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A Bayesian Approach for Solving Goal Programs Having Probabilistic Priority Structure

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Abstract

This paper concerns with the case of having a goal program with no preassigned deterministic ranking for the goals. The priority ranking in this case depends on the states of nature which are random variables. The Bayesian approach is performed to obtain the nondominated set of rankings.

1. Introduction

Consider the general form of the goal program :

$$\text{Min } Z = \sum_{k=1}^{K} P_k g_k (d_i^-, \ d_i^+) \tag{1.1}$$

$$\text{S.T. } f_i(X) + d_i^- - d_i^+ = b_i, \quad i=1,2,\cdots,m \tag{1.2}$$

$$X \geqq 0, \ d_i^-, \ d_i^+ \geqq 0, \quad i=1,2,\cdots,m \tag{1.3}$$

$$d_i^- * d_i^+ = 0, \quad i=1,2,\cdots,m \tag{1.4}$$

where X is the vector of the decision variables, its dimension is $n*1$.

$f_i(X)$ is a real valued function in the decision variables (linear or nonlinear).

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d_i^-, d_i^+ are the negative and positive deviational variables, respectively, for the i-th goal.

$g_k(d_i^-, d_i^+)$ are real valued functions in deviational variables of the goal that has a priority level k.

K is the number of priority levels.

To solve the program (1.1)−(1.4) when $g_k(d_i^-, d_i^+)$ and $f_i(X)$ are linear functions one can use any of the modified simplex method (5) or the iterative approach (1, 3, 4), if $g_k(d_i^-, d_i^+)$ and/or $f_i(X)$ are nonlinear then we can use the available methods in (1, 2, 3, 8). In all these solution methods it is assumed that the decision maker can assign a priority level for each goal of his conflict goals. Lee (5) presented a multiple comparison approach to assist the decision maker choosing his ranking. Ignizio (3) studied the sensitivity analysis of the priority structure. In this paper a Bayesian approach is presented to treat the case of having a goal program with random priority structure.

Assuming that the decision maker is not sure of the rankings of his K goals; instead his rankings for these goals is dependent on the states of nature, which state of nature will occur? the decision maker does not know; but he knows a prior probability mass function of the states of nature, also he can draw a sample from the states of nature and get a posterior mass function for the states of nature. The following two questions will be answered in this papr;

(1) what is the best rank for goals in the case of prior information only?

(2) what is the best rank for goals in the case of posterior information?

Thes two questions will be answered in the following two sections, section 2 and section 3, in section 4 an illustrative example is given to clarify the idea behind the suggested approach.

2. The Best Rank of Goals Using Prior Information

Assuming that we have R states of nature ($S=\{s_1, s_2, \ldots, s_r, \ldots, s_R\}$) with prior mass function $f(S)$, N combinations of rankings of the K goals ($D=\{d_1, d_2, \ldots, d_j, \ldots,$

$d_N\}$) and for every state of nature S_r ($r=1, 2, \ldots, R$) the decision maker can determine the preferred rank of goals (the decision maker assigns for every element in the set S only one element in the set D), suppose that this rerationship between S and D is represented by function G:

$$G: S \rightarrow D = 0, \text{ if } d_j \text{ is not preferred for } s_r$$
$$= 1, \text{ if } d_j \text{ is preferred for } s_r$$
$$(j=1, 2, \ldots, N), \ r=(1, 2, \ldots, R) \qquad (2.1)$$

Thus G could be represented in a matrix as follow:

G =

	s_1	s_2	s_r	......	s_R
d_1	g_{11}	g_{12}	g_{1r}	......	g_{1R}
d_2	g_{21}	g_{22}	g_{2r}	......	g_{2R}
.	.	.	.		.
.	.	.	.		.
d_j	g_{j1}	g_{j2}	g_{jr}	......	g_{jR}
.	.	.	.		.
.	.	.	.		.
d_N	g_{N1}	g_{N2}	g_{Nr}	......	g_{NR}

Each element is either 0 or 1 and the sum of each column is 1. To get the best (nondominated) rank in the case of prior information, the following technique is suggested:

Step 1: Solve N goal programs, the first with the objective (achievement) function built according to the d_1 ranking, the second with the objective (achievement) function built according to d_2, and so on. In this step one should use the sensitivity analysis of goal programming (the case of reordering priority levels) to reduce the computation efforts.

Step 2: let the value of $g_k(d_i^-, d_i^+)$ ($k=1, 2, \ldots, K$) for the optimal solution in the case of using the rank d_j ($j=1, 2, \ldots, N$) to be $G_k^*(j)$, and define the vector $A(j)$ as:

$$A(j) = (g_1{}^*(j),\ g_2{}^*(j),\ \cdots,\ g_k{}^*(j),\ \cdots,\ g_K{}^*(j))$$
$$j = 1,\ 2,\ \cdots,\ N \qquad (2.2)$$

Step 3 : define each of $N*R$ vectors contains K elements and let the first vector to be $h(1, 1)$ which represents the losses in the case of using d_1 then s_1 occurs, the elements of $h(1, 1)$ will be all zeros in case of having d_1 as the suitable rank for s_1, if d_1 is not the preferred rank for s_1 then $h(1, 1) = A(1)$. In general $h(j, r)$ could be written as :

$$\begin{aligned} h(j, r) &= 0 \qquad \text{if } g_{jr} = 1 \\ &= A(j) \ \text{ if } g_{jr} = 0\ ,\ j=1,\ 2,\ \cdots,\ N \\ &\qquad\qquad\qquad\quad r=1,\ 2,\ \cdots,\ R \end{aligned} \qquad (2.3)$$

The unique exception for this definition is that when two or more rankings have the same optimal solution, then they must have the same losses function.

Hence the expected value of the risk when we use the d_j rankings could be written as :

$$B(j) = \sum_{r=1}^{R} h(j, r) * f(s_r),\ \ j=1, 2,\ \cdots,\ N \qquad (2, 4)$$

Step 4 : choose the one(s) with $B(j)$ is nondominated from set D, this(these) is(are) the best ranking(s).

Definition : $B(j)$ is said to be nondominated if there does not exist $B(l)$, $l=1,\ 2,\ \ldots,$ N such that $B(l) < B(j)$ and at least one element in $B(l)$ does not equal its corresponding element in $B(j)$

In case of having more than one best ranking the decision maker could pick up any one of them arbitrarily or according to his risk function, let the preferred nondominated $B(j)$ to be B^*. Since B^* represents the expected value of losses when using the best rank (i.e., when taking the best action), it also represtents the expected value of perfect information (EVPI).

3. The Best Rank of Goals Using Posterior Information

Assuming that the decision maker is not confident in the prior information, also he can pay for drawing a sample from S to know about S. Let the set $V(V=\{v_1, v_2, \ldots, v_w, \ldots, v_R\})$ represent results of the sample (V is an estimate for S from the sample observations) and $p(v_w, s_r)$ ($w=1, 2, \ldots, R$ and $r= 1, 2, \ldots, R$) is the conditional probability mass function of v_w given s_r (i, e., the indicator of the reliablity of the sample results).

The posterior probability mass function of s_r is :

$$f_1(s_r|v_w) = \sum_{w=1}^{R} p(v_w|s_r) \ / \ p(v_w) \tag{3.1}$$

where $p(v_w) = \sum_{r=1}^{R} p(v_w|s_r) \ * \ f(s_r)$, $w = 1, 2, \cdots, R$

$r = 1, 2, \cdots, R$

The same steps of determining the best rank in case of prior information (as presented in section 2) must be followed to get the best rank in the case of getting a posterior information except using $f_1(s_r|v_w)$ instead of $f(s_r)$ in equation (2.4).

A most important point to be studied, after the previous analysis is the posterior analysis, that is the analysis that makes the decision maker be able to know if the information obtained from sample is worthy to pay for it. To know how extent the sampling process is beneficial the expected value of the sampling information (EVSI) must be calculated and it is up to the decision maker to decide if this value is worthy. Before drawing the sample one can not know either of v_w ($w=1, 2, \ldots, R$) will be observed, so he also can not guess the preferred rank of goals or its associated EVPI (B^*). Since B^* here is dependent on v_w, let it to be $B^*(w)$. It is well known that EVSI equals EVPI−expected value of losses in case of drawing the sample, so :

$$EVSI=EVPI-B^*(W) * P(V_w) = B^* - B^{**} \tag{3.2}$$

4. An Illustrative Numerical Example

This example is taken—with modification—from Lee' book (6) about product mix problem, also the MICRO MANAGER Package is used to solve the goal programs generated by applying the first step of the suggested approach.

A company of electronics produces color TV sets. The company has two production lines, the production rates of these two lines are 2 and 1.5 sets per hour respectively. The production capacity is 40 hours per week for each line. The manager wants to determine the number of hours to run each line during a week to achieve the following goals :

(1) Meet a production of 240 sets for a week.

(2) Avoid the underutilization of regular working hours for both lines.

(3) Limit the sum of overtime operation for both lines.

The manager is not sure of the states of market, so he is not sure of his rankings for these goals, but if the demand will be high he prefers the goals to be ranked as they are appeared in the sequence above. If the demand is moderate he prefers the ranking to be, the second goal has the first priority, the third goal has the second priority, and the first goal has the third priority. If the demand is low he prefers the goals to be ranked according to their priorities as the third then the second then the first. The prior distribution of the demand shows that it is high with the probability of 0.3, it is moderate with the probability of 0.5 and it is low with the probability of 0.2.

The manager can apply a consultation from a marketing survey office for $10,000 charge about the predicted state of the demand. The predictions of this office is not always true. The following table shows the reliability of the predictions of this office (i.e., how extent these predictions were true).

	High	Moderate	Low
High	0.7	0.3	0.1
Moderate	0.2	0.6	0.4
Low	0.1	0.1	0.5

Now the manager wants to know the number of working hours for each line by using the prior information only and where it is profitable to apply the consultation from marketing survey office or not.

The Solution···

Here we have 3 possible rankings, i. e. :

$D = \{d_1\ d_2\ d_3\}$

where $d_1 = \{1\ 2\ 3\}$; $d_2 = \{2\ 3\ 1\}$; $d_3 = \{3\ 2\ 1\}$ and 3 states of nature :

S = {high moderate low}

According to the information in the example the matrix G is :

G =

	High	Moderate	Low
d_1	1	0	0
d_2	0	1	0
d_3	0	0	1

Now by applying the suggested four steps in this paper, we can see :

Step 1 : the following table shows the solutions of the 3 goal programs.

D	Optimal solution	Achievement function
d_1	(90 , 40)	(0 0 50)
d_2	(40 , 40)	(0 0 100)
d_3	(40 , 40)	(0 0 100)

Step 2 : in this step we rewrite the values of the achievement function by rearranging its elements to be in the following disciplines : the first element is assigned for the underachievement in the first gial, the second element represents the underachievement in the second goal and the third element is to represent the overachievment in the third goal, $A(1) = (0\ 0\ 50)$, $A(2) = (100\ 0\ 0)$ and $A(3) = (100\ 0\ 0)$.

Step 3 : in this step we present the loss function (the opportunity cost), it is a vector function $(h(j, r))$ represents the cost of using the j−th ranking in the r−th state of nature.

This cost is 0 if the j—th ranking is the preferred one for this state of nature, so :

$$h(1,1) = (0\ 0\ 0),\ h(1,2) = (0\ 0\ 50),\ h(1,3) = (0\ 0\ 50)$$
$$h(2,1) = (100\ 0\ 0),\ h(2,2) = (0\ 0\ 0),\ h(2,3) = (0\ 0\ 0)$$
$$h(3,1) = (100\ 0\ 0),\ h(3,2) = (0\ 0\ 0),\ h(3,3) = (0\ 0\ 0)$$

Note that, since d_2 and d_3 have same optimal solution then they have the same losses function. Also in this step the expected value of the losses for each ranking—that is B(j)—is calculated by using the formula (2.4) :

$B(1) = (0\ 0\ 35)$, $B(2) = (30\ 0\ 0)$ and $B(3) = (30\ 0\ 0)$

These expected values of the losses is presented to the manager to choose the best one, now if the manager chooses B(2) — according to some criteria or according to his satisfication with this result and his ability to accept an average losses in his sales (30 units) more than overutilization in line's work hours (35 hours in average)—then : $B^* = (30\ 0\ 0) = EVPI$, and $x^* = (40\ 40)$ is the optimal solution in case of using prior information only.

Now the manager wants to know whether to buy the Marketing Survey Office's consultation or not, so we are going to make a preposterior analysis for this problem. By using the information about prior probability mass function of the demand and the conditional probability function of the predictions of the office, it is easy to get the following results : (1) In the case of getting the consultation from the office and the prediction of the office for the demand is "High" then the posterior probability mass function for the demand by using the formula (3.1) is :

S	High	Moderate	Low
$f_1(s)$	0.55	0.4	0.05

and by using these posterior probabilites to calculate the expected losses for each optimal solution of this problem we get the following B(j)'s :

$B(1) = (0\ 0\ 22.5)$, $B(2) = (55\ 0\ 0)$ and $B(3) = (55\ 0\ 0)$

Again, assume that the manager prefers B(1) then :

$B^*(1) = (0\ 0\ 22.5)$ and $X^* = (90\ 40)$

(2) In case of getting the consultation of the office and the prediction is "Moderate", by using the same approach as in (1) we get:

$B^*(2) = (14\ 0\ 0)$ and $X^* = (40\ 40)$

(3) Finally, if the manager gets the consultation from the office and the prediction is "Low" then:

$B^*(3) = (17\ 0\ 0)$ and $X^* = (40\ 40)$

By calculating the probability mass function of the predictions of the office we get:

Prediction (w)	High	Moderate	Low
P(w)	0.38	0.44	0.18

and, the expected value of losses in case of getting the consultation of the office is:

$B^{**} = B^*(w) * p(w) = (9.22\ 0.\ 8.55)$

and, the expected value of getting the consultation is :

$EVSI = B^* - B^{**} = (20.78\ 0.\ -8.55)$

It is up to the manger to decide whether this EVSI is worthy to pay $10,000 for the office or not.

5. Conclusion

This paper is the first trial in discussing the case of having random priority structure. The paper covers the discrete case, i.e., the case when the states of nature are discrete random variables. The case of having continuous random variables has not touched up yet. The author intends to complete the work in this field and to try it in some real applications.

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