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INCLUSIONS FOR CLASSES OF LACUNARY SETS

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1. Introduction

A sequence, $a_1 < a_2 < a_2 < \ldots$, of positive integers is called *lacunary* if the difference seguence $d_n = a_{n+1} - a_n$ tends to infinity as $n \longrightarrow \infty$.

In several recent papers we have made use of these sequences in analysis and combinatorics. In [6] we show that the class L of all sets which are either finite or the range of a lacunary sequence is "full" in the sense that if (t_k) is a real sequence and $\sum_{k \in \mathrm{L}} |t_k| < \infty$ for each $\mathrm{L} \in L$, then (t_k) is an l_1 sequence. In [3] the class Z of all finite unions of sets of L is shown to consist of exactly those sets of integers, A, whose characteristic sequence, χ_A, is in the well known summability space $bs + c_0$. More recently, in [1], we study lacunary sequences in connection with the conjecture of P. Erdös that, if a set A of integers satisfies $\sum_{a \in A} 1/a = \infty$, then A contains arbitrarily long arithmetic progressions. It turns out that Erdös' conjecture is true if, and only if, it is true for all sets in L, and that the conjectue is indeed true for all sets in L_1, a certain full subclass of L to be defined below.

In this paper we introduce some natural subclasses of L and prove inclusions among them and among their closures with respect to finite unions and subsets. These subclasses were suggested by the combinatorial and analytical work done in [1] and [3]. Furthermore, the use of lacunary sets goes back as far as the classical contribution of G. G. Lorentz [5]. These statements notwithstanding, the proofs of these inclusions became so demanding, that the results seem to generate an interest in themselves aside from any possible applications.

For a class S of subsets of the natural numbers I we define S^* and $[S]$ to be the "hereditary closure" and closure under finite unions of S res-

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pectively, that is,

$$S^* = \{A : A \subset S \text{ for some } S \in \mathcal{S}\},$$
$$[\mathcal{S}] = \{A : A = S_1 \cup S_2 \cup \ldots \cup S_k \text{ for some } S_i \in \mathcal{S} \text{ and } k \geq 0\}.$$

It is easy to see that $[S^*] = [S]^*$. Moreover, a class of sets A is of the form $[S^*]$ if and only if $A = 2^I$ or A is a "zero-class", that is, the class of sets of zero upper density with respect to some density on I (see [2] and [4]).

We now define the subclasses of L in which we are interested. For an integer $j \geq 0$ define L_{M_j} to be the class of all lacunary sequences for which $s \leq t$ implies that $d_s \leq d_t + j$. Further, we define L_1 to be the "monotone" lacunary sequences L_{M_0}. Finally, define two subclasses of L_1 thus:

$$L_2 = \{A \in L_1 : \textstyle\sum_{a \in A} 1/a = \infty\}, \quad L_3 = L_1 - L_2.$$

2. Inclusions

The remainder of this paper will be devoted to proving the following diagrams. In every case the inclusion itself is a trivial consequence of the definitions. It is in proving the two classes to be equal or unequal, as the case may be, that the real difficulties arise.

$$\begin{matrix} [L_2] \subsetneqq & & \\ & [L_1] \subsetneqq [L_{M_i}] \subsetneqq [L_{M_j}] \subsetneqq [L] & \quad (1) \\ [L_3] \subsetneqq & & \end{matrix}$$

where $1 \leq i < j$ and $[L_2]$ and $[L_3]$ are incomparable. If we remove the closure under finite unions from each of the above classes the same inclusions hold by definition. However we get the following for hereditary closure * of these classes.

$$\begin{matrix} L_2^* \subsetneqq & & \\ & L_1^* \subsetneqq L_{M_i}^* = L^* = L & \quad (2) \\ L_3^* \subsetneqq & & \end{matrix}$$

for all $i \geq 1$. L_2^* and L_3^* remain incomparable. Finally, taking both closures we get

$$[L_3^*] \subsetneqq [L_2^*] = [L_1^*] \subsetneqq [L]. \quad (3)$$

We omit the simple proof of our first proposition.

Proposition 1. *If $A \subset B \subset 2^I$ and A is full, so is B.*

PROPOSITION 2. *$\mathcal{A}\subset 2^I$ is full if and only if $[\mathcal{A}]$ is full if and only if $\mathcal{A}^*$ is full.*

Proof. If $\mathcal{A}$ is full then by Proposition 1, $[\mathcal{A}]$ and $\mathcal{A}^*$ are full.

Supose $[\mathcal{A}]$ is full, and (t_k) is a sequence in R such that $\sum_{k=1}^{\infty}|t_k|=\infty$. Then there exists $A\in[\mathcal{A}]$ such that $\sum_{k\in A}|t_k|=\infty$. Let $A=A_1\cup A_2\cup\ldots\cup A_n$ where $A_i\in\mathcal{A}$ for $i=1, 2, \ldots, n$. Then there exist i such that $\sum_{k\in A_i}|t_k|=\infty$. Hence $\mathcal{A}$ is full.

Suppose that $\mathcal{A}^*$ is full. If $\sum_{k=1}^{\infty}|t_k|=\infty$, there exists $A\in\mathcal{A}^*$ such that $\sum_{k\in A}|t_k|=\infty$. Let $A\subset B$ where $B\in\mathcal{A}$. Then obviously $\sum_{k\in B}|t_k|=\infty$ and $B\in\mathcal{A}$. Therefore $\mathcal{A}$ is full.

PROPOSITION 3. *L, L_1, L_2 are full.*

Proof. Since $L_2\subset L_1\subset L$, we only need to show that L_2 is full. Let (t_k) be a real sequence such that $\sum_{k=1}^{\infty}|t_k|=\infty$. For each n, there exists $b_n\in I$ such that $\sum_{k=1}^{\infty}|t_{(b_n+k2^n)}|=\infty$.

We construct two sequences $(M_n)_{n=2}^{\infty}$, and $(N_n)_{n=1}^{\infty}$ in I with the following properties:

$$N_n<M_{n+1}<N_{n+1} \qquad (n\geq 1) \qquad (4)$$

$$N_n\equiv M_n\equiv b_n \bmod 2^n \qquad (n\geq 2) \qquad (5)$$

$$M_{n+1}\equiv N_n \bmod (2^n+1) \qquad (n\geq 1) \qquad (6)$$

$$M_n>b_n \qquad (n\geq 2) \qquad (7)$$

$$\sum_{a\in B[2^n, M_n, N_n]}|t_a|>1 \qquad (n\geq 2) \qquad (8)$$

$$\sum_{a\in B[2^n, M_n, N_n]}1/a>1 \qquad (n\geq 2) \qquad (9)$$

where $B[s, a, b]=\{a, a+s, a+2s, \cdots, a+[(b-a)/s]s\}$.

Take $N_1=b_1$ and suppose that we have constructed two squences $(M_n)_{n=2}^{m-1}$ and $(N_n)_{n=1}^{m-1}$ such that (4) and (6) are true for $n=1, 2, \cdots, m-2$ and (5), (7), (8) and (9) are true for $n=2, 3, \cdots, m-1$. Since 2^m and $2^{m-1}+1$ are relatively prime, we can find $M_m\in I$ such that

$$M_m\equiv b_m \bmod 2^m,$$
$$M_m\equiv N_{m-1} \bmod (2^{m-1}+1),$$
$$M_m>b_m \text{ and } M_m>N_{m-1}.$$

Since $\sum_{k=1}^{\infty}|t_{(b_m+k2^m)}|=\infty$ and $M_m\equiv b_m \bmod 2^m$ we have $\sum_{k=1}^{\infty}|t_{(M_m+k2^m)}|=\infty$. Clearly $\sum_{k=1}^{\infty}1/(M_m+k2^m)=\infty$.

Now we can take N_m large enough such that

$$N_m\equiv M_m \bmod 2^m,$$

$$\sum_{a\in B[2^m, M_m, N_m]} |t_a| > 1,$$
$$\sum_{a\in B[2^m, M_m, N_m]} 1/a > 1.$$

Let

$$A = \cup_{k=1}^{\infty} (B[2^k+1, N_k, M_{k+1}] \cup B[2^{k+1}, M_{k+1}, N_{k+1}]).$$

Clearly $A \in L_2$ and $\sum_{a\in A} |t_a| = \infty$.

PROPOSITION 4. *The class L_3 is not full. Thus $[L_3^*] \subsetneq [L_1^*]$.*

Proof. The sequence $1/k$ satistfies $\sum_{k=1}^{\infty} 1/k = \infty$. But, for any infinite set A in L_3, $\sum_{a\in A} 1/a < \infty$. The last statement follows since $[L_1^*]$ is full.

PROPOSITION 4 also establishes the corresponding inclusion in diagram (1) and (2).

PROPOSITION 5. $[L_2^*] = [L_1^*]$.

Proof. Obviously $[L_2^*] \subset [L_1^*]$. For $[L_1^*] \subset [L_2^*]$, we only need to show $L_1 \subset [L_2^*]$. In fact we show that, for any infinite set $A = \{a_n\} \in L_1$, $A \subset B_1 \cup B_2$ where B_1, B_2 are members of L_2. For $n \geq 1$, let $d_n = a_{n+1} - a_n$. We know, $d_n \leq d_{n+1}$ for each n and $\lim d_n = \infty$. Thus we can find $s_0, t_1 \in I$ such that $d_1 \leq a_{t_1} - (a_1 + s_0 d_1) < 2d_1 < d_{t_1}$ and $\sum_{j=1}^{s_0} 1/(a_1 + jd_1) > 1$. Suppose that we have thus have constructed $s_0 < s_1 < \cdots < s_{m-1}$, $t_0 = 1 < t_1 < \cdots < t_m$ such that

$$d_{t_{k-1}} \leq a_{t_k} - (a_{t_{k-1}} + s_{k-1} d_{t_{k-1}}) < 2d_{t_{k-1}} < d_{t_k}$$

and

$$\sum_{j=1}^{s_{k-1}} 1/(a_{t_k} + jd_{k-1}) > 1$$

for $k = 1, 2, \cdots, m$. Again, since $d_n \leq d_{n+1}$ for each n and $\lim d_n = \infty$, we can find s_m and t_{m+1} such that

$$s_{m-1} < s_m \text{ and } t_m < t_{m+1}$$
$$d_{t_m} \leq a_{t_{m+1}} - (a_{t_m} + s_m d_{t_m}) < d_{t_{m+1}} \text{ and } \sum_{j=1}^{s_m} 1/(a_{t_m} + jd_{t_m}) > 1.$$

For $n = 1, 2, 3, \ldots$, let

$$P_n = \{a_{t_n}, a_{t_n} + d_{t_n}, \cdots, a_{t_n} + s_n d_{t_n}\}$$
$$W_n = \{a_{t_n}, a_{(t_n+1)}, \cdots, a_{t_{n+1}}\}.$$

Then we have, $\sum_{a\in p_n} 1/a > 1$ and $A = \cup_{n=1}^{\infty} W_n$. Let

$$B_1 = P_1 \cup W_2 \cup P_3 \cup W_4 \cup \cdots \cup P_{2n-1} \cup W_{2n} \cup \cdots$$
$$B_2 = W_1 \cup P_2 \cup W_3 \cup P_4 \cup \cdots \cup W_{2n-1} \cup P_{2n} \cup \cdots.$$

Clearly $B_i \in L_2$ for $i = 1, 2$ and $A \subset B_1 \cup B_2$.

We have shown that $[L_2^*]=[L_1^*]$. We proceed to show that $L_2^* \subsetneq L_1^*$.

Definition 1. (1) Let $a, x_1, x_2, \cdots, x_n$ be positive integers with $a=x_1+x_2+\cdots+x_n$ and $x_1 \le x_2 \le \cdots \le x_n$. Then $(x_1, x_2, \cdots, x_n)$ is called a *monotone partition* of a of length n. (2) Let $(a_1, a_2, \cdots, a_n)$ be any finite sequence of positive integers and let

$$(y_{11}, y_{12}, \cdots, y_{1k_1}, y_{21}, y_{22}, \cdots, y_{2k_2}, \cdots, y_{n1}, \cdots, y_{nk_n}) \qquad (10)$$

be a nondecreasing sequence such that $(y_{i1}, y_{i2}, \cdots, y_{ik_i})$ is a monotone partition of a_i. Then the block (10) is called a *monotone partition* of the sequence $(a_1, a_2, \cdots, a_n)$.

Definition 2. Let (x_n) be a sequence and $(t(n))$ a strictly increasing sequence of positive integers with $t(1)=1$. Then $(x_{t(n)}, x_{t(n)+1}, \cdots, x_{t(n+1)-1})$ is called the n-th *part* of $(x_n)_{n=1}^{\infty}$ with respect to the $(t(n))$.

Lemma 6. *Let $p>2$ be a prime number and let $(a_1, a_2, \cdots, a_p)$ be the sequence with $a_i=p$, for all $i=1, 2, \cdots, p$. Let $(y_{11}, y_{12}, \cdots, y_{1k_1}, y_{21}, \cdots, y_{2k_2}, \cdots, y_{p1}, y_{p2}, \cdots, y_{pk_p})$ be a monotone partition of $(a_1, a_2, \cdots, a_p)$ with $y_{11}>1$. Then $k_p=1$ and $y_{p1}=p$.*

Proof. Suppose that $k_p>1$. Then $y_{pk_p}<p$ and since p is a prime, $y_{p1}<y_{pk_p}$. It follows that $k_i>1$ for all $i<p$ since if $k_i=1$, then $y_{i1}=p>y_{bk_p}$, which is a contradiction. Furthermore, $y_{i1}<y_{ik_i}$ since $a_i=p$ is a prime. Therefore $1<y_{11}<y_{21}<\cdots<y_{p1}<p$ which is a contradiction.

Proposition 7. $L_2^* \subsetneq L_1^*$.

Proof. We construct $A \in L_1^* - L_2^*$. Let p_m be the m-th prime number. Let $D_m=(p_m, p_m, \cdots, p_m)$ be p_m repetitions of p_m. Let $\{d_n\}=(D_1, D_2, \cdots, D_m, \cdots)$ and finally let the sequence $A=(a_n)$ be defined such that $a_1=1$ and $a_{n+1}=a_n+d_n$.

Clearly $A \in L_1 \subset L_1^*$. Suppose that $A \in L_2^*$ and so $A \subset B=\{b_n\}$, where $B \in L_2$. Let $e_u=b_{u+1}-b_u$ for $u \ge 1$. Since B is lacunary there exists N such that, for any $k \ge N$, $e_k>1$. If $t(m)=1+\sum_{i=1}^{m-1} p_i$ then $\{a_{t(m)}, a_{t(m)+1}, \cdots, a_{t(m+1)-1}\}$ is the m-th part of A corresponding to the m-th part D_m of $\{d_n\}$. Take m such that $b_N \le a_{t(m)}$. For each i, since $A \subset B$, some part of $\{e_u\}$ is a monotone partition of D_i. Then $b_N \le a_{t(m)}=b_s$, for some s, and thus $N \le s$ and $e_s>1$. By Lemma 6, if $a_{t(m+1)} \le b_u < b_{u+1} \le a_{t(m+2)-1}$, then $p_m \le e_u \le p_{m+1}$. By Bertrand's postulat (i. e., $p_{j+1}<2p_j$)

we get $(1/2)\ p_{m+1} < p_m \le e_u$. Hence $e_u + e_{u+1} > p_{m+1}$. This implies that $e_u = p_{m+1}$. Thus A and B are asymtotically equal. Hence $B \in L_2$ implies $A \in L_2$. But the following computation shows that $A \in L_2$. For each n,

$$
\begin{aligned}
\textstyle\sum_{k=t(m)+1}^{t(m+1)} 1/a_k = \sum_{k=1}^{p_m} 1/\{a_{t(m)} + (k-1)p_m\} &< \int_0^{p_m} 1/\{a_{t(m)} + xp_m\}\, dx \\
&= \frac{1}{p_m} \log \frac{a_{t(m+1)}}{a_{t(m)}} \\
&= \frac{1}{p_m} \log \left\{1 + \frac{p_1^2 + \cdots + p_m^2}{1 + p_1^2 + \cdots + p_{m-1}^2}\right\} \\
&= \frac{1}{p_m} \log \left\{1 + \frac{p_m^2}{1 + p_1^2 + \cdots + p_{m-1}^2}\right\} \\
&\le \frac{1}{p_m} \cdot \frac{p_m^2}{1 + p_1^2 + \cdots + p_{m-1}^2} \\
&\le \frac{p_m}{1 + p_1^2 + \cdots + p_{m-1}^2} \\
&< \frac{p_m}{1 + \sum_{k=1}^{m-1} k^2}
\end{aligned}
$$

Thus, using the Prime Number Theorem,

$$
\begin{aligned}
\textstyle\sum_{a \in A} 1/a = 1 + \sum_{m=1}^{\infty} \left(\sum_{k=t(m)+1}^{k=t(m+1)} 1/a_k\right) &< 1 + \sum_{m=1}^{\infty} \frac{p_m}{1 + \sum_{k=1}^{m-1} k^2} \\
&< r + s \sum_{m=1}^{\infty} \frac{m \log m}{m^3} \\
&< r + s \sum_{m=1}^{\infty} \frac{\log m}{m^2} < \infty
\end{aligned}
$$

where r and s are positive constants.

PROPOSITION 8. *$L_2^* \not\subset L_3^*$ and $L_3^* \not\subset L_2^*$.*

Proof. Suppose that $L_2^* \subset L_3^*$. Since L_2^* is full, L_3^* would also be full. This contradicts Proposition 4.

Suppose that $L_3^* \subset L_2^*$, then $L_2^* = L_3^* \cup L_2^* = L_1^*$ which contradicts Proposition 7.

Next we show that $[L_2^*] \subsetneq [L]$. This will establish the corresponding inclusions in diagram (3) and (after Proposition 12 below) in diagram (2). First we present two lemmas.

LEMMA 9. *Let x, u and v be positive integers. Suppose that*

$$x+(x-1)+\cdots+(x-u+1)=d_1+d_2+\cdots+d_\alpha,$$
$$(x-u)+(x-u-1)+\cdots+(x-u-v+1)=d_{\alpha+1}+\cdots+d_{\alpha+\beta},$$
$$d_1\le d_2\le\cdots\le d_{\alpha+\beta} \text{ and } d_1>(1/2)uv(u+v).$$

Then we have $d_1<d_{\alpha+\beta}$.

Proof. Suppose that $d_1=d_2=\cdots=d_{\alpha+\beta}$. Then

$$ux-(1/2)u(u-1)=\alpha d_1$$
$$vx-(1/2)v(2u+v-1)=\beta d_1.$$

It follows that

$$uvx-(1/2)uv(u-1)=\alpha vd_1$$
$$uvx-(1/2)uv(2u+v-1)=\beta ud_1.$$

Substracting, we get $(1/2)uv(u+v)=(\alpha v-\beta u)d_1$. Thus d_1 divides $(1/2)uv(u+v)$, which contradicts the hypothesis.

We omit the proof of the second lemma:

LEMMA 10. Let M_t, H_t $(t=1,2,\cdots,r)$, G and B be given reals which satisfy $H_{t+1}=H_t+M_t$ *for* $t=1,2\cdots,r-1$ and $M_t=(1+G)^{t-1}M_1$ for $t=1,2,\cdots,r$. Then $M_t=G(H_t+B)$ for $t=1,2,\cdots,r$.

PROPOSITION 11. $[L_1^*]\subsetneqq[L]$.

Proof. Containment is clear since $L_1^*\subset L^*=L$. For $m\ge1$, let $D_m=(m^2+m-1, m^2+m-2, \cdots, m)$. The sequence $(d_n)=(D_1, D_2, D_3, \cdots)$ will be the difference for a set $A=\{a_n\}$ with $a_1=1$. It is clear that $A\in L$. We will prove that $A\in[L_1^*]$. Let us assume, otherwise, that $A\subset A_1\cup A_2\cup\cdots\cup A_r$ where each $A_i\in L_1$. For each i, $1\le i\le r$, we write $A_i=\{a_n^i\}$ and $d_n^i=a_{n+1}^i-a_n^i$. Since the A_i are lacunary sets there is an N such that $n\ge N$ implies $d_n^i\ge(3r)^3$ for all i. Take $a^*=\max\{d_N^i : 1\le i\le r\}$.

Consider the part P_m of A corresponding to D_m. That is

$$P_m=\{a_{\alpha(m)}, a_{\alpha(m)+1}, \cdots, a_{\alpha(m+1)}\}$$

where $\alpha(t)=1+\sum_{i=1}^{t-1}i^2=(1/6)(t-1)t(2t-1)+1$ and $(d_{\alpha(m)}, d_{\alpha(m)+1}, \cdots, d_{\alpha(m+1)-1})=D_m$. We consider m large enough so that $a^*\le a_{\alpha(m)}$. Let $M_0=3r$, $B=(1/2)(3r-1)$, $G=9r^3$, $M_1=G(m+3r+B)$ and $M_t=(1+G)^{t-1}M_1$ for $t=1,2,\cdots,r$. Then we have

$$\begin{aligned}M_r+M_{r-1}+\cdots+M_1+M_0&=(1/G)\{(1+G)^r-1\}M_1+M_0\\&=\{(1+G)^r-1\}(m+3r+B)+3r.\end{aligned}$$

Since $M_r+\cdots+M_0$ is thus a polynomial in m of degree 1, we can further choose m such that $m^2>M_r+\cdots+M_0$.

We will partition some of P_m into $r+1$ blocks $L_r, L_{r-1}, \cdots, L_1, L_0$ thus:

$$L_t=\{a_j : \alpha(m+1)-(M_0+M_1+\cdots+M_t)\leq j\leq \alpha(m+1)-(M_0+M_1+\cdots+M_{t-1})\}.$$

Hence L_{t+1} is to the left of L_t with the rightmost point of L_{t+1} and the leftmost point of L_t equal. Furthermore, each L_t has M_t+1 points in it and thus represents M_t differences of A. Finally, since $M_r+M_{r-1}+\cdots+M_0+1\leq m^2=\alpha(m+1)-\alpha(m)$, it follows that $\cup L_t\subset P_m$. Also, the rightmost point of L_0 is $a_{\alpha(m+1)}$.

Let H_t be the smallest difference d_n represented in the block L_t (it occurs at the right hand end of L_t). Since, within P_m, the differences decrease by one at each point we clearly get $H_{t+1}=H_t+M_t$ for $0\leq t<r$. Note that $H_0=m$ so that $H_1=m+3r$. We can apply Lemma 10 and obtain $M_t=G(H_t+B)$ for $t=1, 2, \cdots, r$. Note that M_t is divisbile by $3r$. We now partition L_t into $M_t/3r$ blocks $I_1^t, I_2^t, \cdots, I_{M_t/3r}^t$ thus:

$$I_k^t=\{a_j : \alpha(m+1)-(M_0+\cdots+M_t)+(k-1)3r\leq j\leq \alpha(m+1)-(M_0+\cdots+M_t)+k\cdot 3r\}$$

Here I_k^t is to the left of I_{k+1}^t with one point in common. The number of elements of A in I_k^t is $3r+1$. Since $I_k^t\subset A_1\cup A_2\cup\cdots\cup A_r$, we get that for some i

$$|I_k^t\cap A_i|>3.$$

Let $a_p=a_\delta^i$, $a_{p'}=a_{\delta+\alpha}^i$ and $a_{p''}=a_{\delta+s\alpha+\beta}^i$ be three elements of $I_k^t\cap A_i$. The following equations result:

$$x+(x-1)+\cdots+(x-u)=d_\delta^i+d_{\delta+1}^i+\cdots+d_{\delta+\alpha-1}^i$$
$$(x-u-1)+(x-u-2)+\cdots+(x-u-v)=d_{\delta+\alpha}^i+d_{\delta+\alpha+1}^i+\cdots+d_{\delta+\beta-1}^i$$

where $x=d_p$, $u=p'-p$, $v=p''-p'$. Recall $d_j^i\leq d_{j+1}^i$ and $d_\delta^i>(3r)^3>(1/2)uv(u+v)$ (since $u+v\leq 3r$). We can apply Lemma 9 and get $d_\delta^i<d_{\delta+\alpha+\beta-1}^i$. Thus we conclude that, for any I_k^t, there exists an A_i such that d_n^i strictly increases at least once for elements of A_i in the interval [min I_k^t, max I_k^t].

We first look at L_r, the left most of the L_i. According to the last paragraph, since there are $M_r/3r$ blocks I_k^r in L_r, there are at least $M_r/3r$ increases of the d_n^i among $A_1, A_2, \cdots, A_r$. Thus there exists i_0

such that, for points of A_{i_0} within the interval $[\min L_r, \max L_r]$, $d_n^{i_0}$ increases at least $M_r/3r^2$ times. Let $d_{n_r}^{i_0}$ be the largest difference of A_{i_0} in the interval $[\min L_r, \max L_r]$. Clearly $d_{n_r}^{i_0} > M_r/3r^2$. On the other hand $M_r/3r^2 = (3r)M_r/9r^3 = (3r)M_r/G = (3r)(H_r+B) = (3r)(2H_r+3r-1)/2$. This last number is the diameter of the interval determined by $I_{M_r/3r}^r$. That is,

$$d_{M_r}^{i_0} > \max I_{M_r/3r}^r - \min I_{M_r/3r}^r.$$

Evidently this diameter exceeds any diameter of the interval determined by I_j^t when $t<r$. It follows that $|A_{i_0} \cap I_j^t| \leq 1$ for any j, t where $t<r$. Without loss of generality we may assume $i_0=1$.

Now we look at L_{r-1}. Again, for the $M_{r-1}/3r$ blocks, I_k^{r-1}, there is an A_i such that $|A_i \cap I_k^{r-1}| \geq 3$. Clearly $i \neq 1$ and it follows, as before, that there is an i_1 $(\neq 1)$ such that, for points of A_{i_1} within the interval $[\min L_{r-1}, \max L_{r-1}]$, $d_n^{i_1}$ increases at least $M_{r-1}/3r^2$ times. We may assume $i_1=2$. The largest difference $d_{n_{r-1}}^2$ thus exceeds $M_{r-1}/3r^2$. So that, as before, $|A_2 \cap I_j^t| \leq 1$ for $t<r-1$.

We repeat this process r times and then look at $L_0 = I_1^0$. It follows from the above that $|A_i \cap I_1^0| \leq 1$ for all $i=1, 2, \cdots, r$. But this implies that $3r+1 = |I_1^0| = I_1^0 \cap (A_1 \cup A_2 \cup \cdots \cup A_r)| \leq r$ a contradiction.

Proposition 12. $L_{M_1}^* = L$.

Proof. Let $A = \{a_i\} \in L$ and set $N_0=1$. For any $k \geq 1$, there exists $N_k > N_{k-1}$ such that $d_n > k^2$ whenever $n > N_k$. For each n with $N_k < n \leq N_{k+1}$, we let $d_n = q_n k + r_n$, where $0 \leq r_n < k$. Thus $q_n k = d_n - r_n > k^2 - k = (k-1)k$. Hence $q_n > k-1$ and $d_n = (q_n - r_n)k + (k+1)r_n$ where $q_n - r_n$ is positive. Let $\alpha_n = (\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nq_n})$ be the finite sequence $(k, k, \cdots, k, k+1, k+1, \cdots, k+1)$ where there are $q_n - r_n$ many k and r_n many $k+1$. Let

$$\begin{aligned}(e_m) &= (\alpha_1, \alpha_2, \alpha_3, \cdots) \\ &= (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1q_1}, \alpha_{21}, \cdots, \alpha_{2q_2}, \cdots, \alpha_{n1}, \cdots, \alpha_{nq_n}, \cdots).\end{aligned}$$

It follows from the definition of α_n that, for $n \leq m$, $\alpha_{ni} \leq \alpha_{mj} + 1$ for any i and j. Hence, letting $b_1 = a_1$ and $b_{m+1} = b_n + e_m$, the set $B = \{b_m : m \in I\} \in L_{M_1}$.

For any n, $d_n \alpha_{n1} + \cdots + \alpha_{nq_n}$. Thus, for $a_n \in A$, $a_n = a_1 + \sum_{i=1}^{n-1} d_i = a_1 + \sum_{i=1}^{n-1} (\sum_{j=1}^{q_i} \alpha_{ij}) = b_m$, where $m = 1 + \sum_{i=1}^{n} q_i$. Hence $A \subset B$ and $L \subset L_{M_1}^*$. The reverse inclusion is immediate.

Next we show that $[L_{M_i}]\subsetneq[L_{M_j}]$ for $i<j$. We need a lemma:

LEMMA 13. *Suppose that d, m, s, t, u, v, i and j are nonnegative integers such that $d>m^2+m$, $i<j<m$, $s\leq m$, $1\leq v\leq m$ and $1\leq t\leq m$, then*

1) $v(d+j)\leq t(d+j)+i$ *implies* $v\leq t$ *and* $v(d+j)\leq(d+j)$,

2) $vd\leq td+i$ *implies* $v\leq t$ *and* $vd\leq td$,

3) $v(d+j)\leq s(d+j)+td+i$ *implies* $v<s+t$ *and* $v(d+j)<s(d+j)+td$,

4) $v(d+j)+sd\leq td+i$ *implies* $v+s<t$ *and* $v(d+j)+sd<td$,

5) $vd\leq sd+t(d+j)+i$ *implies* $v\leq s+t$ *and* $vd<sd+t(d+j)$,

6) $vd+s(d+j)\leq t(d+j)+i$ *implies* $v+s\leq t$ *and* $vd+(d+j)<t(d+j)$,

Proof. The proofs of 1), 3), 5) are similar to those of 2), 4), 6) respectively. We prove only 1), 3) and 5):

1) $v(d+j)\leq t(d+j)+i<t(d+j)+d+j=(t+1)(d+1)$. Hence $v<t+1$ so that $v\leq t$.

3) $v(d+j)\leq s(d+j)+td+i<s(d+j)+t(d+j)=(s+t)(d+j)$ which proves the first part. Now $v(d+j)\leq(s+t-1)(d+j)=s(d+j)+td+(t-1)j-d<s(d+j)+td$ since $(t-1)j-d<m^2-(m^2+m)<0$.

5) Since $vd\leq sd+t(d+j)+i$ is equivalent to $-i-tj\leq(s+t-v)d$, we have $-d<-m-m^2<-i-tj\leq(s+t-v)d$. Thus we get $-1<(s+t-v)$ or, $v\leq s+t$. If $vd\geq sd+t(d+j)$ then we have $(v-s)d\geq t(d+j)$. This implies $v-s>t$ which is a contradiction.

PROPOSITION 14. $[L_{M_i}]\subsetneq[L_{M_j}]$ *for* $0\leq i<j$.

Proof. We make the following definitions:

$L_m=(m^3+j, m^3+j, \cdots, m^3+j)$, m repititions of m^3+j,

$R_m=(m^3, m^3, \cdots, m^3)$, m repititions of m^3,

$B_m=(L_m, R_m, L_m, R_m, \cdots, L_m, R_m)$, m repititions of L_m, R_m,

$(d_n)=(B_1, B_2, \cdots, B_m, \cdots)$,

$A=\{a_n\}$ where $a_n=1+d_1+\cdots+d_{n-1}$,

$A[a_m, a_n]=\{a_r : m\leq r\leq n\}$,

$A(a_m, a_n)=\{a_r : m<r<n\}$,

$\alpha(m, t)=1+2(1^2+2^2+\cdots+(m-1)^2)+2(t-1)m$ for $1\leq m$, $1\leq t\leq m+1$,

$\beta(m, t)=\alpha(m, t)+m$.

Note that $\alpha(m+1, 1)=\alpha(m, m+1)$. For $1\leq t\leq m+1$ define

$$A_{Lmt}=A[a_{\alpha(m,t)}, a_{\beta(m,t)}], \qquad A^\circ_{Lmt}=A(a_{\alpha(m,t)}, a_{\beta(m,t)})$$

$$A_{Rmt}=A[a_{\beta(m,t)}, a_{\alpha(m,t+1)}], \qquad A^\circ_{Rmt}=A(a_{\beta(m,t)}, a_{\alpha(m,t+1)}).$$

If we let $A_m=A_{Lm1}\cup A_{Rm1}\cup A_{Lm2}\cup A_{Rm2}\cup\cdots\cup A_{Lmm}\cup A_{Rmm}$, then A_m is the m-th part of A corresponding B_m. It is clear that $A\in L_{M_j}\subset[L_{M_j}]$.

Suppose that $X=\{x_q\}\in L_{M_i}$ and $X\subset A$. We will show that, if $j<m$ and $d=m^3>m^2+m$, then

$$|X\cap A_{Rmm}|\leq 2.$$

Let $\{y_q\}$ be the difference sequence of $\{x_q\}$ and f be the function on I such that $x_q=a_{f(q)}$. Then $f(s+1)-f(s)$ equals the number of terms in the sum $y_s=d_{f(s)}+d_{f(s)+1}+\cdots+d_{f(s+1)-1}$. At first we will consider the following six cases.

(i) If $a_{\alpha(m,t)}\leq x_q<x_{q+1}<x_{q+2}\leq a_{\beta(m,t)}$ (i. e., three consecutive elements of x are in A_{Lmt}), then, since $x\in L_{M_i}$ so that $y_q\leq y_{q+1}+i$, we have

$$x_{q+1}-x_q\leq x_{q+2}-x_{q+1}+i,\quad a_{f(q+1)}-a_{f(q)}\leq a_{f(q+2)}+i,$$
$$(f(q+1)-f(q))(d+j)\leq(f(q+2)-f(q+1))(d+j)+i,\quad \text{where } d=m^3.$$

By the LEMMA 13, case 1), we conclude that

$$f(q+1)-f(q)\leq f(q+2)-f(q+1)\text{ and } y_q\leq y_{q+1}.$$

(ii) Similarly, if $x_q<x_{q+1}<x_{q+2}$ are in the interval A_{Rmt}, then we apply the LEMMA 13 case 2) and we get

$$f(q+1)-f(q)\leq f(q+2)-f(q+1)\text{ and } y_q\leq y_{q+1}.$$

(iii) If $a_{\alpha(m,t)}\leq x_q<x_{q+1}\leq a_{\beta(m,t)}<x_{q+2}\leq a_{\alpha(m,t+1)}$ that is, x_q, x_{q+1} are in A_{Lmt} and x_{q+2} is in A_{Rmt}, then, since $x_{q+1}-x_q\leq x_{q+2}-x_{q+1}+i$, it follows that

$$\begin{aligned}a_{f(q+1)}-a_{f(q)}&\leq a_{f(q+2)}-a_{f(q+1)}+i\\&=a_{\beta(m,t)}-a_{f(q+1)}+a_{f(q+2)}-a_{\beta(m,t)}+i,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(f(q+1)-f(q))(d+j)\leq{}&(\beta(m,t)-f(q+1))(d+j)\\&+(f(q+2)-\beta(m,t))d+i.\end{aligned}$$

Now we apply the LEMMA 13 case 3) and get

$$f(q+1)-f(q)<f(q+2)-f(q+1)\text{ and so } y_q<y_{q+1}.$$

(iv) Similarly if

$$a_{\alpha(m,t)}\leq x_q<a_{\beta(m,t)}\leq x_{q+1}<x_q\leq a_{\alpha(m,t+1)},$$

that is, x_q is in A_{Lmt} and x_{q+1}, x_{q+2} are in A_{Rmt}, then we can apply the LEMMA 13 case 4) and get

$$f(q+1)-f(q)<f(q+2)-f(q+1)\text{ and } y_q<y_{q+1}.$$

(v) If

$$a_{\beta(m,t)} \leq x_q < x_{q+1} \leq a_{\alpha(m,t+1)} < x_{q+2} \leq a_{\beta(m,t+1)}$$

where $t \leq m$, that is, x_q and x_{q+1} are in A_{Rmt} and x_{q+2} is in $A_{Lm(t+1)}$, then we have

$$a_{f(q+1)} - a_{f(q)} \leq a_{f(q+2)} - a_{f(q+1)} + i$$
$$= a_{\alpha(m,t+1)} - a_{f(q+1)} + a_{f(q+2)} - a_{\alpha(m,t+1)} + i$$

or, equivalently,

$$(f(q+1) - f(q))d \leq (\alpha(m, t+1) - f(q+1))d$$
$$+ (f(q+2) - \alpha(m, t+1))(d+j) + i.$$

By the lemma case 5) and we get

$$f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q < y_{q+1}.$$

(vi) Finally, if

$$a_{\beta(m,t)} \leq x_q < a_{\alpha(m,t+1)} \leq x_{q+1} < x_{q+2} \leq a_{\beta(m,t+1)}$$

then we can apply the previous lemma case 6) and obtain

$$f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q < y_{q+1}.$$

Now assume that $|X \cap A_{Rmm}| \geq 3$ and so there exist three consecutive elements x_w, x_{w+1}, x_{w+2} of x in A_{Rmm}. By case (ii)

$$f(w+1) - f(w) \leq f(w+2) - f(w+1)$$

and so

$$2(f(w+1) - f(w)) \leq f(w+1) - f(w) + f(w+2) - f(w+1)$$
$$= f(w+2) - f(w) \leq m.$$

Thus $f(w+1) - f(w) \leq (1/2)m$ and $y_w = (f(w+1) - f(w))d \geq (1/2)md = (1/2)m^4$, the half diameter of A_{Rmm}.

We claim, for any $u < w$ and $x_u \geq a_{\alpha(m,1)}$, that $y_u \leq y_w$.

Proof of claim: Since $X \in L_{M_i}$, we have $y_u \leq y_w + i$. We may write $y_u = t(d+j) + vd$ and $y_w = qd$ and get $t(d+j) + vd \leq qd + i$. If $t > 0$ (resp. $t = 0$), then we apply the previous lemma case 4) (resp. case 2)) and get $y_u \leq y_w$.

By this claim we conclude that for any $u \leq w$ and $x_u \geq a_{\alpha(m,1)}$ we have $y_u \leq y_w \leq (1/2)m^4 = (1/2)$ diameter of $A_{Rmt} \leq (1/2)$ diameter of A_{Lmt} for $t = 1, 2, \ldots, m$.

Hence, for any $t \leq m$, A_{Rmt} and A_{Lmt} each contain at least two elements of X.

Therefore we conclude that:

By cases (iii) and (iv) above, if $x_q \in A^{\circ}_{Lmt}$ and $x_{q+2} \in A^{\circ}_{Rmt}$ then

$f(q+1)-f(q)<f(q+2)-f(q+1)$.

By cases (v) and (vi), if $x_q \in A^{\circ}_{Rmt}$ and $x_{q+2} \in A^{\circ}_{Lm(t+1)}$ then $f(q+1)-f(q) \leq f(q+2)-f(q+1)$.

By cases (i) and (ii) if $x_q, x_{q+1}, x_{q+2} \in A_{Lmt}$ or $x_q, x_{q+1}, x_{q+2} \in A_{Rmt}$, then $f(q+1)-f(q) \leq f(q+2)-f(q+1)$.

Now if we let x_{s_q} be an element of X such that $x_{s_q} \in A^{\circ}_{Rmq}$ and $x_{s_q+2} \in A^{\circ}_{Lmq}$ for $q=1, 2, \cdots, m$. Then we have for $q=1, 2, \cdots, m-1$,

$$f(s_q+1)-f(s_q)<f(s_{q+1}+1)-f(s_{q+1}).$$

Therefore we get

$$1 \leq f(s_1+1)-f(s_1)<f(s_2+1)-f(s_2)<\cdots<f(s_m+1)-f(s_m) \leq f(w+1)-f(w) \leq (1/2)m.$$

Since there are $m-1$ strict inequalities, we get a contradiction. Therefore we conclude that $|X \cap A_{Rmm}| \leq 2$.

Finally we show that $A \notin [L_{M_i}]$. Suppose that $A=X_1 \cup X_2 \cup \cdots \cup X_n$ where $X_s \in L_{M_i}$ for $s=1, 2, \cdots, n$. Since $A_{Rmm}=\cup_{i=1}^{n}(A_{Rmm} \cap X_i)$, for any m with $m^3>m^2+m$ and $m>j$, we have

$$m\,|A_{Rmm}| \leq \sum_{i=1}^{n} |A_{Rmm} \cap X_i| \leq 2n.$$

Thus m is bounded above, a contradiction.

COROLLARY 15. *For all* $i \geq 0$, $[L_{M_i}] \subsetneq [L]$. *In particular* $[L_1] \subsetneq [L]$.

PROPOSITION 16. $[L_2] \subsetneq [L_1]$.

Proof. Obviously $[L_2] \subset [L_1]$. Strictness is proved by observing that $\{n^2\} \in L_1$ but $\{n^2\} \notin [[L_2]$.

At this point we have completed the proofs of all diagrams given at the beginning of this section. Some further interesting inclusions concerning L_1 follow.

PROPOSITION 17. $L_1^* \subsetneq [L_1^*]$.

Proof. Let $A=\{n^2\}$ and $B=\{n^2+1\}$. Then $A \cup B \in [L_1] \subset [L_1^*]$. But $A \cup B$ is not lacunary. Thus $A \cup B \notin L_1^*$.

Finally we will prove $[L_1] \subsetneq [L_1^*]$. First, we define some terms and prove a lemma.

DEFINITION 3. Let $\{a_n\}=A$ be a sequence and $(a_s, a_{s+1}, \cdots, a_{s+r})$ be a part of $\{a_n\}$.

If $(d_s, d_{s+1}, \cdots, d_{s+r-1})$ is a strictly decreasing sequence, where $d_i=$

$a_{i+1}-a_i$, then we say that $(a_s, a_{s+1}, \cdots, a_{s+r})$ is a *consecutive descending wave of length* $r+1$ in A. Further, the d_t are called the *(decreasing) steps* of the wave. (Note that definition of descending wave in [1] is more general.)

LEMMA 18. *There exists a function* $f(n)$ *(depending only on* n*) such that, for any sets* $A_1, A_2, \cdots, A_n \in L_1$, *and for any consecutive descending wave* X *in* $A_1 \cup A_2 \cup \cdots \cup A_n$, $|X| \leq f(n)$.

Proof. We take $f(1)=2$ which clearly works.

Suppose there exists $f(n-1)$ such that for any $A_1, A_2, \cdots, A_{n-1}$ in L_1, and any consecutive descending wave X in $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$, we have $|X| \leq f(n-1)$.

Let $A = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and $B = \{b_u\} = A_n$ where $A_1, A_2, \cdots, A_n \in L_1$. Further let

$$W_u = \{a \in A : b_u < a < b_{u+1}\},$$
$$V_u = \{c \in A \cup B : b_u \leq c \leq b_{u+1}\}.$$

Suppose that X is a consecutive descending wave in $A \cup B$, $V_u \subset X$ and $V_{u+1} \subset X$, then we prove that $|W_u| < |W_{u+1}|$.

Let $e_1 > e_2 > \cdots > e_{q+1} > c_1 > c_2 > \cdots > c_{p+1}$ be the decreasing steps of the consecutive descending wave $V_u \cup V_{u+1}$, where $|W_u| = q$ and $|W_{u+1}| = p$. Since $B \in L_1$, $(q+1)e_{q+1} \leq e_1 + e_2 + \cdots + e_{q+1} = b_{u+1} - b_u \leq b_{u+2} - b_{u+1} = c_1 + c_2 + \cdots + c_{p+1} \leq (p+1)c_1 < (p+1)e_{q+1}$. Therefore $q+1 < p+1$ and so $q < p$.

Next we show, if $X \subset A \cup B$ is a consecutive descending wave then $|X \cap B| \leq f(n-1) + 2$.

Suppose, otherwise, that $|X \cap B| > f(n-1) + 2$. Let $\{b_r, b_{r+1}, \cdots, b_s\} = X \cap B$, where $s \geq r + f(n-1) + 2$. Then $V_k \subset X$ for all $r \leq k \leq s-1$. By the above, $0 \leq |W_r| < |W_{r+1}| < \cdots < |W_{s-1}|$, thus we have $|W_{s-1}| \geq s - r - 1 \geq f(n-1) + 1 > f(n-1)$ which is a contradiction since W_{s-1} is a consecutive descending wave in A.

Finally, let X be a descending wave of $A \cup B$. Again, writing $X \cap B = \{b_r, b_{r+1}, \cdots, b_s\}$, we get $X \subset H \cup V_r \cup \cdots \cup V_{s-1} \cup J$ where H and J are the (possibly empty) consecutive descending waves in $A \cap X$ which come before b_r and after b_s respectively. Thus $|X| \leq |H| + |J| + \sum_{i=r}^{s-1} |V_i| \leq (f(n-1)+3)(f(n-1)+2)$ and so we can set $f(n) = (f(n-1)+2)(f(n-1)+3)$.

PROPOSITION 19. $[L_1] \subsetneqq [L_1^*]$.

Proof. Let

$B_n = (n^2, (n-1)n, (n-2)n, \cdots, 2n, n)$,

$(d_n) = (B_1, B_2, \cdots, B_q, \cdots)$,

$\{a_n\}$ where $a_n = 1 + d_1 + \cdots + d_{n-1}$ for $n = 1, 2, 3, \cdots$,

$W_m = (m, m, \cdots, m)$, with $m(m+1)/2$ repititions of m,

$(y_m) = (w_1, w_2, \cdots, w_p, \cdots) = (1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, \cdots)$,

$\{x_n\}$ where $x_m = 1 + y_1 + y_2 + \cdots + y_{m-1}$ for $m = 1, 2 \cdots$.

Then $\{x_n\} \in L_1$ and $\{a_n\} \subset \{x_n\}$. Thus $\{a_n\} \in L_1^* \subset [L_1^*]$. Since $\{a_n\}$ contains arbitray long consecutive descending waves, by the previous lemma, $\{a_n\} \notin [L_1]$. Thus $[L_1] \subsetneqq [L_1^*]$.

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