J. Korean Math. Soc. 25(1988), No. 2, pp. 175-198

ASYMPTOTIC BEHAVIOR OF ELEMENTARY SOLUTIONS OF PERIODIC GENERALIZED DIFFUSION EQUATIONS

JAE-PIL OH* AND MATSUYO TOMISAKI

1. Introduction

Let $\mathcal{G}=(d(d/ds)-dk)/dm$ be a periodic generalized diffusion operator on the real line $\boldsymbol{R}$, that is, s, m and k are real valued functions on $\boldsymbol{R}$ satisfying the following conditions a)-d), and ds, dm and dk stand for the measures induced by s, m and k, respectively. a) s is continuous and increasing. b) m is non-trivial, right continuous and nondecreasing. c) i) k is right continuous and nondecreasing, or ii) k is right continuous, dk is absolutely continuous with respect to dm and the Radon-Nikodym density is bounded. d) There is a positive ρ such that $s(x+1)-s(y+1)=\rho(s(x)-s(y))$, $m(x+1)-m(y+1)=\rho^{-1}(m(x)-m(y))$, $k(x+1)-k(y+1)=\rho^{-1}(k(x)-k(y))$, for every $x, y\in\boldsymbol{R}$.

Let $p(t, x, y)$ be an elementary solution of the equation

$$\partial u(t, x)/\partial t=\mathcal{G}u(t, x), \quad t>0, \tag{1.1}$$

on $\boldsymbol{R}$ in the sense of McKean [6]. In this paper we study the asymptotic behavior of $p(t, x, y)$ for large t.

The spectrum of $\mathcal{G}$ has been studied in [1], [4], [8]. By using their results, we can see easily that

$$\lim_{t\to\infty} t^{1/2}\exp(\lambda_0 t)p(t, x, y)=\alpha(x, y, \lambda_0), \quad x, y\in\boldsymbol{R}, \tag{1.2}$$

where λ_0 is the principal eigenvalue of $\mathcal{G}$, $\alpha(x, y, \lambda_0)$ is a positive constant depending on x, y and λ_0. The point of our result is the following global asymptotic estimate.

$$\limsup_{t\to\infty} t^{1/2}\exp(\lambda_0 t) \sup_{x,y\in\boldsymbol{R}} \rho^{-(x+y)/2}p(t, x, y)<\infty. \tag{1.3}$$

As will be seen in Lemma 2.1 below.

$$\sup_{x,y\in\boldsymbol{R}} \rho^{-(x+y)/2}\alpha(x, y, \lambda_0)<\infty.$$

Received August 18, 1987.

*This research is partially supported by the research grant of the Ministry of Education, 1987.

Hence we can get the following uniform estimate for $T_t f(x) = \int_{\boldsymbol{R}} p(t, x, y) f(y) dm(y)$,

$$(1.4) \qquad \lim_{t\to\infty} \sup_{x\in\boldsymbol{R}} \rho^{-x/2} \Big| t^{1/2} \exp(\lambda_0 t) T_t f(x) - \int_{\boldsymbol{R}} \alpha(x, y, \lambda_0) f(y) dm(y) \Big| = 0,$$

for every f satisfying

$$(1.5) \qquad \int_{\boldsymbol{R}} \rho^{y/2} |f(y)| dm(y) < \infty.$$

We will also discuss the same problem for the restriction $\mathcal{G}^\gamma$ of $\mathcal{G}$ to the half line $\boldsymbol{R}_+ \equiv [0, \infty)$ with a sticky elastic boundary condition at 0. Here γ stands for some quantity determining the boundary condition, and $\mathcal{G}^\gamma$ and $p^\gamma(t, x, y)$ are a periodic generalized diffusion operator and an elemetary solution of (1.1) on $\boldsymbol{R}_+$, respectively, depending on γ (see §2 for details). In this case, $p^\gamma(t, x, y)$ behaves in a little different way from (1.2). Namely,

$$(1.6) \qquad \lim_{t\to\infty} t^{\delta^\gamma} \exp(\lambda^\gamma t) p^\gamma(t, x, y) = \beta^\gamma(x, y, \lambda^\gamma), \qquad x, y \in \boldsymbol{R}_+.$$

where λ^γ is the principal eigenvalue of $\mathcal{G}^\gamma$ and $\beta^\gamma(x, y, \lambda^\gamma)$, which is positive for $x, y > 0$, is a constant depending on x, y, λ^γ and γ. Further, $\delta^\gamma = 0[1/2$ or $3/2]$ if λ^γ belongs to the point [resp. continuous] spectrum of $\mathcal{G}^\gamma$ according to γ. We will also obtain the following global asymptotic estimates.

$$(1.7) \qquad \lim_{t\to\infty} \sup t^{\delta^\gamma} e^{\lambda^\gamma t} \sup_{x,y\in\boldsymbol{R}_+} \{\rho^{-(x+y)/2} / (1+x)(1+y)\} p^\gamma(t, x, y) < \infty,$$

$$(1.8) \qquad \lim_{t\to\infty} \sup_{x\in\boldsymbol{R}_+} \frac{\rho^{-x/2}}{1+x} \Big| t^{\delta^\gamma} e^{\lambda^\gamma t} T_t^\gamma f(x) - \int_{(0,\infty)} \beta^\gamma(x, y, \lambda^\gamma) f(y) dm(y) \Big| = 0,$$

where $T_t^\gamma f(x) = \int_{(0,\infty)} p^\gamma(t, x, y) f(y) dm(y)$ and f satisfies

$$(1.9) \qquad \int_{(0,\infty)} (1+y) \rho^{y/2} |f(y)| dm(y) < \infty.$$

In the case of that $m(x)$ is regularly varying, the asymptotic behavior of elementary solutions of generalized diffusion equations has been studied in [7]. Our periodic operator $\mathcal{G}$ is reduced to the one in [7] if and only if $k(x) \equiv 0$ and $\rho = 1$, and moreover a sticky boundary condition is set at 0 provided it is dealed with on $\boldsymbol{R}_+$. Then our results coincide

with those in [7]. Namely,

$$\lim_{t\to\infty} t^{1/2}p(t, x, y) = (s(1)/4\pi m(1))^{1/2}, \quad x, y\in \boldsymbol{R}, \tag{1.10}$$

$$\lim_{t\to\infty} t^{1/2}p^{\gamma}(t, x, y) = (s(1)/\pi m(1))^{1/2}. \quad x, y\in \boldsymbol{R}_+, \tag{1.11}$$

for a sticky boundary condition γ. In this case we can also derive from (1.6) that, if we set the absorbing boundary condition at 0, then

$$\lim_{t\to\infty} t^{3/2}p^{\gamma}(t, x, y) = (m(1)/4\pi s(1))^{1/2}s(x)s(y), \quad x, y\in \boldsymbol{R}_+. \tag{1.12}$$

(1.11) and (1.12) show an evident difference in the asymptotic behavior of $p^{\gamma}(t, x, y)$ as $t\to\infty$ between boundary conditions γ, which is well known in the case of Brownian motion.

In §2 we will describe precise definitions and state our main results. In §3 we will give some remarks on the asymptotic behavior of elementary solutions of, not necessarily periodic, generalized diffusion equations. §4 is devoted to the proof of our results. In §5 we will find some examples. Especially we will discuss one of them in detail, because it can't be reduced to a Hill's operator and it presents an interesting phenomenon.

The authors would like to thank Professor Y. Ogura for his valuable comments.

2. Main results

We use the same notations as in [8]. We repeat the definitions for completeness. Let s, m and k be real valued functions on the real line $\boldsymbol{R}$ satisfying the conditions a)-d) mentioned at beginning of §1. We may assume $s(0)=m(0)=k(0)=0$ without loss of generality. $D(\mathcal{G})$ is the class of functions $u\in L^2(\boldsymbol{R}, m)$ having the following two properties. a) There exists the derivative $u^+(x)\equiv\lim_{h\downarrow 0}\{u(x+h)-u(x)\}/\{s(x+h)-s(x)\}$, of bounded variation on compact intervals of $\boldsymbol{R}$. b) There is an $h_u\in L^2(\boldsymbol{R}, m)$ such that

$$\int_{a+}^{b+} h_u(x)\,dm(x) = u^+(b) - u^+(a) - \int_{a+}^{b+} u(x)\,dk(x), \quad a, b\in \boldsymbol{R}. \tag{2.1}$$

where the integral is read as

$$\int_{a+}^{b+} f(x)\,d\xi(x) = \begin{cases} \int_{(a,b]} f(x)\,d\xi(x) & \text{for } a \leq b, \\ -\int_{(b,a]} f(x)\,d\xi(x) & \text{for } a > b, \end{cases}$$

for any Stieltjes measure $d\xi(x)$. The map $\mathcal{G} : u(\in D(\varphi)) \longmapsto h_u$ is called a periodic generalized diffusion operator on $\boldsymbol{R}$. Indeed, $\mathcal{G}$ has the property that $\mathcal{G}(u(\cdot+1))(x) = (\mathcal{G}u)(x+1)$, $x \in \boldsymbol{R}$ for every u such that both u and $u(\cdot+1)$ belong to $D(\mathcal{G})$ (cf. [1]).

In order to define the elementary solution $p(t, x, y)$ of the equation (1.1), we need some arguments similar to those in [2], [6], [10]. We summalize them without proof. Let $\varphi_i(x, y)$, $x \in \boldsymbol{R}$. $\lambda \in \boldsymbol{C}$. $i=1, 2$ be the solutions of the integral equations

$$\begin{aligned} \varphi_1(x, \lambda) &= 1 + \int_{0^+}^{x^+} (s(x) - s(y))\varphi_1(y, \lambda)(-\lambda dm(y) + dk(y)), \\ \varphi_2(x, \lambda) &= s(x) + \int_{0^+}^{x^+} (s(x) - s(y))\varphi_2(y, \lambda)(-\lambda dm(y) + dk(y)). \end{aligned} \tag{2.2}$$

Note that the Wronskian of $\varphi_1(x, y)$ and $\varphi_2(x, \lambda)$ is equal to 1:

$$\varphi_1(x, \lambda)\varphi_2^+(x, \lambda) - \varphi_1^+(x, \lambda)\varphi_2(x, \lambda) = 1. \tag{2.3}$$

For $\lambda \in \boldsymbol{C} \backslash \boldsymbol{R}$, there exist the limits $f_i(\lambda)$, $i=1, 2$:

$$\begin{aligned} f_1(\lambda) &= -\lim_{x \to +\infty} \varphi_1(x, \lambda)/\varphi_2(x, \lambda) = -\lim_{x \to +\infty} \varphi_1^+(x, \lambda)/\varphi_2^+(x, \lambda), \\ f_2(\lambda) &= -\lim_{x \to -\infty} \varphi_1(x, \lambda)/\varphi_2(x, \lambda) = -\lim_{x \to -\infty} \varphi_1^+(x, \lambda)/\varphi_2^+(x, \lambda). \end{aligned} \tag{2.4}$$

We set

$$\begin{aligned} f_{11}(\lambda) &= 1/(f_2(\lambda) - f_1(\lambda)), \\ f_{12}(\lambda) &= f_{21}(\lambda) = f_2(\lambda)/(f_2(\lambda) - f_1(\lambda)), \\ f_{22}(\lambda) &= f_1(\lambda) f_2(\lambda)/(f_2(\lambda) - f_1(\lambda)). \end{aligned}$$

Define the functions $\sigma_{ij}(u)$, $i, j=1, 2$ on $\boldsymbol{R}$ by

$$\sigma_{ij}(u_2) - \sigma_{ij}(u_1) = \frac{1}{\pi} \lim_{v \downarrow 0} \int_{u_1}^{u_2} Im\, f_{ij}(u + \sqrt{-1}v)\, du, \qquad u_1 < u_2,$$

and denote the induced Stieltjes measures on $\boldsymbol{R}$ by $d\sigma_{ij}$. The matrix valued measure $[d\sigma_{ij}]_{i,j=1,2}$ is nonnegative definite and it is called the spectral measure of $\mathcal{G}$.

Now we define the elementary solution $p(t, x, y)$ of the periodic generalized diffusion equation (1.1) on $\boldsymbol{R}$ by

$$p(t, x, y) = \sum_{i,j=1,2} \int_{\boldsymbol{R}} e^{-\lambda t} \varphi_i(x, \lambda)\varphi_j(y, \lambda)\, d\sigma_{ij}(\lambda), \quad t > 0, \quad x, y \in \boldsymbol{R}.$$

As was seen in [8; Theorems 1 and 2], the spectrum of $\mathcal{G}$ is bounded

from below, consists only of the continuous spectrum and coincides with the set S, where S is given by

$$(2.5)\qquad \begin{aligned} &S=\{\lambda\in\boldsymbol{R}: D(\lambda)\leq 0\},\\ &D(\lambda)=\Delta^2(\lambda)-4\rho,\quad \Delta(\lambda)=\varphi_1(1,\lambda)+\rho\varphi_2{}^+(1,\lambda).\end{aligned}$$

We set

$$(2.6)\qquad \lambda_0=\inf S(>-\infty).$$

Then we can see that

$$(2.7)\qquad \begin{aligned} &\Delta(\lambda_0)=2\sqrt{\rho},\quad \Delta'(\lambda_0)<0,\\ &\varphi_1{}^+(1,\lambda_0)\leq 0,\quad \varphi_2(1,\lambda_0)>0,\end{aligned}$$

(see Theorem 1, Lemma 3.3 and Proof of Theorem 3 in [8]). Let

$$(2.8)\qquad \begin{aligned} &\alpha_0=2\rho^{1/4}(-\pi\Delta'(\lambda_0))^{1/2},\quad \alpha_{11}=\varphi_2(1,\lambda_0)/\alpha_0,\\ &\alpha_{12}=\alpha_{21}=\{\rho^{1/2}-\varphi_1(1,\lambda_0)\}/\alpha_0,\quad \alpha_{22}=-\rho\varphi_1{}^+(1,\lambda_0)/\alpha_0,\end{aligned}$$

$$(2.9)\qquad \alpha(x,y,\lambda_0)=\sum_{i,j=1,2}\alpha_{ij}\varphi_i(x,\lambda_0)\varphi_j(y,\lambda_0).$$

First we note

LEMMA 2.1. *$\alpha(x,y,\lambda_0)$ is positive for every $x,y\in\boldsymbol{R}$. Further,*
$$\sup_{x,y\in\boldsymbol{R}}\rho^{-(x+y)/2}\alpha(x,y,\lambda_0)<\infty.$$

The proof of this lemma as well as all the others in this section is postponed untill §4. Now we have the following

THEOREM 1. *The formulas* (1.2) *and* (1.3) *are valid for λ_0 and $\alpha(x,y,\lambda_0)$ defined by* (2.6) *and* (2.9), *respectively. Further* (1.4) *holds for every f satisfying* (1.5).

We next consider the restriction of $\mathcal{Q}$ to the half line $\boldsymbol{R}_+=[0,\infty)$ with the sticky elastic boundary condition at 0. Let Γ be the collection of triples $(\gamma_1,\gamma_2,\gamma_3)$ such that $\gamma_1=\gamma_2=\gamma_3-1=0$ or $\gamma_1\geq 0$, $\gamma_2=1$, $\gamma_3\geq 0$. For each $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\Gamma$ we define the measure $m^\gamma(dx)=\gamma_1\delta_{\{0\}}(dx)+\chi_{(0,\infty)}(x)dm(x)$. $D(\mathcal{Q}^\gamma)$ is the set of all functions $u\in L^2(\boldsymbol{R}_+,m^\gamma)$ having the following properties. a) The derivative $u^+(x)$ exists and is of bounded variation on compact intervals of $\boldsymbol{R}_+$. b) There is an $h_u\in L^2(\boldsymbol{R}_+,m^\gamma)$ such that (2.1) holds for every a, $b\in\boldsymbol{R}_+$ and the following boundary condition (2.10) is satisfied.

$$(2.10)\qquad \gamma_1h_u(0)=\gamma_2u^+(0)-\gamma_3u(0).$$

$\mathcal{G}^\gamma$ is the map $u(\in D(\mathcal{G}^\gamma)) \longmapsto h_u$. The boundary condition (2.10) is nothing more than the reflecting [absorbing] one in the case of that $\gamma_1=\gamma_2-1=\gamma_3=0$ [resp. $\gamma_1=\gamma_2=\gamma_3-1=0$].

Following the arguments as in [2], [6], [10], we also define $p^\gamma(t, x, y)$. For each $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$ we define the functions $\phi_i^\gamma(x, y)$, $x\in\boldsymbol{R}_+$, $\lambda\in\boldsymbol{C}$, $i=1, 2$:

$$(2.11)\quad \begin{aligned} &\phi_1^\gamma(x, \lambda)=\gamma_2\varphi_1(x, \lambda)-(\gamma_1\lambda-\gamma_3)\varphi_2(x, \lambda), \\ &\phi_2^\gamma(x, \lambda)=\{(\gamma_1\lambda-\gamma_3)\varphi_1(x, \lambda)+\gamma_2\varphi_2(x, \lambda)\}/\{|\gamma_1\lambda-\gamma_3|^2+\gamma_2\}. \end{aligned}$$

Then there exist the limits

$$h^\gamma(\lambda)=\lim_{x\to\infty}\phi_2^\gamma(x, \lambda)/\phi_1^\gamma(x, \lambda)=\lim_{x\to\infty}\phi_2^{\gamma+}(x, \lambda)/\phi_1^{\gamma+}(x, \lambda)$$

for $\lambda\in\boldsymbol{CR}$. Define the function σ^γ on $\boldsymbol{R}$ by

$$\sigma^\gamma(u_2)-\sigma^\gamma(u_1)=\frac{1}{\pi}\lim_{v\downarrow 0}\int_{u_1}^{u_2} Imh^\gamma(u+\sqrt{-1}v)\,du,\quad u_1<u_2.$$

We denote by $d\sigma^\gamma$ the induced Stieltjes measure on $\boldsymbol{R}$, which is the spectral measure of $\mathcal{G}^\gamma$.

The elementary solution $p^\gamma(t, x, y)$ of the periodic generalized diffusion equation (1.1) with boundary condition (2.10) is defined by

$$p^\gamma(t, x, y)=\int_{\boldsymbol{R}} e^{-\lambda t}\phi_1^\gamma(x, \lambda)\phi_2^\gamma(y, \lambda)\,d\sigma^\gamma(\lambda),\quad t>0,\quad x, y\in\boldsymbol{R}_+.$$

Let us recall λ_0 defined by (2.6). We put

$$(2.12)\qquad \kappa_0(\varepsilon)=\varepsilon\lambda_0+(\sqrt{\rho}-\varphi_1(1, \lambda_0))/\varphi_2(1, \lambda_0),\quad \varepsilon\geq 0.$$

Given $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$ we can consider the following four cases.

Case I : $\gamma_1\geq 0, \gamma_2=1, 0\leq\gamma_3<\kappa_0(\gamma_1)$.

Case II : $\gamma_1\geq 0, \gamma_2=1, \gamma_3=\kappa_0(\gamma_1)$.

Case III: $\gamma_1\geq 0, \gamma_2=1, \gamma_3>\kappa_0(\gamma_1)$.

Case IV: $\gamma_1=\gamma_2=\gamma_3-1=0$.

It follows from Theorem 6 with the proof in [8] that there is a unique solution ν^γ of the equation $\gamma_1\lambda+\gamma_2 f_1(\lambda)-\gamma_3=0$, $\lambda<\lambda_0$, only in Case I, where f_1 is given by (2.4). This ν^γ belongs to the point spectrum of $\mathcal{G}^\gamma$. Noting [8; Corollary to Theorem 6] we get

$$(2,13)\qquad \lambda^\gamma\ (\text{principal eigenvalue of } \mathcal{G}^\gamma)=\begin{cases}\nu^\gamma & \text{in Case I,}\\ \lambda_0 & \text{otherwise.}\end{cases}$$

Now we put

$$(2.14)\qquad \delta^\gamma=\begin{cases}0 & \text{in Case I,}\\ 1/2 & \text{in Case II,}\\ 3/2 & \text{in Cases III and IV,}\end{cases}$$

$$(2.15)\qquad \beta_0{}^\gamma=\begin{cases}\left(\int_{R_+}\phi_1{}^\gamma(x,\nu^\gamma)^2dm^\gamma(x)\right)^{-1}, & \text{in Case I,}\\ 2\alpha_{11}, & \text{in Case II,}\\ (4\pi\alpha_{11}(\gamma_3-\kappa_0(\gamma_1))^2)^{-1}, & \text{in Case III,}\\ (4\pi\alpha_{11})^{-1}, & \text{in Case IV,}\end{cases}$$

$$(2.16)\qquad \beta^\gamma(x,y,\lambda^\gamma)=\beta_0{}^\gamma\phi_1{}^\gamma(x,\lambda^\gamma)\phi_1{}^\gamma(y,\lambda^\gamma).$$

Then we get

LEMMA 2.2. *$\beta^\gamma(x,y,\lambda^\gamma)$ is positive for $x,y\geq 0$ $[x,y>0]$ in Cases I, II and III [resp. Case IV]. Moreover*

$$\sup_{x,y\geq 0}\rho^{-(x+y)/2}\beta^\gamma(x,y,\lambda^\gamma)<\infty, \qquad \textit{in Case I and II,}$$

$$\sup_{x,y\geq 0}\rho^{-(x+y)/2}\beta^\gamma(x,y,\lambda^\gamma)/(1+x)(1+y)<\infty, \quad \textit{in Case III and IV.}$$

Our second result is

THEOREM 2. *The asymptotic formulas* (1.6), (1.7) *and* (1.8) *hold for λ^γ, δ^γ and $\beta^\gamma(x,y,\lambda^\gamma)$ defined by* (2.13), (2.14) *and* (2.16), *respectively, where f in* (1.8) *is any function satisfying* (1.9).

Finally we state about (1.10), (1.11) and (1.12) more precisely.

COROLLARY. *Assume that $k(x)\equiv 0$ and $\rho=1$. Then* (1.2), (1.3) *and* (1.4) *hold with $\lambda_0=0$ and $\alpha(x,y,\lambda_0)\equiv(s(1)/4\pi m(1))^{1/2}$.* (1.6), (1.7) *and* (1.8) *hold with $\lambda^\gamma=0$ and*

$$\delta^\gamma=\begin{cases}1/2 & \textit{if } \gamma_3=0,\\ 3/2 & \textit{if } \gamma_3>0,\end{cases}$$

$$\beta^\gamma(x,y,\lambda^\gamma)=\begin{cases}(s(1)/\pi m(1))^{1/2}, & \textit{if } \gamma_3=0,\\ (m(1)/4\pi s(1))^{1/2}(\gamma_2/\gamma_3+s(x))(\gamma_2/\gamma_3+s(y)), & \textit{if } \gamma_3>0.\end{cases}$$

3. Elementary solutions of generalized diffusion equations

In this section we give some remarks on the asymptotic behaviors of elementary solutions of, not necessarily periodic, generalized diffusion equations.

Let s, m and k be real valued functions on an interval I with the end points l_1 and l_2, $-\infty\leq l_1<0<l_2\leq\infty$. We assume that s, m and k satisfy the first three conditions a)–c) mentioned at beginning of §1, and $s(0)=m(0)=k(0)$. In the case of that $\lim_{x\to l_i,\, x\in I}(|s(x)|+|m(x)|+|k(x)|)$

$<\infty$, we set a sticky elastic boundary condition at l_i as in (2.10), $i=1,2$. Then the generalized diffusion operator $\mathcal{G}=(d(d/ds)-dk)/dm$ on I and the elementary solution $p(t,x,y)$ of the generalized diffusion equation (1.1) on I are defined in the same way as in §2 (for details see [2], [6], [9] and [10]). In general, $p(t,x,y)$ is expressed in terms of solutions $\varphi_i(x,\lambda), x\in I, \lambda\in\boldsymbol{C}, i=1,2$ of the integral equations (2.2), and a nonnegative definite matrix valued measure $[\xi_{ij}(d\lambda)]_{i,j=1,2}$. Namely,

$$(3.1)\qquad p(t,x,y)=\sum_{i,j=1,2}\int_{\boldsymbol{R}} e^{-\lambda t}\varphi_i(x,\lambda)\varphi_j(y,\lambda)\xi_{ij}(d\lambda),\quad t>0,\ x,y\in I.$$

In particular, if $\lim_{X\to l_i, X\in I}(|s(x)|+|m(x)|+|k(x)|)<\infty$ for $i=1$ or 2, then (3.1) is reduced to the following expression.

$$(3.2)\qquad p(t,x,y)=\int_{\boldsymbol{R}} e^{-\lambda t}\psi(x,\lambda)\psi(y,\lambda)\xi(d\lambda),\quad t>0,\ x,y\in I,$$

where $\psi(x,\lambda)$, $x\in I$, $\lambda\in\boldsymbol{C}$ is a linear combination of $\varphi_i(x,\lambda), i=1,2$, and $\xi(d\lambda)$ is a Radon measure. $\Sigma\equiv\mathrm{Supp}(\xi_{11})\cup\mathrm{Supp}(\xi_{22})=\mathrm{Supp}(\xi)$ is the spectrum of $\mathcal{G}$ and it is bounded from below.

Now we notice the following estimates. Put $U(x,\lambda)=|\lambda||m(x)|+|\int_{0^+}^{x^+}d|k|(y)|$, where $d|k|(y)$ is the measure induced by the total variation of k, Then

$$(3.3)\qquad |\varphi_i(x,\lambda)|\le(1+|s(x)|)\exp(|s(x)|U(x,\lambda)),$$

$$(3.4)\qquad |\varphi_i^+(x,\lambda)|\le(1+U(x,\lambda))\exp(|s(x)|U(x,\lambda)),$$

$$(3.5)\qquad |\varphi_i(x,\lambda)-\varphi_i(x,\mu)|\le|\lambda-\mu|s(x)m(x)(1+|s(x)|)\exp\{|s(x)|(U(x,\lambda)+U(x,\mu))\},$$

for $x\in I, \lambda,\mu\in\boldsymbol{C}$, $i=1,2$. (3.3) follows from (2.2) and Gronwall-Bellman inequality. (2.2) also implies that

$$\varphi_i^+(x,\lambda)=i-1+(2-i)(-\lambda m(x)+k(x))+\int_{0^+}^{x^+}\int_{0^+}^{y^+}\varphi_i^+(z,\lambda)ds(z)(-\lambda dm(y)+dk(y)),$$

$$\varphi_i(x,\lambda)-\varphi_i(x,\mu)=(\mu-\lambda)\int_{0^+}^{x^+}\int_{0^+}^{y^+}\varphi_i(z,\mu)dm(z)ds(y)+\int_{0^+}^{x^+}\int_{0^+}^{y^+}(\varphi_i(z,\lambda)-\varphi_i(z,\mu))(-\lambda dm(z)+dk(z))ds(y).$$

Applying Gronwall-Bellman inequality again, we get (3.4), and (3.5) by (3.3).

Next we observe the asymptotic behavior of $p(t, x, y)$ as $t \to \infty$. We denote by λ_* the principal eigenvalue of $\mathcal{Q}$, that is,

$$\lambda_* = \min \Sigma.$$

PROPOSITION 3.1. 1) *It holds that*

(3.6) $\lim_{t\to\infty} e^{\lambda_* t} p(t, x, y) = \sum_{i,j=1,2} \varphi_i(x, \lambda_*)\varphi_j(y, \lambda_*)\xi_{ij}(\{\lambda_*\}), \quad x, y \in I.$

2) *If there exist an* $\alpha > 0$ *and* $A_{ij} \in \boldsymbol{R}$, $i, j = 1, 2$ *such that*

(3.7) $$\lim_{\varepsilon \downarrow 0} \varepsilon^{-\alpha}\xi_{ij}([\lambda_*, \lambda_* + \varepsilon)) = A_{ij},$$

then

(3.8) $\lim_{t\to\infty} t^\alpha e^{\lambda_* t} p(t, x, y) = \Gamma(\alpha+1) \sum_{i,j=1,2} A_{ij}\varphi_i(x, \lambda_*)\varphi_j(y, \lambda_*), \quad x, y \in I.$

Proof. 1) (3.6) is obvious. 2) Assume (3.7). Then there is an $\varepsilon_0 > 0$ such that $A_{ij} - 1 \le \varepsilon^{-\alpha}\xi_{ij}([\lambda_*, \lambda_* + \varepsilon)) \le A_{ij} + 1$, $0 < \varepsilon < \varepsilon_0$, $i, j = 1, 2$. Fix an $\varepsilon \in (0, \varepsilon_0)$ and $x, y \in I$ arbitrarily. We decompose each term in the expression (3.1) as follows.

(3.9)
$$\begin{aligned}
&\int_\Sigma e^{-\lambda t}\varphi_j(y, \lambda)\xi_{ij}(d\lambda) \\
&= \varphi_i(x, \lambda_*)\varphi_j(y, \lambda_*) \int_{[\lambda_*, \lambda_* + \varepsilon)} e^{-\lambda t}\xi_{ij}(d\lambda) \\
&\quad + \int_{[\lambda_*, \lambda_* + \varepsilon)} e^{-\lambda t}\{\varphi_i(x, \lambda) - \varphi_i(x, \lambda_*)\}\varphi_j(y, \lambda)\xi_{ij}(d\lambda) \\
&\quad + \varphi_i(x, \lambda_*) \int_{[\lambda_*, \lambda_* + \varepsilon)} e^{-\lambda t}\{\varphi_j(y, \lambda) - \varphi_j(y, \lambda_*)\}\xi_{ij}(d\lambda) \\
&\quad + \int_{[\lambda_* + \varepsilon, \infty)} e^{-\lambda t}\varphi_i(x, \lambda)\varphi_j(y, \lambda)\xi_{ij}(d\lambda) \\
&\equiv J^\varepsilon_{ij1}(t) + J^\varepsilon_{ij2}(t) + J^\varepsilon_{ij3}(t) + J^\varepsilon_{ij4}(t).
\end{aligned}$$

Note that $\xi_{ij}(\{\lambda_*\}) = 0$, $i, j = 1, 2$. Integrating by parts, we see

$$\begin{aligned}
&\int_{[\lambda_*, \lambda_* + \varepsilon)} e^{-\lambda t}\xi_{ij}(d\lambda) \\
&= e^{-(\lambda_* + \varepsilon)t}\xi_{ij}([\lambda_*, \lambda_* + \varepsilon)) + e^{-\lambda_* t}\int_0^{\varepsilon t} e^{-\lambda}\xi_{ij}([\lambda_*, \lambda_* + \lambda/t))\,d\lambda.
\end{aligned}$$

Since $|t^\alpha \xi_{ij}([\lambda_*, \lambda_* + \lambda/t))| \le (|A_{ij}| + 1)\lambda^\alpha$, $0 < \lambda < \varepsilon t$, $t > 0$, we get by Lebesgue's dominated convergence theorem

(3.10) $$\lim_{t\to\infty} t^\alpha e^{\lambda_* t} \int_{[\lambda_*, \lambda_* + \varepsilon)} e^{-\lambda t}\xi_{ij}(d\lambda)$$

$$=\int_0^\infty e^{-\lambda}\lim_{t\to\infty} t^\alpha \xi_{ij}([\lambda_*, \lambda_*+\lambda/t))d\lambda = A_{ij}\int_0^\infty e^{-\lambda}\lambda^\alpha d\lambda$$
$$=A_{ij}\Gamma(\alpha+1).$$

Therefore we obtain

$$\lim_{t\to\infty} t^\alpha e^{\lambda_* t} J^\varepsilon_{ij1}(t) = \Gamma(\alpha+1)A_{ij}\varphi_i(x, \lambda_*)\varphi_j(y, \lambda_*). \tag{3.11}$$

In view of (3.3) and (3.5), there are some positive constants c_1 and c_2, independent of t and ε, so that

$$|J^\varepsilon_{ij2}(t)+J^\varepsilon_{ij3}(t)| \le c_1\varepsilon \prod_{k=i,j}\left(\int_{[\lambda_*, \lambda_*+\varepsilon)} e^{-\lambda(t-c_2)}\xi_{kk}(d\lambda)\right)^{1/2}.$$

In the same way as in (3.10), we get

$$\lim_{t\to\infty} t^\alpha e^{\lambda_* t}\int_{[\lambda_*, \lambda_*+\varepsilon)} e^{-\lambda(t-c_2)}\xi_{kk}(d\lambda)<\infty, \quad k=1, 2.$$

Consequently,

$$\lim_{\varepsilon\downarrow 0}\lim_{t\to\infty} t^\alpha e^{\lambda_* t}|J^\varepsilon_{ij2}(t)+J^\varepsilon_{ij3}(t)|=0. \tag{3.12}$$

We next choose a $c_3\in(0, 1)$ such that $c_3(\lambda_*+\varepsilon)\le\varepsilon/2$. Since $\lambda_*+\varepsilon\le\lambda$ implies $c_3\lambda\le\lambda-\lambda_*-\varepsilon/2$, we then find by (3.3) that

$$e^{(\lambda_*+\varepsilon/2)t}|J^\varepsilon_{ij4}(t)|$$
$$\le \prod_{k=i,j}\left(\int_{[\lambda_*+\varepsilon, \infty)} e^{-(\lambda-\lambda_*-\varepsilon/2)t}\varphi_k(x, \lambda)^2\xi_{kk}(d\lambda)\right)^{1/2}$$
$$\le \prod_{k=i,j}\left(\int_{[\lambda_*+\varepsilon, \infty)} e^{-c_3\lambda t}\varphi_k(x, \lambda)^2\xi_{kk}(d\lambda)\right)^{1/2}.$$

Noting that

$$\int_{[\lambda_*+\varepsilon, \infty)} e^{-\lambda t}\varphi_k(x, \lambda)^2\xi_{kk}(d\lambda)<\infty, \quad t>0, \quad x\in I,$$

we get

$$\lim_{t\to\infty} e^{(\lambda_*+\varepsilon/2)t}|J^\varepsilon_{ij4}(t)|=0. \tag{3.13}$$

Combining (3.13) with (3.9), (3.11) and (3.12), we have (3.8).

Remark 3.2. When $p(t, x, y)$ is expressed as (3.2), the assertion of Propopsition 3.1 is rewritten as follows.

$$\lim_{t\to\infty} e^{\lambda_* t}p(t, x, y)=\phi(x, \lambda_*)\phi(y, \lambda_*)\xi(\{\lambda_*\}), \quad x, y\in I. \tag{3.14}$$

If it holds that

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-\alpha}\xi([\lambda_*, \lambda_*+\varepsilon))=A, \tag{3.15}$$

for some $A\in\boldsymbol{R}$ and $\alpha>0$, then

$$\lim_{t\to\infty} t^\alpha e^{\lambda_* t}p(t, x, y)=\Gamma(\alpha+1)A\phi(x, \lambda_*)\phi(y, \lambda_*), \quad x, y\in I. \tag{3.16}$$

We turn to a global asymptotic estimate of $p(t, x, y)$. In the following $\xi(d\lambda\,;\,x, y)$ stands for $\sum_{i,j=1,2}\varphi_i(x,\lambda)\varphi_j(y,\lambda)\xi_{ij}(dy)$, or $\psi(x,\lambda)\psi(y,\lambda)\xi(d\lambda)$ in the case of (3.2).

PROPOSITION 3.3. *Assume that there exist a positive real valued function g on I and a Radon measure ξ_0 whose support includes $\sum$ such that*

$$\int_{\Sigma} e^{-\lambda t}\xi_0(d\lambda)<\infty, \quad \textit{for some } t>0, \tag{3.17}$$

$$\xi(E\,;\,x, x)\leq g(x)^2\xi_0(E), \quad \textit{for } x\in I \textit{ and Borel sets } E\subset I. \tag{3.18}$$

Then

$$\overline{\lim_{t\to\infty}}\sup_{x,y\in I}\,(p(t,x,y)/g(x)g(y))/K_\varepsilon(t)<\infty, \quad \varepsilon>0, \tag{3.19}$$

where

$$K_\varepsilon(t)=\int_{[\lambda_*,\lambda_*+\varepsilon)} e^{-\lambda t}\xi_0(d\lambda). \tag{3.20}$$

Proof. First we note that

$$\lim_{t\to\infty} e^{(\lambda_*+\varepsilon_1)t}K_{\varepsilon_2}(t)=\infty, \quad \varepsilon_1, \varepsilon_2>0.$$

Indeed, for $0<\delta<\min\{\varepsilon_1, \varepsilon_2\}$.

$$K_{\varepsilon_2}(t)\geq e^{-(\lambda_*+\delta)t}\xi_0([\lambda_*, \lambda_*+\delta))>0,$$

and hence

$$e^{(\lambda_*+\varepsilon_1)t}K_{\varepsilon_2}(t)\geq e^{(\varepsilon_1-\delta)t}\xi_0([\lambda_*, \lambda_*+\delta))\to\infty \text{ as } t\to\infty.$$

On the other hand, by the same argument as for (3.13),

$$\lim_{t\to\infty} e^{(\lambda_*+\varepsilon/2)t}\int_{[\lambda_*+\varepsilon,\infty)} e^{-\lambda t}\xi_0(d\lambda)=0.$$

Therefore

$$\overline{\lim_{t\to\infty}}\sup_{x\in I}\,(p(t,x,x)/g(x)^2)/K_\varepsilon(t)\leq\lim_{t\to\infty}\int_{\Sigma} e^{-\lambda t}\xi_0(d\lambda)/K_\varepsilon(t)=1.$$

Since $p(t, x, y)\leq\{p(t, x, x)p(t, y, y)\}^{1/2}$ (see [7 ; (3.13)]), we have (3.19).

4. Proof of Theorems

By the same argument as the standard one for Hill's equations, the equation

$$r^2-\Delta(\lambda)r+\rho=0$$

has two distinct solutions $r_i(\lambda)$, $i=1,2$ with $0<|r_1(\lambda)|<|r_2(\lambda)|$ for $\lambda\in\boldsymbol{C}\backslash S$ (cf. [4 ; §2]). We note that the functions $r_i(\lambda)$ are both analytic in $\boldsymbol{C}\backslash S$. For $\lambda\in S$, we put $r_i(\lambda)=\lim_{v\downarrow 0} r_i(\lambda+\sqrt{-1}v)$,

$i=1, 2$ conventionally. Then we can show two identities in [4 ; (2. 10)], that is,

$$\begin{aligned}\varphi_1(x+n) &= \{(r_1^{n+1}-r_2^{n+1})/(r_1-r_2)-\rho\varphi_2^+(1)(r_1^n-r_2^n)/\\ &\qquad (r_1-r_2)\}\varphi_1(x)+\{\rho\varphi_1^+(1)(r_1^n-r_2^n)/(r_1-r_2)\}\varphi_2(x).\\ \varphi_2(x+n) &= \{\varphi_2(1)(r_1^n-r_2^n)/(r_1-r_2)\}\varphi_1(x)\\ &\qquad +\{\rho\varphi_2^+(1)(r_1^n-r_2^n)/(r_1-r_2)-\rho(r_1^{n-1}-r_2^{n-1})/\\ &\qquad (r_1-r_2)\}\varphi_2(x).\end{aligned} \tag{4. 1}$$

where $x\in\boldsymbol{R}$, $n\in\boldsymbol{Z}$, $\varphi_i(\cdot)=\varphi_i(\cdot, \lambda)$, $\varphi_i^+(\cdot)=\varphi_i^+(\cdot, \lambda)$, $r_i=r_i(\lambda)$, $\lambda\in\boldsymbol{C}$, $i=1, 2$. Further $(r_1^n-r_2^n)/(r_1-r_2)$ is understood to be nr_1^{n-1} in case of $r_1=r_2(=\pm\sqrt{\rho})$.

Now we will give

Proof of Lemma 2. 1. We set

$$\Phi(x)=\varphi_2(1, \lambda_0)^{1/2}\varphi_1(x, \lambda_0)+(-1)^{\varepsilon}(-\rho\varphi_1^+(1, \lambda_0))^{1/2}\varphi_2(x, \lambda_0),\quad x\in\boldsymbol{R},$$

where $\varepsilon=0$ [$=1$] if $\varphi_1(1, \lambda_0)\leq\sqrt{\rho}$ [resp. $\varphi_1(1, \lambda_0)>\sqrt{\rho}$]. Note that $\varphi_1^+(1, \lambda_0)\leq 0$ by (2. 7). It follows from (2. 3) and (2. 7) that

$$(\sqrt{\rho}-\varphi_1(1, \lambda_0))^2=-\rho\varphi_1^+(1, \lambda_0)\varphi_2(1, \lambda_0). \tag{4. 2}$$

Hence $\alpha(x, y, \lambda_0)$ defined by (2. 9) is written as

$$\alpha(x, y, \lambda_0)=\Phi(x)\Phi(y)/\alpha_0,\quad x, y\in\boldsymbol{R}.$$

Since λ_0 is the principal eigenvalue of $\mathcal{Q}$ and Φ satisfies the generalized diffusion equation $\mathcal{Q}\Phi(x)=-\lambda_0\Phi(x)$, $x\in\boldsymbol{R}$, we find by virtue of [3] that $\Phi(x)>0$, $x\in\boldsymbol{R}$, whence $\alpha(x, y, \lambda_0)>0$, $x, y\in\boldsymbol{R}$.

On the other hand, by means of (4. 1) and (4. 2),

$$\Phi(x+n)=\rho^{n/2}\Phi(x),\qquad x\in\boldsymbol{R},\ n\in\boldsymbol{Z}.$$

Therefore by the continuity of $\Phi(x)$,

$$\sup_{x\in\boldsymbol{R}}\rho^{-x/2}\Phi(x)=\sup_{0\leq x\leq 1}\rho^{-x/2}\Phi(x)<\infty.$$

From this we obtain the second assertion of the lemma.

As was noted in § 2, the spectrum of $\mathcal{Q}$ is continuous and coincides with S. Further, the spectral measures $d\sigma_{ij}$, $i, j=1, 2$ are absolutely continuous with respect to the Lebesgue measure in S ([8 ; Theorem 2]). Denote the densities by ρ_{ij}. Let $\mathring{S}$ be the interior of S. Each ρ_{ij} is continuous in $\mathring{S}$ and satisfies

$$(4.3)\qquad \begin{aligned} &\rho_{11}(\lambda)>0,\ \ \rho_{22}(\lambda)>0,\ \ \lambda\in\mathring{S},\\ &\rho_{11}(\lambda)\rho_{22}(\lambda)\geq\rho_{12}(\lambda)\rho_{21}(\lambda)\geq 0,\ \ \lambda\in S,\end{aligned}$$

(cf. [8 ; (2.17)]). Also, note that

$$(4.4)\qquad \int_S e^{-\lambda t}|\rho_{ij}(\lambda)|d\lambda<\infty,\quad t>0,\ \ i,j=1,2.$$

Now $p(t,x,y)$ is rewritten as

$$p(t,x,y)=\sum_{i,j=1,2}\int_S e^{-\lambda t}\varphi_i(x,\lambda)\varphi_j(y,\lambda)\rho_{ij}(\lambda)d\lambda.$$

As was seen in [8 ; Theorem 3],

$$(4.5)\qquad \lim_{\lambda\downarrow\lambda_0}\rho_{ij}(\lambda)(\lambda-\lambda_0)^{-\delta_{ij}}=C_{ij},\quad i,j=1,2,$$

where C_{ij} and δ_{ij} are given by

$$\begin{aligned} &C_{11}=\alpha_{11}/\sqrt{\pi},\ \ \delta_{11}=-1/2;\\ &C_{ij}=\alpha_{ij}/\sqrt{\pi},\ \ \delta_{ij}=-1/2,\ \ \textit{if}\ \ i+j\in\{3,4\}\ \text{ and }\ \varphi_1{}^+(1,\lambda_0)<0;\\ &C_{ij}=(\sqrt{\rho}/\alpha_0\sqrt{\pi})\int_{0+}^{1+}\varphi_{3-i}(x,\lambda_0)\varphi_{3-j}(x,\lambda_0)dm(x),\ \ \delta_{ij}=1/2,\\ &\qquad\qquad \textit{if}\ \ i+j\in\{3,4\}\ \text{ and }\ \varphi_1{}^+(1,\lambda_0)=0.\end{aligned}$$

Proof of Theorem 1. First we note that, if $\varphi_1{}^+(1,\lambda_0)=0$, then $\varphi_1(1,\lambda_0)=\sqrt{\rho}$ by (2.3). Hence by the definition (2.8) of α_{ij},

$$\alpha_{ij}=0,\qquad \text{if } i+j\in\{3,4\}\ \text{ and }\ \varphi_1{}^+(1,\lambda_0)=0.$$

This coupled with (4.5) leads us to

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1/2}\int_{[\lambda_0,\lambda_0+\varepsilon)\cap S}\rho_{ij}(\lambda)d\lambda=2\alpha_{ij}/\sqrt{\pi},\ \ i,j=1,2.$$

Therefore (1.2) follows from Proposition 3.1.

By means of (4.1) and [8 ; (2.17)],

$$\begin{aligned}&\sum_{i,j=1,2}\varphi_i(x+n,\lambda)\varphi_j(y+n,\lambda)\rho_{ij}(\lambda)\\ &\quad=\rho^n\sum_{i,j=1,2}\varphi_i(x,\lambda)\varphi_j(y,\lambda)\rho_{ij}(\lambda),\ \ x,y\in\boldsymbol{R},\ \ n\in\boldsymbol{Z},\ \ \lambda\in\mathring{S}.\end{aligned}$$

By virture of (3.3) we have two constants c_1 and c_2 such that

$$\sup_{0\leq x\leq 1}|\varphi_i(x,\lambda)|\leq c_1e^{c_2\lambda},\quad \lambda\geq\lambda_0,\ \ i=1,2.$$

Since (4.3) yields that $\sum_{i,j=1,2}|\rho_{ij}(\lambda)|\leq(\sqrt{\rho_{11}(\lambda)}+\sqrt{\rho_{22}(\lambda)})^2\leq 2(\rho_{11}(\lambda)+\rho_{22}(\lambda))$, there are positive constants c_3 and c_4 such that

$$\begin{aligned}&\sup_{x\in\boldsymbol{R}}\rho^{-x}\sum_{i,j=1,2}\varphi_i(x,\lambda)\varphi_j(x,\lambda)\rho_{ij}(\lambda)\\ &\quad=\sup_{0\leq x\leq 1}\rho^{-x}\sum_{i,j=1,2}\varphi_i(x,\lambda)\varphi_j(x,\lambda)\rho_{ij}(\lambda)\end{aligned}$$

$$\leq c_3 e^{c_4\lambda}(\rho_{11}(\lambda)+\rho_{22}(\lambda)), \quad \lambda\in\mathring{S}.$$

Putting $\xi_0(E)=c_3\int_{E\cap S} e^{c_4\lambda}(\rho_{11}(\lambda)+\rho_{22}(\lambda))d\lambda$ for Borel sets E, we get by (4.4)

$$\int_S e^{-\lambda t}\xi_0(d\lambda)\leq c_3\sum_{i=1,2}\int_S e^{-\lambda t/2}\rho_{ii}(\lambda)d\lambda<\infty, \quad t\geq 2c_4.$$

Therefore the assumption of Proposition 3.3 is satisfied for this ξ_0 and $g(x)\equiv\rho^{x/2}$. Further by (4.5)

$$\lim_{t\to\infty} t^{1/2}e^{\lambda_0 t}\int_{[\lambda_0,\lambda_0+\varepsilon)} e^{-\lambda t}\xi_0(dy)<\infty, \quad \varepsilon>0.$$

This with (3.19) gives us (1.3). (1.4) follows from (1.3) and Lebesgues's dominated convergence theorem.

We turn to the evaluations of $p^\gamma(t, x, y)$. As was seen in [8; Theorems 4 and 7], the spectrum Σ^γ of $\mathcal{Q}^\gamma$ consists of the continuous spectrum Σ_c^γ and the point spectrum Σ_p^γ. Σ_c^γ coincides with S. Σ_p^γ is a countable set of $\boldsymbol{R}$ and bounded from below. We note the following.

LEMMA 4.1. *For each* $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\Gamma$, *there exist positive constants* C_i, $i=1,2$ *such that*

$$(4.6)\quad \sup_{x\geq 0}\rho^{-x/2}(1+x)^{-1}|\phi_1^\gamma(x,\lambda)| \leq C_1\{|\phi_1^\gamma(1,\lambda)|+|\phi_1^{\gamma+}(1,\lambda)|\}e^{C_2\lambda}, \quad \lambda\in S,$$

$$(4.7)\quad \sup_{x\geq 0}\rho^{-x/2}|\phi_1^\gamma(x,\lambda)|\leq C_1\{|\phi_1^\gamma(1,\lambda)|+|\phi_1^{\gamma+}(1,\lambda)|\}e^{C_2\lambda}, \quad \lambda\in\Sigma_p^\gamma.$$

Proof. Fix a triplet $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\Gamma$ arbitrarily. $\phi_1^\gamma(x,\lambda)$, $\lambda\in\boldsymbol{C}$ satisfies the integral equation

$$\phi_1^\gamma(x,\lambda)=\phi_1^\gamma(1,\lambda)-\phi_1^{\gamma+}(1,\lambda)(s(1)-s(x)) + \int_{x+}^{1+}(s(y)-s(x))\phi_1^\gamma(y,\lambda)(-\lambda dm(y)+dk(y)), \quad x\in\boldsymbol{R}_+.$$

Therefore by Gronwall-Bellman inequality, there are some positive constants c_i, $i=1,2$ satisfying

$$(4.8)\quad \sup_{0\leq x\leq 1}|\phi_1^\gamma(x,\lambda)|\leq c_1\{|\phi_1^\gamma(1,\lambda)|+|\phi_1^{\gamma+}(1,\lambda)|\}e^{c_2|\lambda|}, \quad \lambda\in\boldsymbol{C}.$$

On the other hand, since $\varphi_i(x+1,\lambda)=\varphi_i(1,\lambda)\varphi_1(x,\lambda)+\rho\varphi_i^+(1,\lambda)\varphi_2(x,\lambda)$, $i=1,2$, we find that

$$(4.9)\quad \phi_1^\gamma(x+1,\lambda)=\phi_1^\gamma(1,\lambda)\varphi_1(x,\lambda)+\rho\phi_1^{\gamma+}(1,\lambda)\varphi_2(x,\lambda), x\in\boldsymbol{R}, \lambda\in\boldsymbol{C}.$$

If $\lambda\in S$, then $|r_1(\lambda)|=|r_2(\lambda)|=\sqrt{\rho}$ and hence by (4.1)

$$
\begin{aligned}
&|\phi_1{}^\tau(x+n,\ \lambda)| \\
&\le |\phi_1{}^\tau(1,\lambda)|\,|\varphi_1(x+n-1,\lambda)|+\rho|\phi_1{}^{\tau+}(1,\lambda)|\,|\varphi_2(x+n-1,\lambda)| \\
&\le n\rho^{n/2}[\,|\phi_1{}^\tau(1,\ \lambda)|\,\{(\rho^{-1/2}+|\varphi_2{}^+(1,\ \lambda)|)\,|\varphi_1(x,\lambda)|+|\varphi_1{}^+(1,\lambda)| \\
&\quad |\varphi_2(x,\lambda)|\}+|\phi_1{}^{\tau+}(1,\lambda)|\,\{|\varphi_2(1,\lambda)|\,|\varphi_1(x,\lambda)| \\
&\quad +(\rho^{1/2}+\rho|\varphi_2{}^+(1,\lambda)|)\,|\varphi_2(x,\lambda)|\}\,],
\end{aligned}
$$

for $x\in\boldsymbol{R}_+$ and $n\in\boldsymbol{N}$. Since S is bounded from below, in view of (3.3) and (3.4) we have two positive constants c_3 and c_4 such that

$$
(4.10)\quad \sup_{x\ge1}\ \rho^{-x/2}x^{-1}|\phi_1{}^\tau(x,\ \lambda)|\le c_3\{|\phi_1{}^\tau(1,\ \lambda)|+|\phi_1{}^{\tau+}(1,\ \lambda)|\}\,e^{c_4\lambda},\quad \lambda\in S.
$$

Next let $\lambda\in\sum_p{}^\tau$. Then, by virtue of [8 ; Lemma 5.1], $\lambda\in\boldsymbol{R}$ and it holds that

$$
\begin{aligned}
&(\gamma_1\lambda-\gamma_3)\phi_1{}^\tau(1,\lambda)+\rho\phi_1{}^{\tau+}(1,\lambda)=\phi_1{}^\tau(1,\lambda)-r_1(\lambda)=0, && \text{if } \gamma_2=1,\\
&\phi_1{}^\tau(1,\lambda)=\rho\phi_1{}^{\tau+}(1,\lambda)-r_1(\lambda)=0, && \text{if } \gamma_2=0.
\end{aligned}
$$

This with (4.9) implies that

$$
(4.11)\quad \phi_1{}^\tau(x+n,\lambda)=r_1(\lambda)^n\phi_1{}^\tau(x,\lambda),\qquad x\in\boldsymbol{R}_+,\ n\in\boldsymbol{Z}_+,\ \lambda\in\sum_p{}^\tau,
$$

where $\boldsymbol{Z}_+=\boldsymbol{Z}\cap[0,\infty)$. Since $0<|r_1(\lambda)|<\sqrt{\rho}$ for $\lambda\in\sum_p{}^\tau$, we get

$$
(4.12)\quad \sup_{x\ge0}\ \rho^{-x/2}|\phi_1{}^\tau(x,\lambda)|\le\sup_{0\le x\le1}\ \rho^{-x/2}|\phi_1{}^\tau(x,\lambda)|,\quad \lambda\in\sum_p{}^\tau.
$$

(4.8), (4.10) and (4.12) give us (4.6) and (4.7).

Remark 4.2. By (4.1) we also have

$$
\begin{aligned}
\varphi_1(x+n,\lambda_0)&=\rho^{n/2}\varphi_1(x,\lambda_0)+n\rho^{(n-1)/2}\{(\varphi_1(1,\lambda_0)-\sqrt{\rho})\varphi_1(x,\lambda_0)\\
&\qquad+\rho\varphi_1{}^+(1,\lambda_0)\varphi_2(x,\lambda_0)\},\\
\varphi_2(x+n,\lambda_0)&=\rho^{n/2}\varphi_2(x,\lambda_0)+n\rho^{(n-1)/2}\{\varphi_2(1,\lambda_0)\varphi_1(x,\lambda_0)\\
&\qquad-(\varphi_1(1,\lambda_0)-\sqrt{\rho})\varphi_2(x,\lambda_0)\},
\end{aligned}
$$

for $x\in\boldsymbol{R}$, $n\in\boldsymbol{Z}$. Hence,

$$
(4.13)\quad \phi_1{}^\tau(x+n,\lambda_0)=\rho^{n/2}\phi_1{}^\tau(x,\lambda_0)+n\rho^{(n-1)/2}\Psi^\tau(x),\quad x\in\boldsymbol{R}_+,\ n\in\boldsymbol{Z}_+,
$$

where $\Psi^\tau(x)$ is given by

$$
\begin{aligned}
\Psi^\tau(x)=&\{\gamma_2(\varphi_1(1,\lambda_0)-\sqrt{\rho})-(\gamma_1\lambda_0-\gamma_3)\varphi_2(1,\lambda_0)\}\,\varphi_1(x,\lambda_0)\\
&+\{\gamma_2\rho\varphi_1{}^+(1,\lambda_0)+(\gamma_1\lambda_0-\gamma_3)\,(\varphi_1(1,\lambda_0)-\sqrt{\rho})\}\,\varphi_2(x,\lambda_0).
\end{aligned}
$$

In particular, if $\gamma_3=\kappa_0(\gamma_1)$, then $\Psi^\tau(x)\equiv0$ by (4.2). Hence,

$$
\begin{aligned}
&\phi_1{}^\tau(x+n,\lambda_0)=\rho^{n/2}\phi_1{}^\tau(x,\lambda_0),\quad x\in\boldsymbol{R}_+,\ n\in\boldsymbol{Z}_+,\\
(4.14)\quad &\sup_{x\ge0}\ \rho^{-x/2}\phi_1{}^\tau(x,\lambda_0)=\sup_{0\le x\le1}\ \rho^{-x/2}\phi_1{}^\tau(x,\lambda_0)<\infty,\qquad \text{in Case II.}
\end{aligned}
$$

Now we will show Lemma 2.2.

Proof of Lemma 2.2. λ^τ is the principal eigenvalue of $\mathcal{G}^\tau$ and

$\phi_1{}^\tau(x, \lambda^\tau)$ satisfies the generalized diffusion epuation $\mathcal{G}^\tau\phi_1{}^\tau(\cdot, \lambda^\tau))(x) = -\lambda^\tau\phi_1{}^\tau(x, \lambda^\tau)$, $x \in \boldsymbol{R}_+$ and $\phi_1{}^\tau(0, \lambda^\tau) = \gamma_2$. Then, with the aid of [3], we get that $\phi_1{}^\tau(x, \lambda^\tau) > 0$ for $x \geq 0$ [$x > 0$] if $\gamma_2 = 1$ [resp. $\gamma_2 = 0$]. Therefore $\beta^\tau(x, y, \lambda^\tau) > 0$ for $x, y \geq 0$ [$x, y > 0$] in Cases I, II and III [resp. Case IV].

Since λ^τ belongs to $\sum_p{}^\tau$ in Case I and it coincides with λ_0 ($\in S$) in the other cases, the second assertion of the lemma follows from Lemma 4.1 and Remark 4.2.

The spectral measure $d\sigma^\tau$ is absolutely continuous with respect to the Lebesgue measure in S. Denote by ρ^τ and $\sigma^\tau(\{\lambda\})$ the density function and the mass of spectral measure at $\lambda \in \sum_p{}^\tau$, respectively. Then by virtue of [8 ; (2.22)]

$$\rho^\tau(\lambda) > 0, \quad \lambda \in \mathring{S}.$$

Now we have the following expression.

$$p^\tau(t, x, y) = \int_S e^{-\lambda t}\phi_1{}^\tau(x, \lambda)\phi_1{}^\tau(y, \lambda)\rho^\tau(\lambda)d\lambda + \sum_{\lambda \in \sum_p^\tau} e^{-\lambda t}\phi_1{}^\tau(x, \lambda)\phi_1{}^\tau(y, \lambda)\sigma^\tau(\{\lambda\}).$$

We should notice that

$$\begin{aligned}(4.15)\quad & \int_{\Sigma^\tau} e^{-\lambda t}\{\phi_1{}^\tau(1, \lambda)^2 + \phi_1{}^{\tau+}(1, \lambda)^2\}d\sigma^\tau(\lambda) \\ &= \int_S e^{-\lambda t}\{\phi_1{}^\tau(1, \lambda)^2 + \phi_1{}^{\tau+}(1, \lambda)^2\}\rho^\tau(\lambda)d\lambda \\ &\quad + \sum_{\lambda \in \sum_p^\tau} e^{-\lambda t}\{\phi_1{}^\tau(1, \lambda)^2 + \phi_1{}^{\tau+}(1, \lambda)^2\}\sigma^\tau(\{\lambda\}) < \infty, \quad t > 0.\end{aligned}$$

We will observe the mass $\sigma^\tau(\{\lambda\})$ of spectral measure at $\lambda \in \sum_p{}^\tau$

LEMMA 4.3. *Let* $\lambda \in \sum_p{}^\tau \neq \phi$. *Then*

$$\begin{aligned}(4.16)\quad \sigma^\tau(\{\lambda\}) &= \left(\gamma_1 + (|r_2(\lambda)|/\sqrt{D(\lambda)})\int_{0+}^{1+}\phi_1{}^\tau(x, \lambda)^2 dm(x)\right)^{-1} \\ &= \left(\int_{\boldsymbol{R}_+}\phi_1{}^\tau(x, \lambda)^2 dm^\tau(x)\right)^{-1}.\end{aligned}$$

Proof. Fix $\nu \in \sum_p{}^\tau$ arbitrarily. First we assume $\gamma_2 = 1$. Then it follows from [8 ; (2.20) and (2.21)] that $\sigma^\tau(\{\nu\})$ is the residue of the function $(\gamma_1\lambda - \gamma_3 + f_1(\lambda))^{-1}$ at $\lambda = \nu$. By [8 ; (5.4)] this is identical with

$$(\gamma_1 + r_1'(\nu))^{-1} = \left(\gamma_1 + (|r_2(\nu)|/\sqrt{D(\nu)})\int_{0+}^{1+}\phi_1{}^\tau(x, \nu)^2 dm(x)\right)^{-1}.$$

Suppose next that $\gamma_2=0$. Then by [8 ; Lemma 5.1], $\varphi_2(1,\nu)=0$ and $r_1(\nu)=\rho\varphi_2^+(1,\nu)\neq r_2(\nu)=\varphi_1(1,\nu)$. Further, [8 ; (2.20) and (2.21)] implies that $\sigma^\tau(\{\nu\})$ is the residue of the function $-f_1(\lambda)$ at $\lambda=\nu$. Therefore [8 ; (3.1) and (4.5)] leads us to

$$-(r_1(\nu)-\varphi_1(1,\nu))/\varphi_2'(1,\nu)$$
$$=(r_2(\nu)-r_1(\nu))/r_2(\nu)\int_{0+}^{1+}\varphi_2(x,\nu)^2dm(x).$$

Since $r_i(\nu)=\{\Delta(\nu)+(-1)^{i+\varepsilon}\sqrt{D(\nu)}\}/2$, $i=1,2$ with $\varepsilon=0$ or 1 according to $\Delta(\nu)>2\sqrt{\rho}$ or $\Delta(\nu)<2\sqrt{\rho}$, we get first equality in (4.16).

Incidentally, by the definition of dm^τ and the property of $m(x)$, and by (4.11)

$$\int_{\boldsymbol{R}_+}\psi_1^\tau(x,\nu)^2dm^\tau(x)=\gamma_1+\sum_{n=0}^{\infty}\int_{0+}^{1+}\psi_1^\tau(x+n,\nu)^2dm(x+n)$$
$$=\gamma_1+\sum_{n=0}^{\infty}r_1(\nu)^{2n}\rho^{-n}\int_{0+}^{1+}\psi_1^\tau(x,\nu)^2dm(x).$$

By means of $\rho=r_1(\nu)r_2(\nu)$,

$$\sum_{n=0}^{\infty}r_1(\nu)^{2n}\rho^{-n}=\sum_{n=0}^{\infty}(r_1(\nu)/r_2(\nu))^n$$
$$=r_2(\nu)/(r_2(\nu)-r_1(\nu))=|r_2(\nu)|/\sqrt{D(\nu)}.$$

Thus we arrive at the second epuality in (4.16).

We are ready to prove Theorem 2.

Proof of Theorem 2. In Case I, λ^τ belongs to $\sum_p^\tau$ and, by (2.15) and Lemma 4.3, $\sigma^\tau(\{\lambda^\tau\})=\beta_0^\tau$. (3.14) asserts that (1.6) holds with $\delta^\tau=0$. In Cases II, III and IV, it follows from [8 ; Theorem 5] that

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-\delta^\tau}\int_{[\lambda_0,\lambda_0+\varepsilon)\cap S}\rho^\tau(\lambda)d\lambda=\beta_*^\tau/\delta^\tau, \tag{4.17}$$

where

$$\beta_*^\tau=\begin{cases}\beta_0^\tau/\sqrt{\pi}, & \text{in Case II,}\\ 2\beta_0^\tau/\sqrt{\pi}, & \text{in Cases III and IV.}\end{cases}$$

Thus (3.15) is satisfied with $\alpha=\delta^\tau$ and $A=\beta_*^\tau/\delta^\tau$. Since λ_0 is the principal eigenvalue of $\mathcal{Q}^\tau$ in these cases, (1.6) follows from (3.16).

In order to show (1.7), we put $g(x)=\rho^{x/2}(1+x)$, $x\in\boldsymbol{R}_+$, and

$$\xi_0(E)=C_1^2\int_{E\cap\Sigma}\{|\psi_1^\tau(1,\lambda)|+|\psi_1^{\tau+}(1,\lambda)|\}^2e^{2C_2\lambda}d\sigma^\tau(\lambda),$$

for Borel sets E. By virtue of (4.15),

$$\int_{\boldsymbol{R}} e^{-\lambda t}\xi_0(d\lambda)$$
$$=C_1^2\int_{\Sigma^r} e^{-\lambda(t-2C_2)}\{|\phi_1^r(1,\lambda)|+|\phi_1^{r+}(1,\lambda)|\}^2 d\sigma^r(\lambda)$$
$$\leq C_1^2\int_{\Sigma^r} e^{-\lambda t/2}\{|\phi_1^r(1,\lambda)|+|\phi_1^{r+}(1,\lambda)|\}^2 d\sigma^r(\lambda)<\infty,$$

for $t\geq 4C_2$. This with (4.6) and (4.7) tells us that all the conditions of Proposition 3.3 are fulfilled. Further, for $K_\varepsilon(t)\equiv\int_{[\lambda^r,\lambda_r+\varepsilon)} e^{-\lambda t}\xi_0(d\lambda)$,

$$\lim_{t\to\infty} e^{\lambda_0 t}K_\varepsilon(t)=\xi_0(\{\lambda^r\})<\infty, \qquad \text{in Case I;}$$

by means of (4.17) or [8 ; Theorem 5],

$$\lim_{t\to\infty} t^{\delta^r}e^{\lambda_0 t}K_\varepsilon(t)<\infty, \qquad \text{in Cases II, III and IV.}$$

Thus (1.7) is derived from (3.19). (1.8) follows from (1.7) and Lebesgue's dominated convergence theorem.

Proof of Corollary. First we note that, under the condition $k(x)\equiv 0$, $\rho\neq 1$ if and only if $\lambda_0>0$. This is proved by the same method as in [1 ; Lemma 2.3], or by using [5 ; Theorem 3 in Appendix I].

Now we assume that $k(x)\equiv 0$ and $\rho=1$. Therefore $\lambda_0=0$ and

$$\varphi_1(x,0)\equiv 1,\quad \varphi_2(x,\ 0)=s(x),\quad x\in\boldsymbol{R}.$$

Further, $\varphi_2^+(1,0)=1$ by $\Delta(0)=2$, and hence $\varphi_1^+(1,0)=0$ by (2.3). Also, by means of [8 ; (3.1) and (3.2)],

$$\Delta'(0)=-\varphi_2(1,0)\int_{0+}^{1+}\varphi_1^2(x,0)\,dm(x)=-s(1)m(1).$$

Accordingly,

$$\alpha_{11}=(s(1)/4\pi m(1))^{1/2},\quad \alpha_{12}=\alpha_{21}=\alpha_{22}=0,$$
$$\alpha(x,y,\lambda_0)=\alpha_{11}\varphi_1(x,0)\varphi_1(y,0)=\alpha_{11}.$$

In view of Theorem 1, we get (1.2), (1.3) and (1.4) with $\lambda_0=0$ and the above $\alpha(x,y,\lambda_0)$.

Since $\kappa_0(\varepsilon)\equiv 0$ for $\varepsilon\geq 0$, Case I does not occur and Case II [Cases III and IV] is reduced to the case $\gamma_3=0$ [resp. $\gamma_3>0$]. If $\gamma_3=0$ (i.e. Case II), then $\delta^r=1/2$ and

$$\beta^r(x,y,\lambda^r)=2\alpha_{11}\varphi_1(x,0)\varphi_1(y,0)=(s(1)/\pi m(1))^{1/2}.$$

If $\gamma_3>0$ (i.e. Cases III and IV), then $\delta^r=3/2$ and

$$\beta^r(x,y,\lambda^r)=(1/4\pi\alpha_{11}\gamma_3^2)\phi_1^r(x,0)\phi_1^r(y,0)$$
$$=(m(1)/4\pi s(1))^{1/2}(\gamma_2/\gamma_3+s(x))(\gamma_2/\gamma_3+s(y)).$$

Therefore the assertion of the corollary follows from Theorem 2.

5. Examples

EXAMPLE 1. First we consider a second order differential operator with constant coefficients. Let

$$\mathcal{G}=d^2/dx^2-b\ d/dx-k,$$

where b and k are real numbers. Then

$$\rho=e^b,\ \ ds(x)=e^{bx}dx,\ \ dm(x)=e^{-bx}dx,\ \ dk(x)=ke^{-bx}dx,$$

As was observed in [8 ; Example 1],

$$S=[\lambda_0, \infty),\ \ \lambda_0=b^2/4+k.$$

By using the results there, we see that

$$\alpha(x, y, \lambda_0)=(1/2\sqrt{\pi})e^{b(x+y)/2}.$$

In this case we can get the precise formura of $p(t, x, y)$:

$$\begin{aligned} p(t, x, y) &= \sum_{i,j=1,2}\int_{\lambda_0}^{\infty} e^{-\lambda t}\varphi_i(x, \lambda)\varphi_j(y, \lambda)\rho_{ij}(\lambda)d\lambda \\ &= \int_{\lambda_0}^{\infty} e^{-\lambda t}e^{b(x+y)/2}(2\pi\sqrt{\lambda-\lambda_0})^{-1}\cos(\sqrt{\lambda-\lambda_0}(x-y))d\lambda \\ &= (1/2\sqrt{\pi t})\exp\{-\lambda_0 t-(x-y)^2/4t+b(x+y)/2\}, \\ &\quad t>0,\ \ x, y\in \boldsymbol{R}. \end{aligned}$$

We thus find that (1. 2) and (1. 3) are valid.

Let $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$. $\sum_p^{\gamma}$ is not empty if and only if $\gamma_2=1$ and $0\leq\gamma_3<\kappa_0(\gamma)$, where $\kappa_0(\gamma_1)=\gamma_1\lambda_0+b/2$. Then $\sum_p^{\gamma}$ consists only of the single point ν^{γ}, which is given by

$$\nu^{\gamma}=\lambda_0-(\kappa_0(\gamma_1)-\gamma_3)^2\Big(\gamma_1(\kappa_0(\gamma_1)-\gamma_3)+1/2 + \{\gamma_1(\kappa_0(\gamma_1)-\gamma_3)+1/4\}^{1/2}\Big)^{-1}.$$

Further we see the following:

Case I : $\gamma_1\geq 0$, $\gamma_2=1$ and $0\leq\gamma_3<\kappa_0(\gamma_1)$.

$$\begin{aligned} &\beta^{\gamma}(x, y, \lambda^{\gamma}) \\ &=2(\lambda_0-\nu^{\gamma})^{1/2}\{1+2\gamma_1(\lambda_0-\nu^{\gamma})^{1/2}\}^{-1}\exp[\{-(\lambda_0-\nu^{\gamma})^{1/2}+b/2\} \\ &\quad (x+y)], \end{aligned}$$

Case II: $\gamma_1\geq 0$, $\gamma_2=1$ and $\gamma_3=\kappa_0(\gamma_1)\geq 0$.

$$\beta^{\gamma}(x, y, \lambda^{\gamma})=(1/\sqrt{\pi})e^{b(x+y)/2}.$$

Case III : $\gamma_1\geq 0$, $\gamma_2=1$ and $\gamma_3>\kappa_0(\gamma_1)$, $\gamma_3\geq 0$.

$$\beta^{\gamma}(x, y, \lambda^{\gamma})=\frac{\{(\gamma_3-\kappa_0(\gamma_1))x+1\}\ \{(\gamma_3-\kappa_0(\gamma_1))y+1\}}{2\sqrt{\pi}(\gamma_3-\kappa_0(\gamma_1))^2}e^{b(x+y)/2}.$$

Case IV : $\gamma_1=\gamma_2=\gamma_3-1=0$.

$$\beta^\gamma(x, y, \lambda^\gamma)=(1/2\sqrt{\pi})xye^{b(x+y)/2}.$$

EXAMPLE 2. We next consider a periodic difference operator. Let $s(x)$ be a function satisfying the conditions a) and d) at beginning of §1, that is, $s(x)$ is continuous and increasing on $\boldsymbol{R}$, and satisfies $s(x+1)-s(y+1)=\rho(s(x)-s(y))$, x, $y\in\boldsymbol{R}$ for some $\rho>0$. We may assume that $s(0)=0$ and $s(1)=1$ without loss of generality. Let

$$m(x)=\sum_{i\in Z} m_i\chi_{[i,i+1)}(x),\ k(x)=km(x),\quad x\in\boldsymbol{R}.$$

where for each $i\in\boldsymbol{Z}$, $m_i=m_1(1-\rho^{-i})/(1-\rho^{-1})$ if $\rho\neq 1$, $=m_1 i$ if $\rho=1$; k is a real number; $\chi_E(x)=1$ if $x\in E$, $=0$ if $x\notin E$. Then the operator $\mathcal{Q}u(x)=(du^+(x)-u(x)dk(x))/dm(x)$ is reduced to the periodic difference operator

$$\begin{aligned}\mathcal{Q}u(i)&=\{u^+(i)-u^-(i)\}/\{m(i)-m(i-1)\}-ku(i),\qquad i\in\boldsymbol{Z}.\\ u^\pm(i)&=\{u(i\pm 1)-u(i)\}/\{s(i\pm 1)-s(i)\},\end{aligned}$$

This operator with $k=0$ was also observed in [8 ; Example 2]. In the same way as there, we can get the following. Set $\mu=\rho m_1$.

$$\begin{aligned}&S=[\lambda_0, \lambda_1],\ \lambda_0=(\sqrt{\rho}-1)^2/\mu+k,\ \lambda_1=(\sqrt{\rho}+1)^2/\mu+k,\\ &\alpha(x+i, y+\lambda_0)\\ &=(\rho^{i+j-1/2}/4\pi\mu)^{1/2}\{(\sqrt{\rho}-1)s(x)+1\}\{(\sqrt{\rho}-1)s(y)+1\},\\ &\qquad\qquad 0\leq x, y<1,\ i, j\in\boldsymbol{Z}.\end{aligned}$$

Let $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$. $\sum_p^\gamma$ consists of at most two points. The principal eigenvalue belongs to $\sum_p^\gamma$, i. e. $\lambda^\gamma=\nu^\gamma$, if and only if $\gamma_2=1$ and $0\leq\gamma_3<\kappa_0(\gamma_1)$, where $\kappa_0(\gamma_1)=\gamma_1\lambda_0+\sqrt{\rho}-1$. Then,

$$\begin{aligned}\nu^\gamma=\lambda_0-(\kappa_0(\gamma_1)-\gamma_3)^2[&(\gamma_1-\mu/2)(\kappa_0(\gamma_1)-\gamma_3-\sqrt{\rho})+\gamma_1\sqrt{\rho}\\ &+\Big((\mu/2)^2(\kappa_0(\gamma_1)-\gamma_3-\sqrt{\rho})^2+\gamma_1\rho^{1/2}\mu(\kappa_0(\gamma_1)-\gamma_3)\Big)^{1/2}]^{-1}.\end{aligned}$$

In the following each cases, $0\leq x, y<1$ and $i, j\in\boldsymbol{Z}_+$.

Case I: $\gamma_1\geq 0$, $\gamma_2=1$ and $0\leq\gamma_3<\kappa_0(\gamma_1)$.

$$\begin{aligned}&\beta^\gamma(x+i, y+j, \lambda^\gamma)\\ &=r_1(\nu^\gamma)^{i+j}\Big(\gamma_1+r_1(\nu^\gamma)\{(\lambda_0-\nu^\gamma+4\sqrt{\rho}/\mu)\}^{-1/2}\Big)^{-1}\\ &\quad\times\{(r_1(\nu^\gamma)-1)s(x))+1\}\{(r_1(\nu^\gamma)-1)s(y)+1\},\end{aligned}$$

where $r_1(\nu^\gamma)=1-\gamma_1\nu^\gamma+\gamma_3$.

Case II: $\gamma_1\geq 0$, $\gamma_2=1$ and $\gamma_3=\kappa_0(\gamma_1)\geq 0$.

$$\begin{aligned}\beta^\gamma(x+i, y+j, \lambda^\gamma)=\rho^{i+j-1/2}/\pi\mu)^{1/2}&\{(\sqrt{\rho}-1)s(x)+1\}\\ &\times\{(\sqrt{\rho}-1)s(y)+1\}.\end{aligned}$$

Case III : $\gamma_1 \geq 0$, $\gamma_2 = 1$ and $\gamma_3 > \kappa_0(\gamma_1)$, $\gamma_3 \geq 0$.

$$\begin{aligned}
&\beta^r(x+i, y+j, \lambda^r) \\
&= (\rho^{i+j-3/2}\mu/4\pi)^{1/2}(\gamma_3 - \kappa_0(\gamma_1))^{-2} \\
&\times\Big(\{i(\gamma_3-\kappa_0(\gamma_1))(\sqrt{\rho}-1)+\sqrt{\rho}(\gamma_3-\kappa_0(\gamma_1)+\sqrt{\rho}-1)\}s(x) \\
&+i(\gamma_3-\kappa_0(\gamma_1))+\sqrt{\rho}\Big)\times\Big(\{j(\gamma_3-\kappa_0(\gamma_1))(\sqrt{\rho}-1) \\
&+\sqrt{\rho}(\gamma_3-\kappa_0(\gamma_1)+\sqrt{\rho}-1)\}s(y)+j(\gamma_3-\kappa_0(\gamma_1))+\sqrt{\rho}\Big).
\end{aligned}$$

Case IV : $\gamma_1 = \gamma_2 = \gamma_3 - 1 = 0$.

$$\begin{aligned}
\beta^r(x+i, y+j, \lambda^r) = (\rho^{i+j-3/2}\mu/4\pi)^{1/2}(\{i(\sqrt{\rho}-1)+\sqrt{\rho}\}s(x)+i) \\
\times(\{j(\sqrt{\rho}-1)+\sqrt{\rho}\}s(y)+j).
\end{aligned}$$

Example 3. Finally we will consider an operator with discontinuous coefficients which can not be transformed to a Hill's one. Let $b(x+n) = b(0 \leq x < p)$, $= -b(p \leq x < 1)$, $n \in \boldsymbol{Z}$ with $0 < p < 1$, $b \neq 0$, and

$$\mathcal{Q} = d^2/dx^2 - b(x)d/dx - k,$$

where k is a real number. In this case, $\rho = \exp\{b(2p-1)\}$ and

$$ds(x) = \exp\{\int_0^x b(y)dy\}dx, \quad dm(x) = \exp\{-\int_0^x b(y)dy\}dx,$$
$$dk(x) = kdm(x).$$

Let $\delta(\lambda)$ be the square root of the discriminant of the equation

$$\xi^2 - b\xi + \lambda - k = 0.$$

Set $q = 1-p$, $\delta_\pm(\lambda) = (b \pm \delta(\lambda))/2$ and $\lambda^\circ = b^2/4 + k$. Now the solutions $\varphi_i(x, \lambda)$, $x \in \boldsymbol{R}$, $\lambda \in \boldsymbol{C}$, $i = 1, 2$ of (2.2) are given as follows. For $\lambda \neq \lambda^\circ$ and $n \in \boldsymbol{Z}$.

$$\varphi_i(x+n, \lambda) = \begin{cases} a_{in}\exp\{\delta_+(\lambda)x\} + b_{in}\exp\{\delta_-(\lambda)x\}, & 0 \leq x < p, \\ c_{in}\exp\{-\delta_+(\lambda)(x-p)\} + d_{in}\exp\{-\delta_-(\lambda)(x-p)\}, & p \leq x < 1, \end{cases}$$

where

$$\begin{bmatrix} a_{i,n+1} \\ b_{i,n+1} \end{bmatrix} = A\begin{bmatrix} c_{in} \\ d_{in} \end{bmatrix} = AB\begin{bmatrix} a_{in} \\ b_{in} \end{bmatrix},$$

$$\begin{bmatrix} a_{10} \\ b_{10} \end{bmatrix} = \frac{1}{\delta(\lambda)}\begin{bmatrix} -\delta_-(\lambda) \\ \delta_+(\lambda) \end{bmatrix}, \quad \begin{bmatrix} a_{20} \\ b_{20} \end{bmatrix} = \frac{1}{\delta(\lambda)}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$A = \frac{1}{\delta(\lambda)}\begin{bmatrix} -\delta_-(\lambda) & 1 \\ \delta_+(\lambda) & -1 \end{bmatrix}\begin{bmatrix} \exp\{-\delta_+(\lambda)q\} & \exp\{-\delta_-(\lambda)q\} \\ -\delta_+(\lambda)\exp\{-\delta_+(\lambda)q\} & -\delta_-(\lambda)\exp\{-\delta_-(\lambda)q\} \end{bmatrix},$$

$$B = \frac{1}{\delta(\lambda)}\begin{bmatrix} -\delta_-(\lambda) & -1 \\ \delta_+(\lambda) & 1 \end{bmatrix}\begin{bmatrix} \exp\{\delta_+(\lambda)p\} & \exp\{\delta_-(\lambda)p\} \\ \delta_+(\lambda)\exp\{\delta_+(\lambda)p\} & \delta_-(\lambda)\exp\{\delta_-(\lambda)p\} \end{bmatrix};$$

for $\lambda = \lambda^\circ$ and $n \in \boldsymbol{Z}$.

$$\varphi_i(x+n,\lambda^\circ)=\begin{cases}(a_{in}{}^\circ+b_{in}{}^\circ x)e^{bx/2}, & 0\leq x<p,\\ (c_{in}{}^\circ+d_{in}{}^\circ(x-p))e^{-b(x-p)/2}, & p\leq x<1,\end{cases}$$

where

$$\begin{bmatrix}a^\circ{}_{i,n+1}\\ b^\circ{}_{i,n+1}\end{bmatrix}=e^{-bq/2}\begin{bmatrix}1 & q\\ -b & 1-bq\end{bmatrix}\begin{bmatrix}c_{in}{}^\circ\\ d_{in}{}^\circ\end{bmatrix}$$

$$=e^{b(p-q)/2}\begin{bmatrix}1 & q\\ -b & 1-bq\end{bmatrix}\begin{bmatrix}1 & p\\ b & 1+bq\end{bmatrix}\begin{bmatrix}a_{in}{}^\circ\\ b_{in}{}^\circ\end{bmatrix},$$

$$\begin{bmatrix}a_{10}{}^\circ\\ b_{10}{}^\circ\end{bmatrix}=\begin{bmatrix}1\\ -b/2\end{bmatrix},\quad \begin{bmatrix}a_{20}{}^\circ\\ b_{20}{}^\circ\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

Hence we have for $\lambda\neq\lambda^\circ$,

(5.1) $\quad \Delta(\lambda)=(\sqrt{\rho}/\delta^2(\lambda))\{-4(\lambda-k)C_+(\lambda)+b^2D_+(\lambda)\}$,

where

$$C_\pm(\lambda)=e^{\delta(\lambda)/2}\pm e^{-\delta(\lambda)/2},$$

(5.2) $\quad D_\pm(\lambda)=\exp\{\delta(\lambda)(p-q)/2\}\pm\exp\{-\delta(\lambda)(p-q)/2\}$;

and for $\lambda=\lambda^\circ$.

(5.3) $\quad \Delta(\lambda^\circ)=\sqrt{\rho}\,(2-b^2pq)$.

First we note that the set S is expressed as

(5.4) $$S=\bigcup_{n=1}^{\infty}[\mu_{n-1}{}^{(2)},\mu_n{}^{(1)}].$$

Here the sequence $\{\mu_n{}^{(i)}\}_{i=1,2,n\in N}$ has the following properties.

(5.5) $\quad -\infty<\mu_0{}^{(2)}<\mu_1{}^{(1)}\leq\mu_1{}^{(2)}<\cdots<\mu_n{}^{(1)}\leq\mu_n{}^{(2)}<\cdots\uparrow$,

(5.6) $\quad \Delta(\mu_n{}^{(i)}=(-1)^n2\sqrt{\rho},\ \ n\geq0,\ \ i=1,2,$

(5.7) $\quad \lambda^\circ-b^2/4\leq\lambda_0=\mu_0{}^{(2)}<\lambda^\circ$,

(5.8) $\quad \lambda_0<\mu_1{}^{(1)}<\lambda^\circ+\pi^2<\mu_1{}^{(2)}<\lambda^\circ+4\pi^2$,

(5.9) $\quad \mu_1{}^{(1)}\lesseqgtr\lambda^\circ$ according to $4-b^2pq\gtreqless0$,

(5.10) $\quad \lambda^\circ+(n-1)^2\pi^2<\mu_n{}^{(1)}\leq\lambda^\circ+n^2\pi^2\leq\mu_n{}^{(2)}<\lambda^\circ+(n+1)^2\pi^2,\ \ n\geq2,$

(5.11) $\quad \mu_n{}^{(1)}=\mu_n{}^{(2)}=\lambda^\circ+n^2\pi^2$ if and only if p is rational with the irreducible expression $p=s/t$, t being a divisor of n,

(5.12) $\quad \mu_n{}^{(i)}=\lambda^\circ+n^2\pi^2+(-1)^i\sqrt{2}\,|b|\,(1-\cos 2np\pi)^{1/2}+O(1/n)$, as $n\to\infty$, $i=1,2$.

Indeed, it is known that S has the expression (5.4) with (5.5) and (5.6))cf. [8 ; Theorem 1]). By the change of variables $\lambda=\lambda^\circ\pm\xi^2$, $\xi>0$, we have by (5.1),

(5.13) $\quad \Delta(\lambda^\circ-\xi^2)=2\sqrt{\rho}\cosh\xi-(\sqrt{\rho}\,b^2/2\xi^2)\{\cosh\xi-\cosh(p-q)\xi\}$,

$\quad \Delta(\lambda^\circ+\xi^2)=2\sqrt{\rho}\cos\xi-(\sqrt{\rho}\,b^2/2\xi^2)\{\cos\xi-\cos(p-q)\xi\}$.

(5.3) and (5.13) give us (5.7)-(5.11). To see (5.12), notice that $|\Delta(\lambda^\circ+n^2\pi^2)-(-1)^n 2\sqrt{\rho}\,|\to\infty$. Further, $\eta_n^{(i)}\equiv|\mu_n^{(i)}-\lambda^\circ|^{1/2}$ solves by (5.13)

$$(5.14)\quad 2\sqrt{\rho}\cos\eta_n^{(i)}+(\sqrt{\rho}\,b^2/2(\eta_n^{(i)})^2)\{\cos\eta_n^{(i)}-\cos(p-q)\eta_n^{(i)}\}$$
$$=(-1)^n 2\sqrt{\rho}. \qquad (n-1)\pi<\eta_n^{(1)}\le n\pi\le\eta_n^{(2)}<(n+1)\pi,\quad n\ge 2.$$

Hence it follows that

$$(5.15)\quad \zeta_n^{(i)}\equiv\eta_n^{(i)}-n\pi\to 0 \text{ as } n\to\infty.$$

Substituting (5.15) into (5.14), we have

$$(-1)^n(1-\cos\zeta_n^{(i)})=b^2\{\cos n\pi-\cos(p-q)n\pi+o(1)\}/4(\eta_n^{(i)})^2$$
$$=(-1)^n b^2(1-\cos 2np\pi)/4n^2\pi^2+o(n^{-2}).$$

This implies

$$\zeta_n^{(i)}=(-1)^i|b|(1-\cos 2np\pi)^{1/2}/\sqrt{2}\,n\pi+o(n^{-1}).$$

Repeating the above argument, we obtain

$$(5.16)\quad \zeta_n^{(i)}=(-1)^i|b|(1-\cos 2np\pi)^{1/2}/\sqrt{2}\,n\pi+o(n^{-2}).$$

(5.12) now follows from (5.15) and (5.16).

Let $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\Gamma$. Then by virtue of [8 ; Theorem 6], $\sum_p^\gamma$ is a countable infinite set if one of the following conditions (i)-(iii) is satisfied. (i) $p\neq 1/2$. (ii) $p=1/2$, $b>0, \gamma_1=0$, $\gamma_2=1$ and $\gamma_3>0$. (iii) $p=1/2$, $b<0$, $\gamma_1>0$, $\gamma_2=1$ and $\gamma_3\ge 0$. $\sum_p^\gamma$ is a finite set if $p=1/2$, $b>0$, $\gamma_1>0$, $\gamma_2=1$ and $\gamma_3\in[0,\gamma_1\lambda_0)\cup(\gamma_1\mu_1^{(1)},\infty)$. $\sum_p^\gamma$ is empty otherwise. In particular, $\sum_p^\gamma\cap(-\infty,\lambda_0)=\{\nu^\gamma\}$ if and only if $\gamma_2=1$ and $0\le\gamma_3<\kappa_0(\gamma_1)$. Here $\kappa_0(\gamma_1)=\gamma_1\lambda_0+(\sqrt{\rho}-\varphi_1(1,\lambda_0))/\varphi_2(1,\lambda_0)$ and ν^γ is the solution of the equation $(\gamma_1(\lambda)-\varphi_1(1,\lambda))/\varphi_2(1,\lambda)=-\gamma_1\lambda+\gamma_3$, $\lambda<\lambda_0$.

Now we have for $\lambda\neq\lambda^\circ$.

$$\varphi_1(1,\lambda)=(\sqrt{\rho}/2\delta^2(\lambda))\{-4(\lambda-k)C_+(\lambda)+b^2D_+(\lambda)-b\delta(\lambda)D_-(\lambda)\},$$
$$\varphi_1^+(1,\lambda)=((\lambda-k)/\sqrt{\rho}\,\delta^2(\lambda))\{bC_+(\lambda)-\delta(\lambda)C_-(\lambda)-bD_+(\lambda)\},$$
$$\varphi_2(1,\lambda)=(\sqrt{\rho}/\delta^2(\lambda))\{bC_+(\lambda)+\delta(\lambda)C_-(\lambda)-bD_+(\lambda)\},$$

where $C_\pm(\lambda)$ and $D_\pm(\lambda)$ are given by (5.2). Thus we can get $\alpha(x,y,\lambda_0)$ and $\beta^\gamma(x,y,\lambda^\gamma)$ by using (4.11), (4.13) and (4.14).

References

1. N. Ikeda, K. Kawazu, and Y. Ogura, *Branching one-dimensional periodic diffusion processes*, Stochastic Anal. Appl., **19**(1985), 63-83.
2. K. Itô, and H.P. Mckean, *Diffusion processes and their sample paths*,

Springer-Verlag, Berlin-Heidelberg-New York, 1974.

3. J. Kaneko, Y. Ogura, and M. Tomisaki, *On the principal eigenvalue for one-dimensional generalized diffusion operators*, to appear in Rep. Fac. Engrg., Saga Univ. Math., **15**(1987).
4. K. Kawazu, and Y. Ogura, *A limit theorem for branching one-dimensional periodic diffusion processes*, J. Multivariate Anal., **14**(1984), 360-375.
5. S. Kotani, and S. Watanabe, *Krein's spectral theory of strings and generalized diffusion processes, Functional analysis in Markov processes* (M. Fukushima, ed.), Lecture Notes in Math. 923, 235-259, Springer, Berlin-Heidelberg-New York, 1982.
6. H. P. Mckean, Jr., *Elementary solutions for certain parabolic partial differential equations*, Trans. Amer. Math. Soc. **82**(1956), 519-548.
7. N. Minami, Y. Ogura, and M. Tomisaki, *Asymptotic behavior of elementary solutions of one-dimensional generalized diffusion equations*, The Annals of Probability, **13**(1985), 698-715.
8. M. Tomisaki, *The spectrum of periodic generalized diffusion operators*, (preprint).
9. S. Watanabe, *On time inversion of one-dimensional diffusion processes, Z.* Wahrsch. verw. Gebiete **31**(1975), 115-124.
10. K. Yoshida, *Lectures on differential and integral equations*, Interscience Publishers, New York, 1960.

Kangweon National University
Chuncheon 200-701, Korea
and
Saga University
Saga 840, Japan