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A GENERALIZATION OF FIXED POINT THEOREMS

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§1. Introduction

In 1976, J. Caristi [3] obtained the following interesting result which generalizes the Banach contraction principle.

THEOREM A. *Let* (V, d) *be a complete metric space, and* f *is a selfmap of* V. *If there exists a l. s. c. function* ϕ *from* V *into the nonnegative real numbers such that*

$$d(x, fx) \leq \phi(x) - \phi(fx), \quad x \in V,$$

then f *has a fixed point.*

F. E. Browder [1] gave a remark that if f is continuous, without assuming the lower semicontinuity of ϕ, for any $x \in V$, the iteration $f^n x$ converges to a fixed point of f.

Let (V, d) be a complete metric space. Let $\phi : V \to \boldsymbol{R}^+ = [0, \infty)$, and $f : V \to V$ a function not necessarily continuous such that

$$d(x, fx) \leq \phi(x) - \phi(fx), \quad x \in V. \qquad (*)$$

Given a sequence of functions f_i, $1 \leq i \leq \infty$, set

$$\prod_{i=1}^{\infty} f_i(x) = \lim_{i \to \infty} f_i f_{i-1} \cdots f_1(x)$$

if it exists, call it the countable composition of the f_i. Let $\mathfrak{f}$ be the family of selfmaps f of V satisfying $(*)$.

Note that $\mathfrak{f}$ is closed under composition and that if ϕ is l. s. c., then $\mathfrak{f}$ is closed under countable composition.

In [6], J. Siegel obtained the following generalization of Theorem A.

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THEOREM B. (Siegel [6]) *Let $\mathfrak{g}\subset\mathfrak{f}$ be closed under composition. Let $x_0\in V$.*

(*a*) *If $\mathfrak{g}$ is closed under countable composition, then there exists a $g\in\mathfrak{g}$ such that $\bar{x}=gx_0$ and $f\bar{x}=\bar{x}$ for each $f\in\mathfrak{g}$.*

(*b*) *If each $f\in\mathfrak{g}$ is continuous, then there exists a sequence $\{f_i\}\subset\mathfrak{g}$ and a point $\bar{x}=\lim_{i=\infty} f_i f_{i-1}\cdots f_1(x_0)$ such that $f\bar{x}=\bar{x}$ for each $f\in\mathfrak{g}$.*

In [5], J. J. Moreau obtained a case where the iteration of a nonexpansive map at a point in a Hilbert space converges to a fixed point of that map. He also applied this result to some periodic evolution problems. However, his main result is incorrect.

In this paper, we give a corrected version of Moreau's result and its possible generalizations. Consequently, we obtain another generalization of Theorem A.

Our tools in this paper are generalizations of Theorem A given by J. Siegel [6] and some related results of R. E. Bruck [2].

§2. Moreau's proposition

Let H be a real Hilbert space, D a subset of H, and $S: D\to D$ a nonexpansive map. The following is the main result of Moreau in [5].

PROPOSITION [5]. *If the fixed point set of S has a nonempty interior, then for every $u_0\in D$, the sequence $S^n u_0$, $n\in \boldsymbol{N}$, converges to a fixed point of S.*

The author used the following in [5].

LEMMA [5]. *Let b be a center of a closed ball with radius $\rho>0$, contained in the fixed point set of S. Then for every $u\in D$, we have*

$$\|u-Su\|\leq\frac{1}{2\rho}(\|b-u\|^2-\|b-Su\|^2).$$

Under the hypotheses of Proposition, according to F. E. Browder's remark on Theorem A (see p. 1), the lemma guarantees only that for every $u_0\in D$, the sequence $S^n u_0$ converges to a point of H. Further, if the limit belongs to D, e. g., D is closed, then the limit is fixed under S.

The following is a counterexample to the proposition:

Let $H=\boldsymbol{R}$, $D=(-2,-1)\cup\left\{\frac{1}{n}\right\}_{n\in\boldsymbol{N}}$, and

$$Sx=\begin{cases} x & \text{if } x\in(-2,-1) \\ \dfrac{1}{n+1} & \text{if } x=\dfrac{1}{n} \end{cases}$$

Consequently, the following is a correct version of Moreau's proposition:

THEOREM 2.1. *Let H be a real Hilbert space, D a closed suset of H, and $S: D\to D$ a nonexpansive. If the fixed point set of S has a nonempty interior, then for every $u_0\in D$, the sequence $S^n u_0$, $n\in \boldsymbol{N}$, converges to a point of S.*

§ 3. Generalizations of Moreau's proposition

In this section, we consider possible generalizations of Theorem 2.1. Let $G: V\to \boldsymbol{R}^+$ be a continuous function, $h: \boldsymbol{R}^+\to \boldsymbol{R}^+$ an increasing function and $F=hG$, Let $\mathfrak{f}$ be the family of self-maps f of V satisfying

$$d(x,fx)\leq F(x)-F(fx),\quad x\in V. \tag{1}$$

Then the following holds:

LEMMA 3.1. *$\mathfrak{f}$ is closed under composition.*

Proof. Let f_1 and f_2 be in $\mathfrak{f}$. Then

$$\begin{aligned} d(x,f_1f_2x) &\leq d(x,f_1x)+d(f_1x,\ f_1f_2x) \\ &\leq F(x)-F(f_1x)+F(f_1x)-F(f_1f_2x) \\ &= F(x)-F(f_1f_2x). \end{aligned}$$

LEMMA 3.2. *Let $\{x_i\}$ be a sequence in V such that*

$$d(x_i,x_{i+1})\leq F(x_i)-F(x_{i+1}) \text{ for each } i.$$

Then $\lim_{i\to\infty} x_i=\bar{x}$ exists and

$$d(x_i,\bar{x})\leq F(x_i)-F(\bar{x}) \text{ for each } i.$$

Proof. Since $\{F(x_i)\}$ is decreasing and bounded from below and $d(x_i,x_j)\leq F(x_i)-F(x_j)$ for $i\leq j$, $\{x_i\}$ is a Cauchy sequence in V, and hence, converges to some $\bar{x}=\lim_{i\to\infty} x_i$ in V. On the other hand,

$$\begin{aligned} d(x_i,\bar{x}) &= \lim_{j\to\infty} d(x_i,x_j) \\ &\leq F(x_i)-\lim_{j\to\infty} F(x_j). \end{aligned}$$

Since

$$hG(x_i)=F(x_i)\geq F(x_{i+1})=hG(x_{i+1})$$

and h is increasing, we have $G(x_i) \geq G(x_{i+1})$. Since G is continuous and $x_i \to \bar{x}$, we must have $Gx_i \perp G\bar{x}$. Therefore, we have

$$F(x_i) = hG(x_i) \geq hG(\bar{x}) = F(\bar{x}),$$

and hence,

$$\lim_{j\to\infty} F(x_j) \geq F(\bar{x}).$$

Consequently,

$$\begin{aligned} d(x_i, \bar{x}) &\leq F(x_i) - \lim_{j\to\infty} F(x_j) \\ &\leq f(x_i) - F(\bar{x}). \end{aligned}$$

LEMMA 3.3. $\mathfrak{f}$ *is closed under countable composition.*

Proof. Given a sequence $\{f_i\}$ in $\mathfrak{f}$ and for each $x \in V$, the sequence $\{x_i\}$ given by $f_i f_{i-1} \cdots f_1 x$ satisfies the conditions of Lemma 3.2.

Hence, from Theorem B (Siegel) we obtain the following:

THEOREM 3.1. *Let $\mathfrak{f}$ be the family of selfmaps of a complete metric space V satisfying* (1), *where either* (i) *F is l.s.c. or* (ii) *$F=hG$, $h: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ an increasing function, and $G: V \to \boldsymbol{R}^+$ a continuous function. Then for any $x_0 \in V$,*

(*a*) *there exists a $g \in \mathfrak{f}$ such that $\bar{x} = gx_0$ and $f\bar{x} = \bar{x}$ for each $f \in \mathfrak{f}$.*

(*b*) *If $\mathfrak{g} \subset \mathfrak{f}$ is closed under composition and each $f \in \mathfrak{g}$ is continuous, then there exist a sequence $\{f_i\} \subset \mathfrak{g}$ and a point $\bar{x} = \lim_{i\to\infty} f_i f_{i-1} \cdots f_1 x_0$ such that $f\bar{x} = \bar{x}$ for each $f \in \mathfrak{g}$.*

The following is a generalization of Theorem A.

COROLLARY 3.1. *Let f be a selfmap of a complete metric space V satisfying* (1), *where either* (i) *F is l.s.c. or* (ii) *$F=hG$ as in Theorem* 3.1. *Then*

(*a*) *f has a fixed point in V,*

(*b*) *If f is continuous, then for any $x_0 \in V$, $\{f^n x_0\}$ converges to a fixed point of f.*

Corollary 3.1 generalizes Theorem 2.1.

Another line of generalizations of Moreau's proposition can be obtained by using R. E. Bruck's result [2]. For a metric space (V, d), Buck [2] observed that a special role is played by selfmaps f of V which satisfy, for some $b \in V$ and an increasing function $h: \boldsymbol{R}^+ \to \boldsymbol{R}^+$,

$$d(x, fx) \leq h(d(x, b)) - h(d(fx, b)) \tag{2}$$

for all $x \in V$. Their importance arises from an observation made by Golub et al. [4].

A sequence $\{g_n\}$ of selfmaps of V is said to be iteration normal if for all $x_0 \in V$, the iterates $x_n = g_n x_{n-1}$ $(n \geq 1)$ converges and the same is true for every shift $\{g_k, g_{k+1}, \cdots\}$ of the original sequence.

The following is due to Bruck:

THEOREM 3.2. (Bruck [2]) *Let (V, d) be a complete metric space. If $\{g_n\}$ is iteration normal, each g_n is nonexpansive, and $\{f_n\}$ is a sequence of selfmaps of V which satisfy (2), for some increasing function $h : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ and some $b \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(g_n)$. Then,*

(a) $\{f_n\}$ has a common fixed point, and

(b) the sequence $\{g_1, f_1, g_2, f_2, \ldots\}$ obtained by "shuffling" $\{g_n\}$ with $\{f_n\}$ is also iteration normal.

Note that (a) follows from Theorem 3.1 (a).

Bruck noted that if we take each $g_n = I_V$, (b) asserts that $\{f_n\}$ can be iterated in any order, with the resulting sequence converging.

From this observation, we have the following:

THEOREM 3.3. *Let H be a real Hilbert space, D a closed subset of H, and $\{f_n\}$ a sequence of nonexpansive selfmaps of D. If the common fixed point set $\{f_n\}$ has a nonempty interior, then for every $u_0 \in D$, the sequence $\{u_n\}$ given by $u_n = f_n u_{n-1}$, $n \geq 1$, converges.*

Proof. By putting $V = D$, $g_n = I_D$, b belongs to the interior of the common fixed point set of $\{f_n\}$, and $h(r) = r^2/2\rho$, $r \geq 0$, the conclusion follows from Theorem 3.2 (b).

Note that Theorem 3.3 generalizes Theorem 2.1. In fact, by putting $f_n = f^n$, $\{f^n u_0\}$ converges to a fixed point of f.

REMARK. In [2], it was noted that we do not identify the limit of $\{u_n\}$ in terms of the (common) fixed point sets of $\{f_n\}$. Now we give an example in which the limit of the sequence $\{u_n\}$ is not a common fixed point of $\{f_n\}$ but a fixed point of some f_n.

Let $D = [0, 1]$ be a closed subset of $\boldsymbol{R}$, and $\{f_n\}$ a sequence of nonexpansive selfmaps of D defined by

$$f_1(x)=\begin{cases} x & \text{if } x\in\left[0,\ \frac{1}{2}\right] \\ \frac{1}{2}x+\frac{1}{4} & \text{if } x\in\left[\frac{1}{2},\ 1\right], \end{cases}$$

and

$$f_n(x)=x,\ x\in[0,1] \text{ for } n\geq 2.$$

Then $\bigcap_{i=1}^{\infty}\mathrm{Fix}(f_n)=\left[0,\ \frac{1}{2}\right]$, and all the hypothese of Theorem 3.3 are satisfied. Moreover, for any $u_0\in\left[\frac{1}{2},\ 1\right]$, the sequence $\{u_n\}$ given by $u_n=f_nu_{n-1}$ $(n\geq 1)$ converges to $\bar{u}=\frac{1}{2}u_0+\frac{1}{4}\in D$. However, $\bar{u}$ is not a fixed point of f_1. Therefore, the limit of the sequence $\{u_n\}$ is not contained in the common fixed point set of $\{f_n\}$.

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