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ON THE IDENTITIES OF BOL-MOUFANG TYPE

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Abstract: In this work we study properties of five identities of Bol-Moufang type (1). We establish a necessary and sufficient condition for a loop to be an RC-loop. A corollary to this is the condition for a Bol loop to be an RC-loop. Some properties of a generalization of RC-loops are discussed. We also state the relationship between the supernucleus $M(G)$ of a loop G and the nucleii of loops which do not have the inverse property. Finally, we give two tables of non-associative RC-loops of order 16, which are not Bol loops.

1. Introduction

F. Fenyves [4], studied loops with identities of Bol-Moufang type and pointed out some basic properties of loops satisfying these identities. In this study, we look at five identities viz:

$$x^2 . yz = x^2 y . z \text{(1a)}. \quad yx^2 . z = y . x^2 z \text{(1b)}. \quad yz . x^2 = y . zx^2 \text{(1c)}.$$
$$(yx . x)z = y . x^2 z \text{(1d)}. \quad yx^2 . z = y(x . xz) \text{(1e)}. \qquad (1)$$

F. Fenyves showed that a loop G satisfies the identity (1a) ((1b) or (1c)) if and only if x^2 lies in the left (middle or right) nucleus of G and (1d) or (1e) if and only if x^2 lies in the middle nucleus of G for all x in G. The classes satisfying (1a), (1b) and (1c) contain loops from virtually all classes satisfying the Bol-Moufang type identities. We may refer to the identities (1a), (1b) and (1c) as left square (LS-), square (S-) and right square (RS-) identities respectively; in each case, loops satisfying them may be called LS-, S-, RS-loops. We also refer to (1d) and (1e) as right middle (RM-) and left middle (LM)-identities, loops satisfying them as RM-loops and LM-loops respectively. Identities (1a), (1b), (1c) are special cases of a generalization of the associativity identity viz: $x^k . yz = x^k y . z$; $yx^k . z = y . x^k z$; $yz . x^k = y . zx^k$, where $x \to x^k$ is a mapping of the loop into itself; a loop satisfies any of these identities if and only if x^k lies in the left (middle, right) nucleus respectively.

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A loop $(G, \cdot)$ is called a right centre loop (RC-loop) if and only if,

$$(zy.x)x=z(yx.x) \tag{2}$$

for all $x, y, z \in G$. A loop $(G, \cdot)$ is called a right Bol loop (or Bol loop) if and only if

$$(zy.x)y=z(yx.y). \tag{3}$$

for all $x, y, z \in G$. RC-loops and Bol loops are right alternative [1], [4]. There are examples of loops which are both Bol and RC-loops, especially, all the six Bol loops of order 8 are also RC-loops. A natural question is: what is the condition on a Bol loop to be an RC-loop? Here we prove a more general theorem in which case, the condition is given as a corollary. We prove that a loop is an RC-loop if and only if it is right alternative and the square of its elements belong to the right nucleus. Since Bol loops are right alternative it then follows that if the square of every element of a Bol loop belongs to the right nucleus then it is an RC-loop. Since Bol loops have been extensively studied by a number of authors including Bol [1], Robinson [5], Solarin and Sharma [7], [8] we shall concern ourselves more with those RC-loops which are not Bol loops.

In [3], O. Chein defined supernucleus $M(G)$ of a loop G as set of elements satisfying the identity $(xa.a)y=x(a.ay)$ for all x, y in G. He then proved that, a is in $M(G)$ if and only if a^2 is in the nucleus N of G. We exhibit some properties of $M(G)$ which is a C-ioop, and state its connection with various nucleii of loops which do not have the inverse property.

Loops satisfying the identical relations

$$(x.xy)z=x(x.yx) \tag{4}$$

$$(yx.x)z=y(x.xz) \tag{5}$$

for all $x, y, z \varepsilon G$ are called left C-loops (LC-loops) and C-loops respectively. LC-loops are dual to the RC-loops. In Section 4, we discuss the algebraic properties of the classes of loops satisfying the identities

$$(x^k.xy)z=x^k(x.yz) \tag{6}$$

$$(yx.x^k)z=y(x.x^kz) \tag{7}$$

$$(yz.x)x^k=y(zx.x^k) \tag{8}$$

for all $x, y, z \in G$, where x^k is the image of x under some mapping k of the loop into itself. Loops satisfying (6), (7) and (8) are called GLC-loops, GC-loops and GRC-loops respectively.

The reader may consult Bruck [2] for the definitions of the following terms:

left nucleus N_λ, middle nucleus N_u, right nucleus N_ρ, nucleus N, centrum C, Centre Z, left multiplication $L(x)$, right multiplication $R(x)$, left inverse x^λ, right inverse x^ρ for x in a loop G. All loops are written multiplicatively.

2. We define the following new properties of a loop which are considered in this section.

DEFINITION 2.1. A loop G satisfies the left (right) square-alternative property if $x^4y=x^2.x^2y$ $(yx^4=yx^2.x^2)$ for all x, $y\in G$. It satisfies the square alternative property if it satisfies both left and right square alternative properties.

DEFINITION 2.2. A loop G satisfies the left (right) square-inverse property if $y^2=x^\lambda.xy^2(y^2x.x^\rho=y^2)$, for all x, $y\in G$. It satisfies the square-inverse preperty if it satisfies both left and right square-inverse properties.

DFINITION 2.3. A loop G satisfies the left (right)-inverse squared property if $(x^2)^\lambda.x^2y=y(yx^2.(x^2)^\rho=y)$, for all x, $y\in G$. It satisfies the inverse squared property if it satisfies both left and right-inverse squared properties.

DEFINITION 2.4. A loop G satisfies the square flexible property if $xy^2.x=x.y^2x$, for all x, $y\in G$.

THEOREM 2.1 *If G is an RS-loop, then for all $x\in G$,*

(i) G has the right square-alternative property;

(ii) G has the left square-inverse property;

(iii) $(x^2)^\rho=(x^\rho)^2$; $(x^2)^\lambda=(x^\lambda)^2$; $(x^2)^\rho=(x^2)^\lambda$;

(iv) G has tee right-inverse squared property;

(v) x^2 is in the right nucleus N_ρ of G.

PROOF. Let G be an RS-loop, satisfying the identity

$$yz.x^2=y.zx^2 \text{ for all } x,\ y,\ z\in G.$$

(i) Substituting $z=x^2$ in the RS-identity gives

$$yx^2.x^2=yx^4$$

for all x, $y\varepsilon G$. Hence G has the right square-alternative property.

(ii) Substituting $y=z^\lambda$ in the RS-identity, we obtain $z^\lambda z.x^2=z^\lambda.zx^2$. Since $z^\lambda z=1$, it follows that $x^2=z^\lambda.zx^2$ for all x, $z\in G$. Hence G has the left square-inverse property.

(iii) In the RS-identity, let $y=x$, $z=x$, $x=x^{\rho}$. Then $x^2.(x^{\rho})^2=x(x.(x^{\rho})^2)=x(x.x^{\rho}x^{\rho})=1$. This implies that $(x^{\rho})^2=(x^2)^{\rho}$. Also let $y=x^{\lambda}$, $z=x^{\lambda}$, then $(x^{\lambda})^2.x^2=x^{\lambda}(x^{\lambda}.xx)=1 \Longrightarrow (x^2)^{\lambda}=(x^{\lambda})^2$. Moreover, let $y=x^2$, $z=(x^2)^{\lambda}$, then

$$[x^2.(x^2)^{\lambda}]x^2=x^2[(x^2)^{\lambda}.x^2]=x^2 \Longrightarrow x^2.(x^2)^{\lambda}=1.$$

Therefore $(x^2)^{\lambda}=(x^2)^{\rho}$

(iv) In the RS-identity, let $x=x^{\rho}$, $z=x^2$. Then

$$yx^2.(x^{\rho})^2=y.x^2(x^{\rho})^2=y.x^2(x^2)^{\rho}=y.$$

Also $yx^2.(x^{\rho})^2=yx^2.(x^2)^{\rho}$, therefore $yx^2.(x^2)^{\rho}=y$ for all x, $y\in G$. Hence G has the right-inverse squared property.

(v) The RS-identity and the definition of right nucleus

$$N_{\rho} \Longrightarrow x^2\in N_{\rho} \text{ for all } x\in G.$$

The LS-identity is dual to the RS-identity. Therefore analogous results to Theorem 2.1 hold for LS-loops.

THEOREM 2.2. *If G is an LS-loop, then for all $x\in G$,*

(i) G has the left square-alternative property;

(ii) G has the right square-inverse property;

(iii) $(x^2)^{\rho}=(x^{\rho})^2$; $(x^2)^{\lambda}=(x^{\lambda})^2$; $(x^2)^{\rho}=(x^2)^{\lambda}$;

(iv) G has the left-inverse squared property;

(v) x^2 is in the left nucleus N_{λ} of G.

PROOF. The proof is similar to the proof of Theorem 2.1.

THEOREM 2.3. *If G is an s-loop; then for all $x\in G$,*

(i) G has the square-alternative property;

(ii) G has the inverse squared property;

(iii) $x^{\rho}=x^{\lambda}$.

(iv) G has the square flexibility property

(v) x^2 is in the middle nucleus of G.

PROOF. Let G be an S-loop, satisfying the identity

$$yx^2.z=y.x^2z \text{ for all } x, y, z\in G.$$

(i) Substituting $y=x^2$ in the S-identity gives

$$x^4z=x^2.x^2z, \text{ for all } x, z\in G,$$

also substituting $z=x^2$ gives $yx^2.x^2=yx^4$, for all x, $y\in G$. Hence G has the square-alternative property.

(ii) Substituting $z=(x^2)^{\rho}$ in the S-identity gives $yx^2.(x^2)^{\rho}=y.x^2(x^2)^{\rho}=y$; for all $x, y \in G$. Also substituting $y=(x^2)^\lambda$ gives $(x^2)^\lambda . x^2 z = z$; for all $x, z \in G$. Hence G has the inverse squared property.

(iii) Substituting $y=x^\lambda$, $z=(x^\rho)^2$ in the S-identity gives

$$(x^\lambda . x^2)(x^\rho)^2 = x^\lambda . (x^2 . (x^\rho)^2) \Longrightarrow x^\rho = x^\lambda.$$

(iv) Substituting $z=y$ in the S-identity gives

$$yx^2 . y = y . x^2 y, \text{ for all } x, y \in G.$$

Therefore G satisfies the square flexible property.

(v) The S-identity and the definition of the middle nucleus

$$N_\mu \Longrightarrow x^2 \in N_\mu \text{ for all } x \in G.$$

This completes the proof of Theorem 2.3.

COROLLARY 2.1. *Every finite RS (S, LS)-loop of odd order whose elements have odd order is a group.*

PROOF. If the order of a loop G is odd, then for every $y \in G$ there exists $x \varepsilon G$ such that $y=x^2$, since order of y is odd.

(i) If G is an RS-loop, then $x^2 \varepsilon N_\rho$ for all $x \in G \Longrightarrow y \in N_\rho$ for all $y \in G$. Therefore $N_\rho = G$, but N_ρ is a group. Hence G is a group. Similarly, if G is an LS (S)-loop, then

$$x^2 \in N_\lambda(N_\mu) \text{ for all } x \in G, \Longrightarrow y \in N_\lambda(N_\mu)$$

for all $y \in G$. Thus $N_\lambda(N_\mu) = G$, but $N_\lambda(N_\mu)$ is a group, therefore G is a group. This completes the proof of Corollary 2.1.

In the next Theorem we consider the properties of RM-loops. They satisfy properties similar to those of RC-loops and Bol loops.

THEOREM 2.4. *If G is an RM-loop, then for all $x, y \in G$,*

(a) G has the right alternative property;

(b) G has the right inverse property;

(c) G has the square flexibility property;

(d) $x^\lambda = x^\rho$;

(e) x^2 is in the middle nucleus of G;

(f) (i) $xy^n = xy^{n-1} . y = xy . y^{n-1}$.

(ii) For any integer m, $xy^m . y^n = xy^{m+n}$, where $n \varepsilon Z$.

PROOF. Let G be an RM-loop, satisfying the identity

$$(yx . x)z = y . x^2 z \text{ for all } x, y, z \in G.$$

(a) Substituting $z=1$ in the RM-identity gives

$$yx^2=yx.x$$

Hence G has the right alternative property.

(b) Since G is a quasigroup, let $y,\ b\in G$. Then there exists $x\in G$ such that $yx=b$. Substituting $z=x^{\rho}$ in the RM-identity gives $(yx.x)x^{\rho}=yx$. Putting $yx=b$ we obtain $bx.x^{\rho}=b$ for all $x,\ b\in G$. Hence G has the right inverse property.

(c) In the RM-identity, letting $z=y$ gives

$$(yx.x)y=y.x^2y \text{ also } (yx.x)y=yx.x)y=yx^2.y$$

(right alternative property). Therefore $yx^2.y=y.x^2y$ for all $x,\ y\in G$. Hence G satisfies the square flexible property.

(d) Since $x^{\lambda}x=1$, we have $x^{\lambda}x.x^{\rho}=x^{\rho}$, but $x^{\lambda}x.x^{\rho}=x^{\lambda}$. (by right inverse property). Therefore $x^{\lambda}=x^{\rho}$. This proof is general for all right (left) inverse property loops.

(e) Applying the right alternative property to the RM-identity, we obtain $yx^2.z=y.x^2z$ for all $x,\ y,\ z\in G$. Therefore $x^2\in N_{\mu}$ for all $x\varepsilon G$.

(f) (i) We shall consider two cases viz:

$$\text{(a) } n\in Z^{+},\ \text{(b) } n\in Z^{-}.$$

When $n=0$ (i) reduces to the right inverse property. The case $n=1$ is trivial.

(a) $n\in Z^{+}$. We shall establish this by induction. Assume that (i) holds for $k>1$, that is,

$$xy^{k}=xy^{k-1}.y=xy.y^{k-1}$$

In particular, puttjng $x=1$ yields

$$y^{k}=y^{k-1}.y=y.y^{k-1} \text{ for all } y\in G.$$

Now $\quad xy^{k+1}=x(yyy^{k-1})$
$\qquad =(xy.y)y^{k-1}$ (by RM-identity)
$\qquad =xy.y^{k}$ (by induction hypotehsis).

Also $\quad xy^{k}.y=(x.yyy^{k-2})y$
$\qquad =((xy.y)y^{k-2})y$ (by RM-identity)
$\qquad =(xy.y)y^{k-1}$ (by induction hypothesis)
$\qquad =xy.y^{k}$

Therefore $xy^{k+1}=xy.y^{k}=xy^{k}.y$. Therefore (i) holds for $k+1$ whenever it holds for k. Hence it holds for all $n\in Z^{+}$.

(b) Let $k\in Z^{+}$, then $xy^{k+1}=xy^{k}.y$ from (i). Substiting $y=y^{-1}$ in this equation gives $x(y^{-1})^{k+1}=x(y^{-1})^{k}.y^{-1}$

$$xy^{-k-1}.y=(xy^{-k}.y^{-1})y=xy^{-k} \text{ (by right inverse property)}$$

also substituting $x=xy$ and $y=y^{-1}$ in $xy^{k+1}=xy.y^k$ gives

$$xy.(y^{-1})^{k+1}=(xy.y^{-1})(y^{-1})^k=(xy^{-1})^k \text{ (by right inverse property)}$$
$$\Longrightarrow xy.y^{-k-1}=xy^{-k}$$

Therefore we obtain $xy^{-k}=xy^{-k-1}.y=xy.y^{-k-1}$. Since $k\in Z^+$, $-k\in Z^-$. This completes the proof of (i) for any integer $n\in Z$.

(ii) The result holds trivially for $n=0$. For $n=1$,

$$(i)\Longrightarrow(ii).$$

For any integer $n>1$ assume that (ii) holds for any integer m and all x, $y\in G$. That is

$$xy^m.y^k=xy^{m+k}$$

Consider
$$\begin{aligned} xy^{m+k+1} &= xy^{m+k}.y && \text{(by (i))} \\ &= (xy^m.y^k)y && \text{(by assumption)} \\ &= xy^m.y^{k+1} && \text{(by (i))} \end{aligned}$$

Therefore (ii) holds for $k+1$ whenever it holds for k. Hence (ii) holds for all $k\in Z^+$ for any integer m for all x, $y\in G$. Also replacing m by $m-n$, gives $xy^{m-n}.y^n=xy^m$.

Therefore
$$(xy^{m-n}.y^n)y^{-n}=xy^m.y^{-n}$$
$$(xy^{m-n}.y^n)y^{-n}=xy^{m-n} \quad \text{(by right inverse property)}$$

Hence $xy^{m-n}=xy^m.y^{-n}$, for all integers $n\geq 0$, any integer m and all x, $y\in G$. This completes the proof of (ii) for all integer $n\in Z$.

This completes the proof of Theorem 2.4

COROLLARY 2.2. *Every finite RM-loop is power-associative.*

PROOF. Letting $x=1$ in f(ii) gives $y^m.y^n=y^{m+n}$ for all $y\in G$ and integers m and n. Therefore y generates a cyclic subgroup of G, for all $y\in G$, since G is finite. Therefore G is power-associative.

THEOREM 2.5. *Let G be an RM-loop. Then the following statement are equivalent*:

(i) G is an C-loop;

(ii) G has the left alternative property;

(iii) G has the inverse property.

PROOF. By Theorems 2 and 4 of Fenyves [4], (i.e. Theorem 4.3 below) every C-loop G has both left alternative and inverse properties. Therefore (i)$\Longrightarrow$(ii) and (iii).

(ii) implies (iii) let $y=x^{-1}$ in the RM-identity, we obtain $xz=x^{-1}.x^2z$ therefore

$$x^{-1}.xz=x^{-1}(x^{-1}.x^2z)=x^{-2}.x^2z \quad \text{(by (ii))}$$

Since $x^2 \varepsilon N_\mu$, middle nucleus of G, therefore .

$$x^{-2}.x^2z=(x^{-2}.x^2)z=z \Longrightarrow x^{-1}.xz=z \text{ for all } y,\ z\in G.$$

Hence G has the left inverse property. Consequently, G has the inverse property.

(iii)$\Longrightarrow$(i). Letting $y=x^{-1}$ in the RM-identity, we obtain

$$xz=x^{-1}x^2z \Longrightarrow x.xz=x(x^{-1}.x^2z)=x^2z \quad \text{(by (iii))}$$

Therefore $x.xz=x^2z$ for all $x,\ z\in G$. Substituting this into the RM-identity yields $(yx.x)z=y(x.xz)$ for all $x,\ y,\ z\in G$, which is the C-identity. Hence G is a C-loop.

This completes the proof of Theorem 2.5.

REMARK. Since any loop G with the property that x^2 is in its left (middle, right) nucleus for all $x\in G$, is an LS (S, RS)-loop, therefore, all RM-and LM-loops are S-loops.

To end this section, we state a theorem connecting Bol loops and RC-loops.

THEOREM 2.6. *A loop G is an RC-loop, if and only if it is right alternative and x^2 is in the right nucleus of G, for all $x\in G$.*

PROOF. Let G be a loop. If G is an RC-loop, then it is right alternative and $x^2\in N_\rho$ of G for all $x\in G$.

Now let G be a loop which is right alternative such that $x^2\in N_\rho$ of G. Then $zy.x^2=z.yx^2$ for all $x,\ y,\ z\in G$. By right alternative property of G, $zy.x^2=(zy.x)x$, $zy.x^2=z(yx.x)$, for all $x,\ y,\ z\in G$. Therefore $(zy.x)x=zy.x^2=z.yx^2=z(yx.x)$ for all $z,\ y,\ x\,\varepsilon\, G$, which is the RC-identity. Hence G is an RC-loop. This completes the proof of Theorem 2.6.

COROLLARY 2.3. *A Bol loop G is an RC-loop if and only if $x^2\in N_\rho$ of G for all $x\in G$.*

PROOF. The assertion follows from Theorem 2.6. since every Bol loop is a right alternative loop.

3. In this section, we consider some properties of GLC (GRC)-loops.

THEOREM 3.1. *If $G(.)$ is a GLC (GRC)-loop, then*
(i) $G(.)$ satisfies the left (right) inverse property.
(ii) $x^{\lambda}=x^{\rho}$ for all $x\in G$
(iii) $N_{\lambda}=N_{\mu}$ $(N_{\mu}=N_{\rho})$
(iv) $L(x^{k}.x)=L(x)\ L(x^{k})$, $(R(x)R(x^{k})=R(x.x^{k}))$, for all $x\in G$.

PROOF. (i)a Let $x=y^{\lambda}$ in (6), then

$$(x^{k}.y^{\lambda}y)z=x^{k}(y^{\lambda}.yz)$$

$$x^{q}(y^{\lambda}.yz)=(x^{k}.y^{\lambda}y)z=x^{k}z \text{ for all } y,\ z\in G.$$

Therefore $y^{\lambda}.yz=z$ for all $y,\ z\in G$
Hence $G(.)$ satisfies the left inverse property.

(i)b Let $x=z^{\rho}$ in (8), then

$$(yz.z^{\rho})x^{k}=y(zz^{\rho}.x^{k})=yx^{k} \text{ for all } y,\ z\in G.$$

Therefore $yz.z^{\rho}=y$ for all $y,\ z\in G$. Hence $G(.)$ satrsfies the right inverse property.

(ii)a Let $z=y^{\rho}$ in $y^{\lambda}.yz=z$, then $y^{\lambda}yy^{\rho}=y^{\rho}$ for all $y\in G$ implies $y^{\lambda}=y^{\rho}$ for all $y\varepsilon G$.

(ii)b Let $y=z^{\lambda}$ in $yz.z^{\lambda}=y$ then $z^{\lambda}z.z^{\rho}=z^{\lambda}$ for all $z\in G$ implies $z^{\rho}=z^{\lambda}$ for all $z\varepsilon G$.

(iii) In a left (right) I.P. loop the left (right) and middle nucleii coincide. Since in Theorem 3.1 (i) we have shown that $G(.)$ is $L(R)$ I.P. then (iii) follows.

(iv)a Let $y=1$ in (6), then $(x^{k}.x)z=x^{k}(xz)$ for all $x,\ z\in G$. Thus $zL(x)L(x^{k})=zL(x^{k}.x)$ implies $L(x)L(x^{k})=L(x^{k}.x)$ for all $x\in G$.

(iv)b Let $z=1$ in (7), then $yx.x^{k}=y(x.x^{k})$ for all $y,\ z\in G$ implies $yR(x)\ R(x^{k})=yR(x.x^{k})$ implies $R(x)\ R(x^{k})=R(x.x^{k})$ for all $x\in G$.

THEOREM 3.2. *If $(G.)$ is a GLC-loop and $x^{k}\in N_{\lambda}$ for all $x\in G$, then $G(.)$ is a group.*

THEOREM: *If $(G(.))$ is a GRC-loop and $x^{k}\in N_{p}$ for all $x\in G$, then $G(.)$ is a group.*

4. LEMMA 4.1. *$M(G)$ is a C-loop.*

PROOF. By definition of $M(G)$, $(xa.a)y=x(a.ay)$ for all $a.\ x,\ y,$ in $M(G)$.

COROLLARY 4.2. *$M(G)=G$ if and only if G is a C-loop. This corollary appears interesting in the light of corollary* 1 *of* 0. *Chein* [3]. *We shall return to it shortly,* (*corollary* 4.3).

Since $M(G)$ is a C-loop contained in G, we may refer to it as the C-loop part of G.

THEOREM 4.3. *If G is a C-loop, then*

(*a*) *G has the inverse property*

(*b*) *G is both left and right alternative*

(*c*) *x^2 is in the nucleus N of G. for all x in G.*

PROOF. Follows from Theorems 2 and 4 of F. Fenyves [4].

The next theorem gives the connection between C-loops and other di-associative loops. At this point we like to remark that the condition that a^2 belong to the nucleus of G is not necessary in O. Chein's lemma in [3], $xa^2=xa.a$ and $a^2y=a.ay$ are left and right alternative properties (Theorem 4.3).

THEOREM 4·4. *A di-associative loop G is a C-loop if and only if x^2 is in the nucleus N of G, for all x in G.*

PROOF. Suppose G is di-associative loop such that x^2 is in N for all x in G, then $(yx.x)z=(yx^2)z=y(x^2z)=y(x.xz)$ for all y, z in G. Therefore G is a C-loop.

COROLLARY 4.1. *A Moufang loop G is a C-loop if and only if x^2 is in the nucleus N of G for all x in G.*

PROOF. Immediate from theorem 4.4 since a Moufang loop has the di-associativity property.

This corollary 4.5 with corollary 1 of 0. Chein [1], indicate that every M^3-loop is a C-loop, and there is no contradiction since every extra loop is a C-loop [4].

The analogous of theorem 4.4 hold for RC-, LC-loops [4] viz: A right (left)-inverse property loop G is an RC, (LC)-loop if and only if x^2 is in the right (left) nucleus $N_\rho(N_\lambda)$ of G, for all x in G.

THEOREM 4.5. *If G is a right (left)-inverse property loop then N_ρ $(N_\lambda)\subseteq M(G)$.*

PROOF. If G is a right inverse property loop and a is in $N_\rho = N_\mu$, $(xa.a)y = xa.ay = x(a.ay)$, therefore a is in $M(G)$. Similar argument holds for left inverse property loop.

We can define RC-, LC-loop parts of a loop G as sets of elements which satisfy the RC-, LC-identities, denoted by .$RC(G)$, $LC(G)$ respetively, then $M(G) = RC(G) \cap LC(G)$.

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Table 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	5	6	7	8	1	10	13	12	15	14	9	16	11
3	4	5	6	7	8	1	2	11	12	13	14	15	16	9	10
4	5	6	7	8	1	2	3	12	15	14	9	16	11	10	13
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	7	8	1	2	3	4	5	14	9	16	11	10	13	12	15
7	8	1	2	3	4	5	6	15	16	9	10	11	12	13	14
8	1	2	3	4	5	6	7	16	11	10	13	12	15	14	9
9	10	11	12	13	14	15	16	3	8	5	2	7	4	1	6
10	11	12	13	14	15	16	9	4	3	6	5	8	7	2	1
11	12	13	14	15	16	9	10	5	2	7	4	1	6	3	8
12	13	14	15	16	9	10	11	6	5	8	7	2	1	4	3
13	14	15	16	9	10	11	12	7	4	1	6	3	8	5	2
14	15	16	9	10	11	12	13	8	7	2	1	4	3	6	5
15	16	9	10	11	12	13	14	1	6	3	8	5	2	7	4
16	9	10	11	12	13	14	15	2	1	4	3	6	5	8	7

Table 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	5	6	7	8	1	12	9	14	11	16	13	10	15
3	4	5	6	7	8	1	2	11	12	13	14	15	16	9	10
4	5	6	7	8	1	2	3	14	11	16	13	10	15	12	9
5	6	7	8	9	2	3	4	13	14	15	16	9	10	11	12
6	7	8	1	2	3	4	5	16	13	10	15	12	9	14	11
7	8	1	2	3	4	5	6	15	16	9	10	11	12	13	14
8	1	2	3	4	5	6	7	10	15	12	9	14	11	16	13
9	10	11	12	13	14	15	16	3	4	5	6	7	8	1	2
10	11	12	13	14	15	16	9	2	3	4	5	6	7	8	1
11	12	13	14	15	16	9	10	5	6	7	8	1	2	3	4
12	13	14	15	16	9	10	11	4	5	6	7	8	1	2	3
13	14	15	16	9	10	11	12	7	8	1	2	3	4	5	6
14	15	16	9	10	11	12	13	6	7	8	1	2	3	4	5
15	16	9	10	11	12	13	14	1	2	3	4	5	6	7	8
16	9	10	11	12	13	14	15	8	1	2	3	4	5	6	7

The two tables above were tested on the computer and were found to satisfy the RC-identity but not Bol identity. Their commutative patterns reveal that they are non-isomorphic.