〈研究論文〉

Journal of the Korean Institute of Industrial Engineers
Vol. 14, No. 2, Dec. 1988

Optimal Allocation of Test Items in an Accelerated Life Test under Model Uncertainty

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Abstract

In accelerated life testing, a relationship is usually assumed between the stress and a parameter of the lifetime distribution. However, the true relationship is not usually known, and therefore, the experimenter may wish to provide protections against the likely departures from the assumed relationship. This paper considers an accelerated life test in which two stress levels are involved, and the lifetime of each test item at a stress level is assumed to have an independent, identical, exponential distribution. For the case where a first order relatioship is assumed while the true one is quadratic, a procedure is developed for allocating test items to stress levels such that the bias and/or the variance of the estimated(log-transformed) mean lifetime at the use condition is minimized.

1. INTRODUCTION

An accelerated life test(ALT) is frequently used in industry to quickly obtain information on the lifetime distribution of highly reliable components, equipment, etc. Acceleration in testing time is achieved by subjecting test items to stress conditions that are severer than the normal use condition. The results obtained at the accelerated conditions are then extrapolated to the use condition according to an assumed relationship between the stress and a parameter of the lifetime distribution.

Im most previous works on the optimal design of an ALT, the assumed relationship between the stress and a parameter of the lifetime distribution is regarded as a trus one(e.g., see[2], [4] – [8], etc.). However, the true relationship is not usually known in practice, and therefore, the experimenter had better provide some protection against the likely departures from the assumed model.

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In this paper, the lifetime of each test item at a stress level is assumed to have an independent, identical, exponential distribution. It is further assumed that the experimenter assumes a first order relationship between the stress and the mean lifetime while the true one is quadratic. Developed in this paper is a procedure for allocating test items to stress levels such that the bias and/or the variance of the estimated(log-transformed) mean lifetime at the use condition is minimized. The weighted least squares method is used for the complete(i.e., uncensored) data.

1.1. Notation

x	stress level
x_0, x_1, x_2	the use, low, and high stress levels, respectively
$\theta(x)$	mean lifetime at stress level x
β_0, β_1, β_2	unknown coefficients in the true relationship between the stress and the mean lifetime
β_0^*, β_1^*	unknown coefficients in the assumed relationship between the stress and the mean lifetime
b_0^*, b_1^*	weighted least squares estimalors of β_0^* and β_1^*, respectively
N	total number of test items given
n_1, n_2	numbers of test items allocated to the low and high stress levels, respectively
t_{ij}	time to failure of the j-th test item at the i-th stress level, $i=1, 2$; $j=1, 2, \ldots, n_i$
t_i	$\sum_{j=1}^{n_i} t_{ij}$, $i=1, 2$
$\Psi(.)$, $\Psi'(\cdot)$	digamma and trigamma functions, respectively
$\hat{y}_0$	estimated(log-transformed) mean lifetime at x_0
B_0	bias of $\hat{y}_0$
V_0	variance of $\hat{y}_0$
MSE(·)	mean square error

2. THE MODEL AND ASSUMPTIONS

This paper considers a case where the lifetime of test item at a stress level is exponentially distributed with mean θ. Suppose that the experimenter assumes

$$\theta(x)=\exp(\beta_0^*+\beta_1^*x), \quad (1)$$

and in planning an ALT he wishes to provide protections against the likely departures from(1), especially against the following.

$$\theta(x)=\exp(\beta_0+\beta_1x+\beta_2x^2). \quad (2)$$

The following additional assumptions are made.

1. The total number of test items are given.
2. The use condition is known.
3. An ALT is conducted at two distinct stress levels specified.
4. All test items are run to failure.
5. The lifetimes of test items at a stress level have independent, identical exponential distributions.

Let $n_i(i=1, 2)$ be the number of test items allocated to $x_i(i=1, 2)$ such that $n_1+n_2=N$, and $t_{ij}(i=1, 2;\ j=1, 2,\ldots, n_i)$ be an observed life. Define

$$t_i=\sum_{j=1}^{n_i} t_{ij},\quad i=1, 2 \qquad (3)$$

It is well known that $t_i/\theta(x_i)$ has an one-parameter gamma distribution(e.g., see [3]). Consider the following transformation.

$$y_i'=\ln t_i,\quad i=1, 2 \qquad (4)$$

Then, y_i' can be written as

$$y_i'=\ln \theta(x_i)+e_i' \qquad (5)$$

where e_i' has an independent log-gamma distribution with mean $\Psi(n_i)$ and variance $\Psi'(n_i)$(e.g., see [3]). Define a new variable y_i such that

$$\begin{aligned} y_i &= y_i'-\Psi(n_i) \\ &= \ln \theta(x_i)+\{e_i'-\Psi(n_i)\} \\ &= \ln \theta(x_i)+e_i \end{aligned} \qquad (6)$$

Then,

$$E(e_i)=0$$
$$Var(e_i)=\Psi'(n_i)$$

Under the assumed model the weighted least squares estimators of β_0^* and β_1^* are respectively determined as b_0^* and b_1^* such that the following is minimized.

$$\sum_{i=1}^{2}(y_i-b_0^*-b_1^*x_i)^2/\Psi'(n_i)$$

It can be shown that

$$b_1^*=(y_1-y_2)/(x_1-x_2) \qquad (7)$$

$$b_0^*=(x_1y_2-x_2y_1)/(x_1-x_2) \qquad (8)$$

Then, the log-transformed mean lifetime at the condition is predicted as

$$\hat{y}_0 = b_0^* + b_1^* x_0 \quad \cdots (9)$$

To evaluate the bias and the variance of $\hat{y}_0$, we first consider those of b_0^* and b_1^*. For instance

$$E(b_1^*) = E\{(y_1 - y_2)/(x_1 - x_2)\}$$
$$= \{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2) - (\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)\}/(x_1 - x_2)$$

$$= \beta_1 + \beta_2(x_1 + x_2) \quad \cdots (10)$$

$$Var(b_1^*) = Var\{(y_1 - y_2)/(x_1 - x_2)\}$$

$$= \{\Psi'(n_1) + \Psi'(n_2)\}/(x_1 - x_2)^2 \quad \cdots (11)$$

Similarly,

$$E(b_0^*) = \beta_0 - \beta_2 x_1 x_2 \quad \cdots (12)$$

$$Var(b_0^*) = \{\Psi'(n_2) x_1^2 + \Psi'(n_1) x_2^2\}/(x_1 - x_2)^2 \quad \cdots (13)$$

$$Cov(b_0^*, b_1^*) = -\{\Psi'(n_2) x_1 + \Psi'(n_1) x_2\}/(x_1 - x_2)^2 \quad \cdots (14)$$

The bias and the variance of $\hat{y}_0$ are then detemined as follows.

$$B = E(\hat{y}_0) - \theta(x_0)$$
$$= E(b_0^* + b_1^* x_0) - (\beta_0 + \beta_1 x_0 + \beta_2 x_0^2)$$
$$= -\beta_2(x_0 - x_1)(x_0 - x_2) \quad \cdots (15)$$

$$V_0 = Var(b_0^*) + x_0^2 Var(b_1^*) + 2x_0 Cov(b_0^*, b_1^*)$$
$$= \{\Psi'(n_1)(x_2 - x_0)^2 + \Psi'(n_2)(x_1 - x_0)^2\}/(x_1 - x_2)^2 \quad \cdots (16)$$

Note that the bias of $\hat{y}_0$ does not depend upond n_1 or n_2 wile the variance of $\hat{y}_0$ does. In the following section, optimal values of n_1 and n_2 are determined such that V_0, or equivalently $MSE(\hat{y}_0)$, is minimized.

3. OPTIMAL ALLOCATION OF TEST IEST ITEMS

Without loss of generality we assume that the stress is standardized such that $x_1 = -1$ and $x_2 = 1$. Under such standardization

$$V_0 = \{\Psi'(n_1)(1 - x_0)^2 + \Psi'(n_2)(1 + x_0)^2\}/4 \quad \cdots (17)$$

Note that $\Psi'(n_1)$ and $\Psi'(n_2)$ are trigamma functions defined as(e. g., see[1])

$$\Psi'(1) = \pi^2/6 \quad \cdots (18)$$

$$\Psi'(1+n)=\Psi'(n)-n^{-2}$$
$$=\Psi'(1)-\sum_{i=1}^{n}(n+1-i)^{-2},\ n=1,2, \quad\cdots\cdots(19)$$

Since $n_1+n_2=N$, V_0 in Eq.(17) can be represented as a function of n_2. That is,

$$f(n_2)=4V_0=\Psi'(n_2)(1+x_0)^2+\Psi'(N-n_2)(1-x_0)^2 \quad\cdots\cdots(20)$$

Since $f(n_2)$ is a discrete function of n_2, an optimal value of n_2 must satisfy

$$f(n_2)-f(n_2+1)=\{\Psi'(n_2)-\Psi'(n_2+1)\}(1+x_0)^2$$
$$+\{\Psi'(N-n_2)-\Psi'(N-n_2-1)\}(1-x_0)^2\leqq 0 \quad\cdots\cdots(21)$$

$$f(n_2)-f(n_2-1)=\{\Psi'(n_2)-\Psi'(n_2-1)\}(1+x_0)^2$$
$$+\{\Psi'(N-n_2)-\Psi'(N-n_2+1)\}(1-x_0)^2\leqq 0 \quad\cdots\cdots(22)$$

From Eq.(19).

$$\Psi'(n_2)-\Psi'(n_2-1)=-(n_2-1)^{-2}$$
$$\Psi'(n_2)-\Psi'(n_2+1)=n_2^{-2}$$
$$\Psi'(N-n_2)-\Psi'(N-n_2-1)=-(N-n_2-1)^{-2}$$
$$\Psi'(N-n_2)-\Psi'(N-n_2+1)=(N-n_2)^{-2}$$

Then, inequality (21) can be reduced to

$$f(n_2)-f(n_2+1)$$
$$=n_2^{-2}(1+x_0)^2-(N-n_2-1)^{-2}(1-x_0)^2\leqq 0 \quad\cdots\cdots(23)$$

and therefore, n_2 must satisfy

$$(N-1)(x_0+1)/(2x_0)\leqq n_2 \quad\cdots\cdots(24)$$

Similarly from(22),

$$f(n_2)-f(n_2-1)$$
$$=-(n_2-1)^{-2}(1+x_0)^2+(N-n_2)^{-2}(1-x_0)^2\leqq 0 \quad\cdots\cdots(25)$$

which implies that

$$n_2\leqq\{N(x_0+1)+x_0-1\}/(2x_0) \quad\cdots\cdots(26)$$

Combining(24) and (26) gives

$$(N-1)(x_0+1)/(2x_0)\leqq n_2\leqq\{N(x_0+1)+x_0-1\}/(2x_0) \quad\cdots\cdots(27)$$

The difference between the upper and lower bounds on n_2 is exactly 1, and therefore, there exist at most two integer values of n_2 which satisfy (27). Since MSE($\hat{y}_0$) is $B_0^2+V_3$ and B_0 does not depend upon n_1 or n_2, the optimal value of n_2 also minimizes MSE($\hat{y}_0$).

As an illustration of the above results, suppose that an accelerated voltage life test is to be conducted to evaluate the time to breakdown of a type of electrical insulating fluid. The use(s_0), low(s_1), and high(s_2) stress levels are 25, 30, and 40kV, respectively. Further, 50 specimens of insulating fluid are available for the test.

To utilize the above development, we first transform the original voltage stress(s) to the standardized stress(x) as follows.

$$x=(s-s_m)/d$$

where

$$s_m=(s_1+s_2)/2$$
$$d=(s_2-s_1)/2$$

Then, s_1 and s_2 are respectively transformed to -1 and 1 as desired. Since $s_m=35$kV and $d=5$kV for the given example, the transformed use stress(x_0) becomes -2.

The optimal number of specimens to be allocated to the high stress level must satisfy (27). That is,

$$12.25\leqq n_2\leqq 13.25$$

Therefore, n_2 becomes 13(or equivalently, $n_1=N-n_2=37$).

4. CONCLUDING REMARKS

Under the uncertainty involved in the relationship between the stress and the mean lifetime, a procedure is developed for optimally allocating test items to each stress level such that the variance(or MSE) of the estimated (log-transformed) mean lifetime at the use condition is minimized. The results obtained in this paper are also applicable to the case of Type II censoring with slight modifications. In Type II censoring the life test at the i-th stress level is terminated as soon as r_i failures are observed. Then, we can simply replace n_i by r_i and use (27) to determine the optimal value of r_i. A fruitful area of future research may include extending the present study to the case where Type I censoring and/or more than two stress levels are considered. In addition, it is also desirable to conduct similar analyses to the present one for the cases of non-exponential lifetime distributions.

Acknowledgements

We are grateful to the referees for helpful comments, which led to an improved version of this article.

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