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# Generalized Replacement Demand Forecasting to Complement Diffusion Models

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**Abstract**

Replacement demand plays an important role to forecast the total demand of durable goods, while most of the diffusion models deal with only adoption data, namely initial purchase demand. This paper presents replacement demand forecasting models incorporating repurchase rate, multi-ownership, and dynamic product life to complement the existing diffusion models. The performance of replacement demand forecasting models are analyzed and practical guidelines for the application of the models are suggested when life distribution data or adoption data are not available.

KEY WORDS: Diffusion model; Initial purchase demand; Replacement demand; Total demand; Stock; Adoption; Life distribution; Repurchase rate; Multi-ownership; Dynamic product life; Average life

Diffusion models have been widely used in the demand forecasting of durable goods, which can be justified for adoption data (initial purchase data). Directly applying the diffusion models to total demand data causes conceptual difficulties due to diffusion property. To be conceptually meaningful, replacement demand forecasting model needs to be supplemented to the diffusion models.

Dodson and Muller (1978), Lilien et al (1981), and Harrell and Taylor (1981) suggested replacement rate method to forecast replacement demand. Brown (1965) presented the method of using death rate by product age. Spencer et al (1961), Lawrence and Lowton (1981), Olson and Choi (1985), Kamakura and Balasubramanian (1987) suggested the

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method based on product life distribution. The limitations of the existing replacement demand forecasting models can be attributed to their assumptions about perfect repurchase rate, single-ownership, and static product life. Some owners do not repurchase when the products-in-use are scrapped. Repurchase rate will change for the non-necessity goods. For some consumer durable goods (e.g. automobiles, television sets, radios, refrigerators, etc.) and nondurable goods, household often owns more than one unit. The life of product can be changed by technological advance and change of habit, etc.

Actual life distribution of product is often not available and hence Spencer et al (1961) and Lawrence and Lowton (1981) assumed normal distribution and single point distribution, respectively. Olson and Choi (1985) estimated the parameters of life distribution by assuming Rayleigh distribution and combining the replacement demand forecasting model with diffusion model. Kamakura and Balasubramanian (1987) complemented Olson and Choi's research for the case of unknown adoption data by assuming h-type distribution. Even though these models combine replacement demand forecasting with diffusion model, they are static and assumed single ownership.

A model incorporating repurchase rate, multi-ownership, and dynamic product life, is presented in this paper. Accuracies by each replacement demand forecasting model and life distribution type are analyzed.

## Replacement Demand Forecasting Model

The procedure to forecast total demand by diffusion model which accomodates replacement demand is as follows.

$$\hat{TD}_t = \hat{ND}_t + \hat{RD}_t \,, \; t = T+1, T+2, \ldots \quad (1)$$
$$\hat{ND}_t = \hat{F}_t - \hat{F}_{t-1} \,, \; t = T+1, T+2, \ldots \quad (2)$$

where

$TD_t$ : total demand at period t

$ND_t$ : new demand or initial purchase demand at period t

$RD_t$ : replacement demand at period t

$\hat{F}_t$ : cumulative initial purchasers or number of adopters at the end of period t, which is forecasted by diffusion models. $\hat{F}_{t-1}$ is the observed value at $t=T+1$

T : present period

Most diffusion models and replacement demand forecasting models assume explicitly or implicitly that the consumer always repurchases the new one when the product-in-use is obsolete, i.e., repurchase rate is 1. This implies that number of adopters $F_t$ becomes the number of owners. They also assume that the consumer owns only one unit, which in turn means that the number of owners becomes the amount of stock $S_t$.

In such case,

$$S_t = F_t \quad (3)$$

where $S_t$ : amount of stock at the end of period t.

Stock data $S_t$ is often available relative to $F_t$ and used to estimate diffusion models.

**Replacement Rate Method**

A portion of total stock at the end of period $(t-1)$ is scrapped during the period t, and some of the owners who have scrapped the product repurchase a new one.

$$\hat{RD}_t=\hat{S}_{t-1}\cdot\hat{R}_t,\ t=T+1,\ T+2,\ \ldots \qquad (4)$$

where

$R_t=P_t\cdot K_t$ : replacement rate at period t

$P_t$ : scrappage rate at period t

$K_t$ : repurchase rate among the owners who have scrapped the product at period t, $0<K_t<1$

If the value of repurchase rate $K_t$ equals to 1, the number of adopters $F_t$ becomes stock $S_t$. The diffusion model forecasts future stock $S_t$, which is used in turn to forecast new demand and replacement demand.

The scrappage rate $P_t$ is usually estimated from the average life $M_t$ (see appendix A).

$$\hat{R}_t=\hat{P}_t\cdot\hat{K}_t=\hat{P}_t=1/\hat{M}_t,\ t=T+1,\ T+2,\ \ldots \qquad (5)$$

If $K_t$ is not equal to 1, we forecast stock $S_t$ as follows.

$$\hat{S}_t=\hat{S}_{t-1}+\hat{ND}_t-\hat{S}_{t-1}\cdot\hat{P}_t\cdot(1-K_t)$$
$$=\hat{S}_{t-1}\,(1-\hat{P}_t\cdot(1-\hat{K}_t))+\hat{ND}_t,\ t=T+1,\ T+2,\ \ldots \qquad (6)$$

where $\hat{S}_{t-1}$ is the observed value at $t=T+1$.

Repurchase rate $K_t$ can be estimated from market research and replacement rate $R_t$ can be derived from $P_t$ and $K_t$.

**Life Distribution Method**

Product introduced to market in a specific period would be scrapped with passage of time, namely life. If we get the scrappage probability by life, life distribution, replacement demand forecasting model by life distribution can be given as follows. We assume that there are no scrappage at introduction period.

$$\hat{RD}_t=\sum_{j=1}^{t-1}TD_{t-j}\cdot\hat{f}_j^p\cdot\hat{K}_t,\ t=T+1,\ T+2,\ \ldots \qquad (7)$$

where

$f_j^p$ : the scrappage probability of product at life j

p : estimation period

$p=t-j$, if life distribution is predicted at introduction period $(t-j)$ (use appendix A)

$p=t$ , if life distribution is estimated at forecasting period t.

**Death Rate Method**

A portion of total stock at the end of period $(t-1)$ is scrapped during the period t, but the scrappage probability of each product depends on its age. If we divide these scrapped products into their age group, we can get the death rate by age.

$$\hat{RD}_t=\sum_{j=1}^{t-1} Q_{t-1}(j-1)\cdot\hat{d}_j\cdot\hat{K}_t,\ t=T+1,\ T+2,\ \ldots \qquad (8)$$

where

$Q_{t-1}(j-1)$ : number whose age is $(j-1)$ at the end of period $(t-1)$

$d_j$ : death rate at age j

Death rate method can be viewed as segmenting the replacement rate method by product age group.

$$S_{t-1}=\sum_{j=1}^{t-1} Q_{t-1}(j-1) \qquad (9)$$

To compare life distribution method and death rate method, we can see from definition of $d_j$ and $Q_{t-1}(j-1)$,

$$\hat{d}_j=\frac{\text{Pro. (the life is j)}}{\text{Pro. (the life is above j)}}=\frac{f_j}{1-\sum_{k=1}^{j-1} f_k} \qquad (10)$$

$$Q_{t-1}(j-1)=TD_{t-j}\,(1-\sum_{k=1}^{j-1} f_k). \qquad (11)$$

From eq. 8,

$$\hat{RD}_t=\sum_{j=1}^{t-1} TD_{t-j}\cdot(1-\sum_{k=1}^{j-1} f_k)\cdot\hat{f}_j/(1-\sum_{k=1}^{j-1} f_k)\cdot\hat{K}_t$$

$$=\sum_{j=1}^{t-1} TD_{t-j}\cdot\hat{f}_j\cdot\hat{K}_t,\ t=T+1,\ T+2,\ \ldots \qquad (12)$$

From equations 7 and 12, it is shown that two methods are basically identical. The defect of death rate method is that its forecasting lead time is restricted to only one period, while diffusion model is intended for long term forecasting.

For multi-ownership forecasting, two situations can be expected: one is that a consumer buys one unit per purchase as for most consumer durable goods; the other is that a consumer buys more than one unit per purchase as for nondurable goods for which growth curves are sometimes used. For the former case, the multi-ownership problem can be solved only by rearranging the saturation level in diffusion models if we change the concept of $F_t$ from the number of adopter to the number of penetration. For example, the saturation level for television sets can be changed from one unit to two or three units per household. For the latter case, initial purchase demand, replacement demand, total demand, and stock are multiplied by multiple buying except number of adopter $F_t$. That is

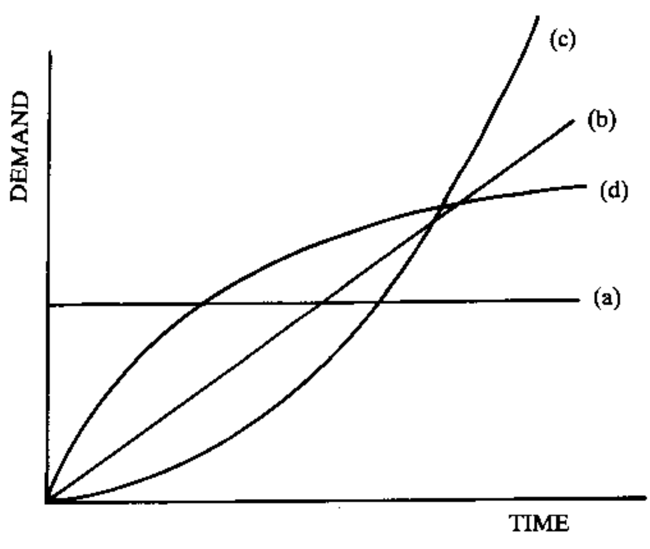
$$\hat{ND}_t=(\hat{F}_t-\hat{F}_{t-1})\cdot E(N),\ t=T+1,\ T+2,\ \ldots \qquad (13)$$

$S_t = F_t \cdot E(N)$ ············································································ (14)

where E (N) : expected number of units bought per purchase.

## Analysis of Replacement Demand Forecasts

The forecasting results by life distribution method and replacement rate method are presented to show the forecasting accuracies of replacement demand forecasting models with respect to different demand patterns. Four typical demand patterns which cover the periods from introduction to growth stages of the most product life cycle (Cox, 1967; Rink and Swan, 1979; Tellis and Crawford, 1981) are used in this paper as shown in Exhibit 1.




(a) : $TD_t = a$
(b) : $TD_t = bt$
(c) : $TD_t = bt^2$
(d) : $TD_t = -b\ (t-a)^2 + ba^2$

Exhibit 1. Four demand patterns from PLC

If life distribution is known, replacement demand and corresponding replacement rate can be derived from total demand patterns by the following equations which are continuous forms of the former descrete type equations.

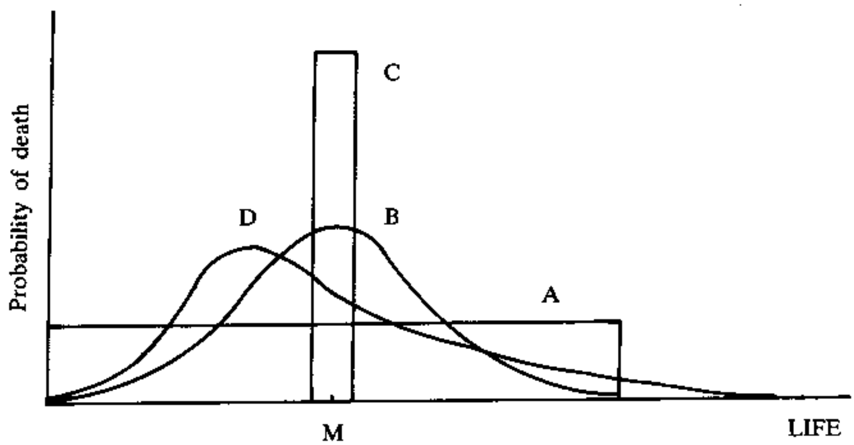
$$RD_t = \int_0^t f(t-T) \cdot TD_T \cdot dT, \; t<L \quad \cdots\cdots (15)$$
$$= \int_{t-L}^{t} f(t-T) \cdot TD_T \cdot dT, \; t>$$

where f (t) : life distribution

$K_t=1$, single-ownership, and static life distribution are assumed

$$R_t = RD_t / S_t = RD_t / \int_0^t ND_t \, dt$$
$$= RD_t / \int_0^t (TD_t - RD_t) \, dt \quad \cdots\cdots (16)$$

For life distribution, three symmetric types (uniform distribution, curtailed normal distribution, and single point distribution) and one nonsymmetric type (Rayleigh distribution) are used as shown in Exhibit 2.




A : UNIFORM DISTRIBUTION
B : NORMAL DISTRIBUTION
C : SINGLE POINT DISTRIBUTION
D : RAYLEIGH DISTRIBUTION

Exhibit 2. Four product life distribution types

For uniform and single point distributions, mathematical derivation is possible using equations 15 and 16. For symmetric distributions like normal distribution, it is shown that the replacement demand forecasts lie between those of uniform and single point distribution (appendix B). For normal and Rayleigh distributions, discretized probability is applied to eq. 7. The resulting replacement demand and replacement rate by each distribution are illustrated in Exhibits 3 and 4, respectively.

Line E in Exhibit 4 corresponds to the value of replacement rate method. If life distribution can be assumed, the replacement demand, stock, and replacement rate can be obtained from total demand. The replacement demand forecasts by replacement rate method can be calculated from stock as follows.

$$\hat{RD}_{t,E}=S_t \cdot \hat{R}_{t,E}=(RD_t/ R_t) \cdot \hat{R}_{t,E}=RD_t \cdot (\hat{R}_{t,E}/ R_t) \quad \cdots\cdots (17)$$

where

$\hat{RD}_{t,E}$, $\hat{R}_{t,E}$ : estimated values by replacement rate method

$\hat{R}_{t,E}=1/ M$ : line E in Exhibit 4

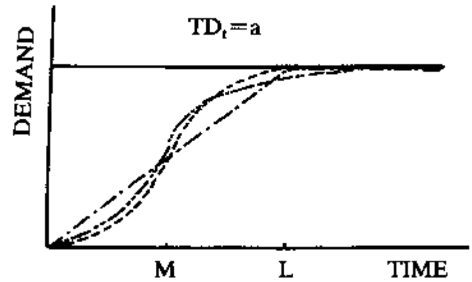
Exhibits 3 shows that the forecasts for replacement demand by uniform distribution (A) and single point distribution (C) composed of upper and lower bounds for the region where the forecasts by each life distribution type lie on. The replacement demand and the stock would probably lie between those by type A and C. The region where $RD_{t,E}$ lies on is shown by the shaded area (see appendix C).

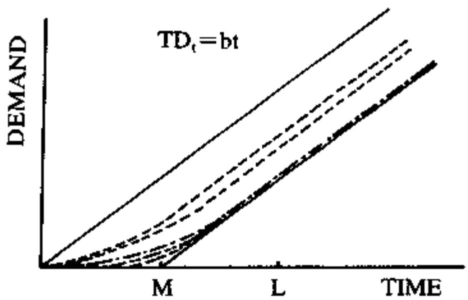
The region of the forecasts by replacement rate method is far from the area between curves A and C. This figure shows that forecasting errors of any types of life distribution are much smaller than those of replacement rate method and that the forecasting results of life distribution method are robust to the distribution type, which is also discussed by Kamakura and Balasubramanian (1987) for various degrees of nonsymmetry. However, the results are sensitive to the change of average life M which plays the role of scale parameter in Exhibits 3 and 4.

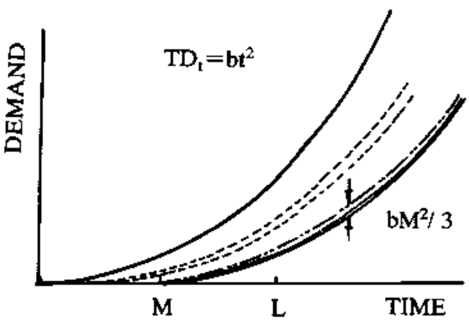
Exhibit 3 shows that the replacement demands by three distribution types (A,B,C) become equal after period L under constant and linear demand patterns. Exhibit 4 demonstrates that life distribution method (A,B,C) and replacement rate method (E) produce similar results under linear demand patterns than under nonlinear demand patterns. It is also shown that the value of replacement rate is small in the early periods and converges to the asymptote 1/ M. This shows that replacement rate method overforecasts especially in the early periods.

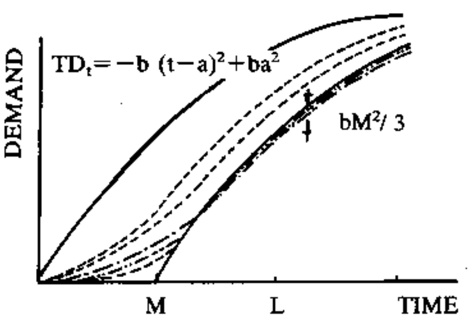
Based on the observations, practical guidelines of applying replacement demand forecasting models can be given as follows. If life distribution is available, apply life distribution method directly. Death rate by age, which is sometimes available especially in reliability analysis, can be used to estimate life distribution by eq. 10 if life distribution is static. Even though life distribution is unavailable, however, life distribution method with simple assumption is preffered. With parameter $M_t$, and distribution type by educated guess can be applied as in Olson and Choi (1985) and Kamakura and Balasubramanian (1987). Single point distribution and uniform distribution can also be applied for combined forecasts.

If adoption data to fit diffusion model are not available, they can be estimated from




**REPLACEMENT DEMAND FUNCTONS**

| Total demand | UNIFORM (A) | | SINGLE POINT (C) | |
|---|---|---|---|---|
| $TD_t=a$ | $\frac{at}{2M}$ | $, t<L$ | $0$ | $, t<M$ |
| | $a$ | $, t>L$ | $a$ | $, t>M$ |
| $TD_t=bt$ | $\frac{bt^2}{4M}$ | $, t<L$ | $0$ | $, t<M$ |
| | $b(t-M)$ | $, t>L$ | $b(t-M)$ | $, t>M$ |
| $TD_t=bt^2$ | $\frac{bt^3}{6M}$ | $, t<L$ | $0$ | $, t<M$ |
| | $b(t-M)^2+bM^2/3$ | $, t>L$ | $b(t-M)^2$ | $, t>M$ |
| $TD_t=-b(t-a)^2+ba^2$ | $b(-t^3+3at^2)/2M$ | $, t<L$ | $0$ | $, t<M$ |
| | $-b(t-a-M)^2+ba^2-bM^2/3$ | $, t>L$ | $-b(t-a-M)^2+ba^2$ | $, t>M$ |

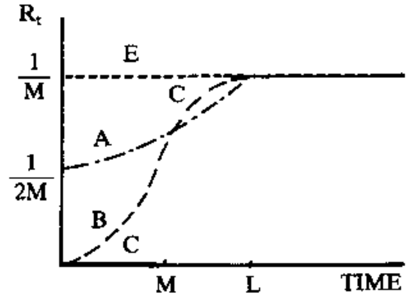
—·—·— UNIFORM (A) —··—··—RAYLEIGH (D) M : AVERAGE LIFE

– – – – NORMAL (B) ======R. R. M. (E) L=2M

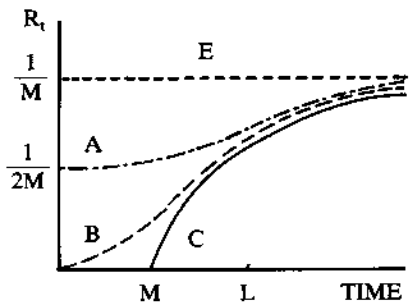
———— SINGLE POINT (C)

Exhibit 3. Replacement demand ($RD_t$) by each method

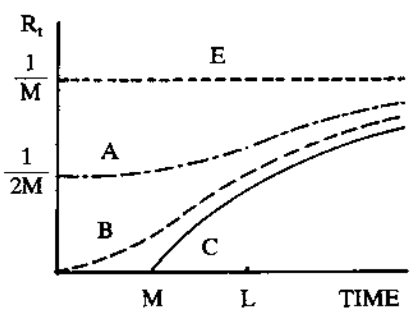
## REPLACEMENT RATE FUNCTIONS



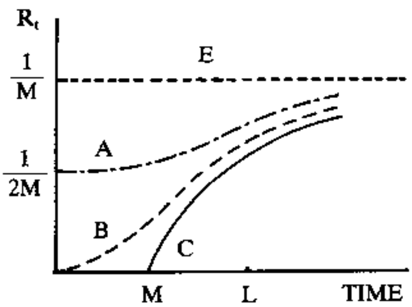

| UNIFORM (A) | SINGLE POINT (C) |
|---|---|
| $\frac{2}{(4M-t)}$ , $t < L$ | $0$ , $t < M$ |
| $\frac{1}{M}$ , $t > L$ | $\frac{1}{M}$ , $t > M$ |




| UNIFORM (A) | SINGLE POINT (C) |
|---|---|
| $\frac{-3}{(t-6M)}$ , $t < L$ | $0$ , $t < M$ |
| $\frac{1}{M}-\frac{1}{(3t-2M)}$ , $t > L$ | $\frac{1}{M}-\frac{1}{(2t-M)}$ , $t > M$ |




| UNIFORM (A) | SINGLE POINT (C) |
|---|---|
| $-4/(t-8M)$ , $t < L$ | $0$ , $t < M$ |
| $\frac{1}{M}-\frac{2t-2M}{3t^2-4Mt+2M^2}$ , $t > L$ | $\frac{1}{M}-\frac{3t-2M}{3t^2-3Mt+M^2}$ , $t > M$ |




| UNIFORM (A) | SINGLE POINT (C) |
|---|---|
| $\frac{-4\,(t-3a)}{(t-2a-4M)^2-4\,(a^2-2Ma+4M^2)}$ , $t < L$ | $0$ , $t < M$ |
| $\frac{9\,(t-a-M)^2-9a^2+3M^2}{M\,\{(3t-3a-2M)^2-9a^2+2M^2\}}$ , $t > L$ | $\frac{1}{M}-\frac{3t-3a-2M}{3t^2-(6a+3M)\,t+3aM+M^2}$ , $t > M$ |

–·–·–·– UNIFORM (A) ——— SINGLE POINT (C) M : AVERAGE LIFE

– – – – – NORMAL (B) -------- R. R. M. (E) L='M

Exhibit 4. Replacement rate ($R_t$) by each method

total demand. Replacement demand can be obtained from total demand, and new demand is driven by subtracting replacement demand from total demand. Integrating new demand produces adoption data.

## Conclusion

A general replacement demand forecasting model which incorporates repurchase rate, multi-ownership, and dynamic product life is proposed in this paper. The performance by each replacement demand forecasting model and life distribution type are analized. The forecasts by the models which use the life distribution of product are robust to the distribution type applied but sensitive to the estimated value of average product life. Replacement rate method overestimates the replacement demand especially in the early periods of product life cycle.

Though actual life distribution data are often not available, life distribution method with guessed distribution type is preferrable to replacement rate method because of the small forecasting errors and robustness with respect to distribution types. If adoption data which are necessary to fit the diffusion models are not available, they can be estimated from total demand data by applying life distribution method.

This paper provides the analysis of replacement demand forecasting models and their forecasting results, by which it extends the applicability of diffusion models and replacement demand forecasting models in practical situations.

## References

Brown, D. A., 'Improving the sales forecast for consumer durables', *Journal of Marketing*, 2 (1965), 229–234.

Cox, Jr. W., 'Product life cycle as marketing models', *Journal of Business*, 40 (1967), 375–384.

Dodson, J. A. and Muller, E., 'Model of new product diffusion through advertising word-of-mouth', *Management Science*, 24 (1978), 1568–1578.

Harrell, S. G. and Taylor, E. D., 'Modeling the product life cycle for consumer durables', *Journal of Marketing*, 45 (1981), 68–75.

Kamakura, W. A. and Balasubramanian, S. K., 'Long term forecasting with innovation diffusion models: the impact of replacement purchase', *Journal of Forecasting*, 6 (1987), 1–19.

Lawrence, K. D. and Lowton, W. H., '*Application of diffusion models: some empirical results in new product forecasting*', Lexington: Y. Wind, V. Mahajan, and R. C. Cardozo (eds) ; Lexington Books, 1981.

Lilien, L., Rao, A. G., and Kalish, S., 'Baysian estimation and control of retailing effort in a repeat-purchase diffusion environment', *Management Science*, 27 (1981), 493–506.

Olson, J. and Choi, S., 'A product diffusion model incorporating repeat purchases', *Technological Forecasting and Social Change*, 27 (1985), 385–397.

Rink, D. R. and Swan, J. E., 'Product life cycle research: a literature review', *Journal of Business Research*, 7 (1979), 219–242.

Spencer, M. H., Clark, C. G., and Hoguet, P. W., '*Business and economic forecasting: an economic approach*', Illinois, Richard D. Irwin, Inc., 1961.
Tellis, G. J. and Crawford, C. M., 'An evolutionary approach to product growth theory', *Journal of Marketing*, 45 (1981), 125–132.

## Appendix A. Average Life M

If life distribution is known, average life M can be simply estimated as follows.

Let f (x) or $f_i$ be the p.d.f. of product life distribution.

$f(x) > 0,\ 0 < x < L$ or $f_i > 0,\ i = 1,2,\ldots,L$
where $\int_0^L f(x) \cdot dx = 1$ or $\sum_{i=1}^{L} f_i = 1$

$$M = \int_0^L x \cdot f(x) \cdot dx \text{ or } M = \sum_{i=1}^{L} i \cdot f_i \quad \cdots\cdots (19)$$

For symmetric life distribution,

$$M = E(x) = L/2 \text{ or } M = E(i) = (L+1)/2. \quad \cdots\cdots (20)$$

If life distribution data are not given directly, two alternative estimations can be possible. One is estimation by market survey, by reliability test, or quality characteristics aimed at design stage. The other is statistical estimation from past total demand and replacement demand data. If life distribution is static with time, the value of M which has the least sum of error or squared error is selected as the estimate. The former criterion is selected because of property that the sum of input into the market (total demand) must be equal to the sum of output from the market (replacement demand) and replacement demand forecasts by life distribution method are exhaustive for total demand.

Select M which satisfies
min $|\Sigma E_t|$ or min $(\Sigma E_t^2)$
where
$E_t = RD_t - \widehat{RD}_t$
$\widehat{RD}_t$ is estimate by life distribution method.

If the mean and the variance of demand data is not stable with time, percentage error ($PE_t$) or weighted least square method may be preferrable.

Whether the life distribution is static or dynamic can be identified from the scatter diagram of $E_t$ or $PE_t$ with time. If they show any increasing or decreasing trend, it reveals that average life has been changing. These relations are illustrated in Exhibit 5. Under the changing life distribution, the value m which cuts the horizontal axis can be the estimate of average life $M_t$ at that cutting period. Three cases of Exhibit 5 is rearranged in Exhibit 6.

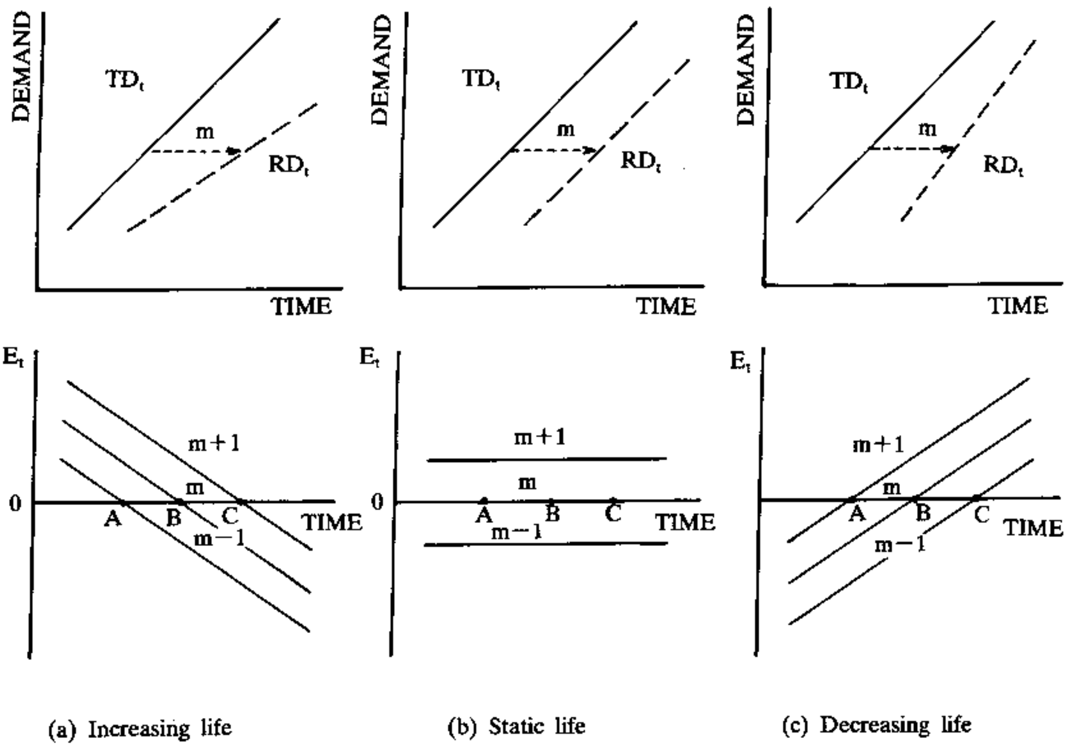



Exhibit 5. Identification of dynamic product life

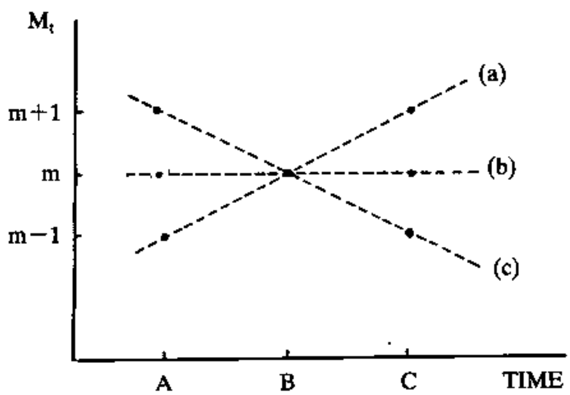



Exhibit 6. Estimation of dynamic product life

**Appendix B. Replacement Demand in Symmetric Life Distribution**

Let f (t) or $f_i$ be the p.d.f. of symmetric life distribution.

1. Under constant demand ($TD_t = TD = a$)

From eq. 17,

$RD_t = \int_0^t f(t-T) \cdot TD \cdot dT = TD \cdot F(t),\ t < L$ ......... (21)

$\quad = \int_{t-L}^{t} f(t-T) \cdot TD \cdot dT = TD,\ t \geqslant L$

where F (t) : distribution function of life t.

2. Under linear demand ($TD_t = b \cdot t$ or $TD_{t+i} = TD_t + b \cdot i$)

From eq. 8,

if $t < L+1$,

$$RD_t = \sum_{j=1}^{t-1} TD_{t-j} \cdot f_j \quad \text{......... (22)}$$

if $t \geqslant L+1$,

$$RD_t = \sum_{j=1}^{t-1} TD_{t-j} \cdot f_j$$

Let $i+j=L+1$

$$RD_t = \sum_{i=1}^{L} TD_{t-L-1+i}\ f_{L+1-i}$$

$$= \sum_{i=1}^{L} (TD_{t-L-1} + bi) \cdot f_i$$

$$= TD_{t-L-1} \cdot \sum_{i=1}^{L} \cdot f_i + b \cdot \sum_{i=1}^{L} i \cdot f_i$$

$$= TD_{t-L-1} + b \cdot M$$

$$= TD_{t-2M} + b \cdot M$$

$$= TD_{t-M} \quad \text{......... (23)}$$

We can see that replacement demand follows total demand with time lag of M. Therefore the result by any symmetric life distribution types becomes the same with that by single point life distribution.

3. Under nonlinear demand

Let's assume the concave function, $TD_t$, as shown in following Exhibit 7.

Let $f_1$ (t) : tangent line to $TD_t$ on the point $TD_{T-M}$

$f_2$ (t), $f_3$(t): line segments connecting the point $TD_{T-M}$ with the points $TD_T$, $TD_{T-L}$

From Exhibit 7,

$TD_t > f_1(t)$ for all t

$RD_T > TD_{T-M}$ ......... (24)

where $TD_{T-M}$ is the replacement demand at T when $f_1$ (t) is the total demand.

We can see that replacement demand by any symmetric life distribution for concave demand pattern is greater than that by single point life distribution, $TD_{T-M}$.

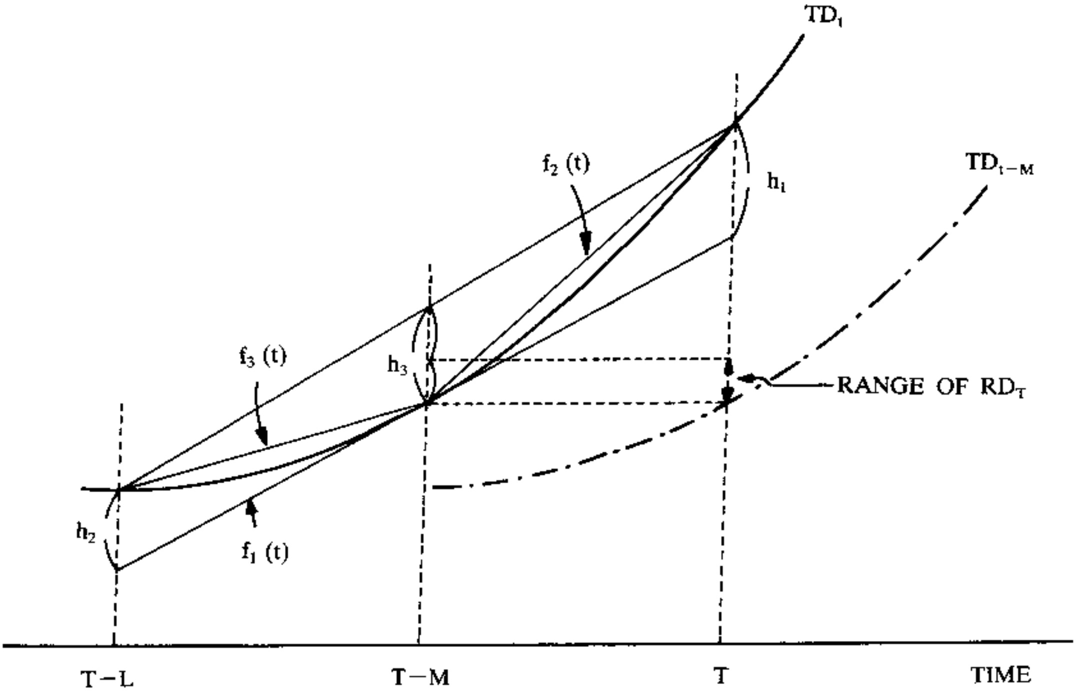



Exhibit 7. Replacement demand under nonlinear demand pattern

If life distribution is a uniform distribution,

$$RD_T = 1/L \int_{T-L}^{T} TD_t \cdot dt$$

$$< 1/L \left( \int_{T-L}^{T-M} f_3(t)\, dt + \int_{T-M}^{T} f_2(t)\, dt \right)$$

$$= 1/L \left\{ \int_{T-L}^{T} f_1(t)\, dt + M(h_1+h_2)/2 \right\}$$

$$= TD_{t-M} + (h_1+h_2)/4$$

$$= TD_{T-M} + \{(TD_T + TD_{T-L})/2 - TD_{T-M}\}/2.$$

$$\therefore RD_T < (TD_T + 2TD_{T-M} + TD_{T-L})/4 \quad \cdots\cdots (25)$$

The forecasts by symmetric life distribution like normal, which is more concentrated around average than uniform distribution, would be closer to the lower limit $TD_{T-M}$ than that by uniform distribution. Therefore the forecasts by uniform distribution become the upper limit. Replacement demand by ordinary symmetric life distribution under concave demand patterns lies between both limits: those by uniform and single point distributions. The results are opposite in case of convex demand pattern.

**Appendix C. Replacement Demand by Replacement Rate Method ($RD_{t,E}$)**

$\hat{RD}_{t,c} < RD_t < \hat{RD}_{t,A}$
$\therefore \hat{ND}_{t,A} < ND_t < \hat{ND}_{T,C}$
$\therefore \hat{S}_{t,A} < S_t < \hat{S}_{t,C}$

From eq. 17,

$\hat{S}_{t,A} / M < \hat{RD}_{t,E} < \hat{S}_{t,C} / M$

where

$\hat{RD}_{t,i}$ : replacement demand forecast by distribution type i
(i = A for uniform distribution and
i = C for single point distribution)
$\hat{ND}_{t,i} = TD_t - \hat{RD}_{t,i}$ : new demand by distribution type i
$\hat{S}_{t,i} = \int_0^t ND_{t,i}\, dt$ : stock by distribution type i.

This is equivalent to mapping the ratio of $\hat{R}_{t,E}$ to true value $R_t$, which is supposed to lie between curves A and C in Exhibit 4, to replacement domain in Exhibit 3.