$ is also in the class $C_p^*(\alpha,\beta,\mu)$.

6. Fractional calculus.

There are many definitions of the fractional calculus, that is, the fractional derivative and the fractional integral. In 1978, Owa [6] gave the following definitions for the fractional calculus.

Definition 1. The fractional integral of order kis defined by

$$D_z^{-k}f(z)=\frac{1}{\Gamma(k)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-k}},$$

where $k>0$, $f(z)$ is an analytic function in a simply connected region of the z–plasne containing the origin and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition 2. The fractional derivative of order kis defined by

$$D_z^k f(z)=\frac{1}{\Gamma(1-k)}\frac{d}{dz}\int_0^z \frac{f(\zeta)\zeta}{(z-\zeta)^k}$$

where $0\leqq k<1$, $f(z)$ is an analytic function in al simply connected region of the z–plane containing the origin and the multiplicity of $(z-\zeta)^{-k}$ is removed by requiring log $(z-\zeta)$ to be real when $(z-\zeta)>0$

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order (n+k) is defined by

$$D_z^{n+k} f(z)=\frac{d^n}{d_z^n} D_z^k f(z),$$

where $0\leqq k<$ and $n\in N\cup\{0\}$

Theorem 12. Let a function

$$f(z)=z^p-\sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} (a_{p+n}\geq 0,\ p\in N)$$

be in the class $S_p^*(\alpha,\beta,\mu)$. Then we have

$$\left|D_z^{-k} f(z)\right|\geq \frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k}\times$$

$$\left\{1-\frac{p+1}{p+1+k}\cdot\frac{(1+\mu)\beta(p-\alpha)}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|\right\}$$

and

$$\left|D_z^{-k} f(z)\right|\leq \frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k}\times$$

$$\left\{1+\frac{p+1}{p+1+k}\cdot\frac{(1+\mu)\beta(p-\alpha)}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|\right\}$$

for $0<k<1$ and $z\in U$ The result is sharp.

Proof. Let

$$F(z)=\frac{\Gamma(p+1+k)}{\Gamma(p+1)}\, z^{-k}D_z^{-k}f(z)$$

$$=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\ \Gamma(p+1+k)}{\Gamma(p+n+1+k)\ \Gamma(p+1)}\, a_{p+n}\ z^{p+n}$$

$$=z^p-\sum_{n=1}^{\infty}A(n)\ a_{p+n}\ z^{p+n},$$

where

$$A(n)=\frac{\Gamma(p+n+1)\Gamma(p+1+k)}{\Gamma(p+n+1+k)\ \Gamma(p+1)}\ (n\geq 1)\cdot$$

Since

$$0<A(n)\leq A(1)=\frac{(p+1)}{(p+1+k)},$$

we have, with the help of Theorem 1,

$$\left|F(z)\right|\geqq|z|^p-A(1)\,|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}$$

$$\geqq|z|^p-\frac{(p+1)}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)}{\{1+\beta[\mu+(1+\mu)(p-\alpha)]\}}\,|z|^{p+1}$$

and

$$\left|F(z)\right|\leqq|z|^p+A(1)\,|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}$$

$$\leqq|z|^p+\frac{(p+1)}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)}{\{1+\beta[\mu+(+\mu)\ (p-\alpha)]\}}\,|z|^{p+1}$$

which prove the inequalities of Theorem 12. Further, equalities are attained for the function

$$D_z^{-k}f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1+k)}\cdot z^{p+k}\times$$

$$\{1-\frac{(p+1)}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)}{\{1+\beta[\mu+(+\mu)\ (p-\alpha)]\}}\,z\},$$

or

$$f(z)=z^{p}+\frac{(1+\mu)\beta(p-\alpha)}{1+\beta[\mu+(+\mu)\ (p-\alpha)]}z^{p+1}$$

Thus we complete the proof of Theorem 12.

Corollary 7. Under the hypotheses of Theorem 12, $D_z^{-k}f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)}\left\{\frac{(p+1+k)\ (1+\beta+\mu)+(1+\mu)\beta(p-\alpha)\ (2p+2+k)}{(p+1+k)\ \{1+\beta[\mu+(1+\mu)\ (p+\alpha)]\}}\right\}$$

Using Theorem 2, we have

Theorem 13. Let a function

$$f(z)=z^{p}-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}(a_{p+n}\geqq 0,\ p\subset N)$$

be in the class $C_p^*(\alpha,\beta,\mu)$. Then we have

$$\left|D_z^{-k}\ f(z)\right|\geq\frac{\Gamma(p+1)}{\Gamma(p+1+)}|z|^{p+k}\times$$

$$\left\{1-\frac{1}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|\right\}$$

and

$$\left|D_z^{-k}\ f(z)\right|\leq\frac{\Gamma(p+1)}{\Gamma(p+1+)}|z|^{p+k}\times$$

$$\left\{1-\frac{1}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|\right\}$$

for $o<k<1$ and $z\in U$ The result is sharp for the function

$$f(z)=z^{p}-\frac{(1+\mu)\beta(p-\alpha)p}{(p+1)\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}z^{p+1}.$$

Corollary 8. Under the conditions of Theorem 13, $D_z^{-k} f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)}\left\{\frac{(p+1+k)\ (1+\beta\mu)+(1+\mu)\beta(p+\alpha)\ (2p+1+k)}{(p+1+k)\ \{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}\right\}$$

Finally, we derive

Theorem 14. Let a function

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\ z^{p+n}\ (a_{p+n}\geq 0,\ \ p\in N)$$

be in the class $C_p^*(\alpha,\beta,\mu)$. Then we have

$$\left|D_z^k f(z)\right|\geq\frac{\Gamma(p+1)}{\Gamma(p+1-k)}\left|z\right|^{p-k}\times$$

$$\left\{1-\frac{1}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}\left|z\right|\right\}$$

and

$$\left|D_z^k f(z)\right|\leq\frac{\Gamma(p+1)}{\Gamma(p+1-k)}\left|z\right|^{p-k}\times$$

$$\left\{1+\frac{1}{(p+1+k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}\left|z\right|\right\}$$

for $0\leq k<1$ and $z\in U$ The result is sharp.

Proof. Let

$$G(z)=\frac{\Gamma(p+1-k)}{\Gamma(p+1)}z^k D_z^k f(z)$$

$$=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\ \Gamma(p+1-k)}{\Gamma(p+n+1-k)\ \Gamma(p+1)}a_{p+n}\ a^{p+n}$$

$$=z^p-\sum_{n=1}^{\infty}(p+n)\ \ B(n)\ \ a_{p+n}\ z^{p+n},$$

where

$$B(n)=\frac{\Gamma(p+n)\ \Gamma(p+1-k)}{\Gamma(p+n+1-k)\ \Gamma(p+1)} \qquad (n\geqq 1).$$

Noting

$$o<B(n)\leq B(1)=\frac{1}{(p+1-k)},$$

with Theorem 2, we have

$$\left|G(z)\right| \geq |z|^{p}-B(1)|z|^{p+1}\sum_{n=1}^{\infty}(p+n)a_{p+n}$$

$$\geq |z|^{p}-\frac{1}{(p+1-k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|^{p+1}$$

and

$$\left|G(z)\right| \leq |z|^{p}+B(1)|z|^{p+1}\sum_{n=1}^{\infty}(p+n)a_{p+n}$$

$$\leq |z|^{p}+\frac{1}{(p+1-k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}|z|^{p+1}$$

which give the inequalities of Theorem 14. Since equalities are attained for the function $f(z)$ defined by

$$D_z^k f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-k)}z^{p-k}\times$$

$$\left\{1-\frac{1}{(p+1-k)}\cdot\frac{(1+\mu)\beta(p-\alpha)p}{\{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}z\right\}$$

that is, by

$$f(z)=z^{p}-\frac{(1+\mu)\beta(p-\alpha)p}{(p+1)\ \{1+\beta[\mu+(1+\mu)\ (p-\alpha)]\}}z^{p+1},$$

we complete the assertion of Theorem 14.

Corollary 9. Under the conditions of Theorem 14, $D_z^k f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1-k)} \times$$

$$\left\{ \frac{(p+1-k)\ (1+\beta\mu)+(1+\mu)+(1+\mu)\beta(p-\alpha)\ (2p+1-k)}{(p+1-k)\ \{1+\beta[\mu+(1+\mu)\ \beta(p-\beta)]\}} \right\}.$$

Acknowledgement.

I would like to thank professor Dr. shigeyoshi Owa for his kind encouragement and valuable remarks in preparing this paper.

References

1. O.P. Ahuja and P.K. Jain, "On starlike and convex functions with missing and negative coefficients", Bull. Malays. Math. Soc. 3(1980), 95–101.
2. R.M. Goel and N.S. Sohi, "Multivalent functions with negative coefficicients", Indian J. Pure Appl. Math. 12(1981), 844–853.
3. V.P. Gupta and P.K. Jain, "Certain classes of univalent functions with negative coefficients II", Bull. Austral. Math. Soc. 15(1976), 467–473.
4. V. Kumar, "Hadamard product of certain starlike functions", J. of Math. Analysis and Appl. 110(1985), 425–28.
5. T.V. Lakshminarasimhan, "On subclasses of functions starlike in the unit disc", J. of the unit disc", J. of the Indian Math. Soc., 41(1977), 233–243.
6. S.Owa, "On the distortion theorems I", Kyungpook Math. J., 18(1978), 53–59.
7. S. Owa, "On the classes of univalent functions with negative coefficients", Math. Japonica 27, No. 4(1982), 409–416.
8. S. Owa, "On certain subclasses of analytic p–valent functions", J. Korean Math. Soc. Vol. 20, No. 1(1983), 41–58.
9. S. Owa and M.K. Aouf, "On subclasses of univalent functions with negative coefficients", The Pusan–Kyongnam Math. J. (To appear).
10. S. Owa and M.K. Aouf, "On subclasses of univalent functions with negative coefficients II", Pure Appl. Math.Sci.(To appear).
11. K. S. Padmanabhan, "On certain classes of starlike functions in he unit disc", J. Indian Math. Soc., Vol. XXXII (1968), 89–103.
12. H. Silverman, "Univalent functions with negative coefficients", Proc. Amer. Math. Soc. 51 (1975), 109–116.
13. H. Silverman and E.M. Silvia, "Convex families of starlike functions", Houstion J. Math. 4(1978), 263–268

14. Silverman and E.M. Silvia, "Prestarlike with negative coefficients", International J. Math Sci. 9(1979), 427–439.
15. R. Singh, "On a class of starlike functions", J. Indian Math, Soc., Vol. XXXII (1969), 207–213.
16. H.M. Srivastava and S.Owa, "An application of the fractional derivative", Math. Japonica 29 (1984), 383–389.

Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura, Egypt