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RESTRICTED FLEXIBLE ALGEBRAS

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1. Introduction

An algebra A with multiplication denoted by xy over a field F is called *flexible* if it satisfies the flexible law $(xy)x=x(yx)$ for all $x, y \in A$. For an element x of A, let L_x and R_x denote the left and right multiplications in A by x. Following Schafer's work [13] on restricted noncommutative Jordan algebras of characteristic $p>2$, we call a flexible algebra of characteristic $p>2$ *restricted* if A is strictly power-associative and satisfies

(1) $$L_{x^p}=L_x{}^p \text{ or } R_{x^p}=R_x{}^p$$

for all $x \in A$. All algebras considered in this note are assumed to be finite-dimensional. Recall that a flexible algebra is called a noncommutative Jordan algebra if it satisfies the Jordan identity $(x^2y)x=x^2(yx)$. Well known noncommutative Jordan algebras are the commutative Jordan and alternative algebras which are shown to be restricted for characteristic >0 [4]. There exist simple flexible power-associative algebras of characteristic >2 which are not noncommutative Jordan ([6] and [8]). An example of a simple flexible algebra of characteristic $p>0$ which is not restricted has been given by Schafer [13].

It is the purpose of this note to extend the results of Schafer [13] for restricted noncommutative Jordan algebras to restricted flexible algebras. Since flexible power-associative algebras of characteristic 0 have a satisfactory structure theory and enjoy those properties constrained by the restricted identity (1), we may regard those algebras of characteristic 0 as restricted algebras.

2. Nodal algebras

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A power-associative algebra A over F with identity element 1 is said to be *nodal* in case every element of A is of the form $\alpha 1+z$, where $\alpha\in F$ and z is nilpotent, and A is not of the form $A=F1+N$ for a nilsubalgebra N of A. It is well known that there are no nodal algebras which are alternative of arbitrary characteristic, commutative Jordan of characteristic $\neq 2$, or noncommutative Jordan of characteristic 0 (Jacobson [3] and McCrimmon [9]). However, nodal noncommutative Jordan algebras of characteristic $p>0$ do exist, and Kokoris [7] gave the first construction of such algebras, called Kokoris algebras, which have also been shown to be useful for the study of simple Lie algebras of prime characteristic (see Schafer [14] and Strade [16], for example). Scribner [15] has constructed a nodal noncommutative Jordan algebra of infinite dimension. Schafer [13] has proven that Kokoris algebras cannot be restricted by showing that there are no nodal, restricted noncommutative Jordan algebras of characteristic $p>2$. We here extend this result to restricted flexible algebras. Attached to an algebra A over a field F of characteristic $\neq 2$ is the commutative algebra denoted by A^+ with multiplication $x\cdot y=\frac{1}{2}(xy+yx)$ defined on the vector space A. We begin with

LEMMA 1. *Let A be a strictly power-associative algebra with identity element 1 over a field F of characteristic $\neq 2$ such that every element of A is of the form $\alpha 1+z$ for $\alpha\in F$ and a nilpotent element z in A. Then A is the vector space direct sum $A=F1+N$ where N is a subspace of nilpotent elements in A and N^+ is the maximal nil ideal of A^+.*

Proof. Let N denote the set of nilpotent elements in A and let M be a maximal ideal of A^+. Then, $M\subseteq N$, since if $M\subsetneq N$, then there is an element $\alpha 1+z$ in M for $\alpha\neq 0$ and a nilpotent element z, and hence $\alpha 1+z$ is invertible by power-associativity. Thus A^+/M is a simple commutative, strictly power-associative algebra of degree one. If the characteristic of F is greater than two, then Oehmke [11] has shown that such an algebra must be a field, and if the characteristic is zero, then the same holds by a result of Albert [1]. Therefore, A^+/M is a field, and it must be that $M=N$, which is a subspace of A.

Extending the known result for commutative Jordan algebras, as an immediate consequence of Lemma 1, we have

COROLLARY 2. *There is no nodal commutative strictly power-associative* *algebra of characteristic* $\neq 2$.

THEOREM 3. *There is no nodal restricted flexible algebra over a field of characteristic* $\neq 2$.

Proof. Suppose that such a nodal algebra A exists. It is readily seen that any homomorphic image of a restricted algebra is also restricted. Note also that any nonzero homomorphic image of a nodal algebra is also nodal (Schafer [12, p. 116]). By Lemma 1, we can write $A=F1+N$ where N is a subspace of A and is the set of all nilpotent elements in A. If M denotes a maximal ideal of A, then as in the proof of Lemma 1 we have $M\subseteq N$. Then, the quotient algebra $\bar{A}=A/M$ is a simple restricted flexible algebra of degree one, and by a result of Kleinfeld and Kokoris [5] $\bar{A}^+$ must be an associative algebra. Hence, $\bar{A}$ is a nodal restricted noncommutative Jordan algebra, which contradicts the result of Schafer [13] that such algebras do not exist.

3. Semisimple algebras

A power-associative algebra A is called *semisimple* if the maximal nil ideal of A is zero. Oehmke [10] has proven that any semisimple flexible strictly power-associative algebra A over F of characteristic $\neq 2$ has an identity element and is the direct sum of simple ideals, and that any simple such algebra of characteristic $\neq 2, 3$ is one of the algebras: a commutative Jordan algebra (for degree ≥ 3); a quasi-associative algebra; a flexible algebra of degree 2; and an algebra of degree one. We make use of this result to prove the following structure theorem.

THEOREM 4. *Any semisimple restricted flexible algebra A over a field F of characteristic* $\neq 2, 3$ *is the direct sum* $A=A_1\oplus\cdots\oplus A_n$ *of simple ideals* A_i *of* A. *If A is simple, then A is one of the following*:

(a) *a simple commutative Jordan algebra of characteristic* $\neq 2$ (*for degree* ≥ 3),

(b) *a simple restricted flexible algebra of degree two,*

(c) *an algebra* $B(\lambda)$ *with multiplication* $x*y=\lambda xy+(1-\lambda)yx$ *defined on a simple associative algebra B over F for a fixed* $\lambda\neq 1/2$ *in the prime field of F, where xy denotes the product in B,*

(d) *a field* (*for degree one*).

Proof. The first part and the algebras described in (a) and (b) follow from the result of Oehmke noted above. If A is a simple quasi-associative algebra over F, then it is shown in [13] that the restricted condition (1) gives the algebra described by (c). Assume then that A is a simple algebra of degree one. Since A cannot be nodal by Theorem 3, by a result of Block [2] we conclude that A^+ is a simple algebra of degree one which must be a field. Here we have used the fact that any scalar extension of a restricted algebra is also restricted (see [13, p. 143]). Therefore, by Lemma 1, A is a field also in this case. This completes the proof.

By the known classification, all algebras but case (b) in Theorem 4 are completely described. It is not known whether there exists a simple restricted flexible algebra (of degree 2) which is not noncommutative Jordan. We note that if A is noncommutative Jordan, then one of the two restricted conditions in (1) implies the other. This is due to the fact that a commutative Jordan algebra is restricted and flexibility is equivalent to the identity $L_xR_x=R_xL_x$. In fact, let $T_x=\frac{1}{2}(L_x+R_x)$ denote the right multiplication in the Jordan algebra A^+ by x. Since any Jordan algebra of characteristic $p>0$ is restricted [4], $\frac{1}{2}(L_{x^p}+R_{x^p})$ $=T_{x^p}=T_x^p=\frac{1}{2}(L_x+R_x)^p=\frac{1}{2}(L_x^p+R_x^p)$, since L_x commutes with R_x. Hence one of the conditions (1) implies the other.

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