J. Korean Math. Soc. 25(1988), No. 1, pp. 67-76

A STABILITY IN TOPOLOGICAL DYNAMICS

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1. Introduction

THEOREM. *Let (X, φ) be a flow whose phase space X is a locally compact metric space. Then a compact invariant subset M of X is asymptotically stable if and only if there exists a continuous nonnegative real valued function f defined on an invariant neighborhood U of M such that f vanishes exactly on M, and that $f(xt)=e^{-t}f(x)$ for all points x of U and real numbers t* [1].

In this paper we introduce the concept of a c-first countable space which is a more general concept than that of a metric space, and extend the above theorem to the case that the phase space X is c-first countable and locally compact. All spaces are assumed to be Hausdorff.

2. C-first countable spaces.

DEFINITION. A space X is said to be *c-first countable* if for each compact subset K of X the quotient space X/K is first countable.

Let X be a c-first countable space. Given any compact subset K of X, there exists a family $\mathcal{U}$ consisting of countably many neighborhoods of K such that every neighborhood of K contains some member of $\mathcal{U}$. Such a family $\mathcal{U}$ will be called a *countable neighborhood base of K*.

THEOREM 2.1 *Every second countable space is c-first countable.*

Proof. Let X be a second countable space. There exists a countable basis $\mathcal{B}$ for X. Given any compact subset K of X, let $\mathcal{U}$ be the family of all neighborhoods of K which are finite unions of members of $\mathcal{B}$. Then $\mathcal{U}$ is a countable neighborhood base of K. Thus X is c-first countable.

Received August 11, 1987.
This research is supported by a grant from the Daewoo Foundation.

The converse of the above theorem is not true as shown by uncountable discrete spaces. Clearly, every c-first countable space is first countable but its couverse does not hold.

EXAMPLE 2.1. Let $X_0=\{(x,0) : x\in \boldsymbol{R}\}$ and $X_1=\{(x,1) : x\in \boldsymbol{R}\}$ be two subsets of the plane $\boldsymbol{R}^2$. We take a basis $\mathscr{B}$ for the topology on the set $X=X_0\cup X_1$ as follow;

$$\mathscr{B}=\{\{(x,1)\} : x\in \boldsymbol{R}\}\cup\{B(x,r) : x\in \boldsymbol{R}, r>0\}$$

where $B(x,r)=\{(y,0) : |x-y|<r\}\cup\{(y,1) : 0<|x-y|<r\}$. It is clear that X is first countable. We claim that X is not c-first countable. Let us choose a compact subset $K=\{(x,0) : x\in I\}$ of X where I is the unit interval. For each neighborhood U of K, let $I(U)=\{x\in I : (x,1)\notin U\}$. Suppose that $I(U)$ is infinite for some neighborhood U of K. $I(U)$ has a cluster point, say y, in I. Since $(y,0)\in K\subset U$, there exists a number $r>0$ such that $B(y,r)\subset U$. Since y is a cluster point of $I(U)$, there is a number $z\in I(U)$ such that $0<|y-z|<r$. Since $(z,1)\in B(y,r)\subset U$, we have a contradiction. Thus $I(U)$ is finite for all neighborhoods U of K. Let $U_1, U_2, U_3, \dots$ be neighborhoods of K. Since $I(U_n)$ is finite for all n, $A=\bigcup_{n=1}^{\infty} I(U_n)$ is countable. Thus there is a number $w\in I-A$. Let $V=X_0\cup\{(x,1) : x\neq w\}$. Then V is a neighborhood of K and $U_n\not\subset V$ for all n. Thus there is no countable neighborhood base of K. Hence X is not c-first countable.

THOEREM 2.2. *Every metric space is c-first countable.*

Proof. Let (X,d) be a metric space. Given any compact subset K of X, it is easy to show that the family $\left\{B\left(K, \frac{1}{n}\right) : n=1,2,3,\dots\right\}$ is a countable neighborhood base of K, where $B\left(K, \frac{1}{n}\right)=\left\{x\in X : d(K,x)<\frac{1}{n}\right\}$. Thus X is c-first countable.

The converse of the above theorem is not true. The following example shows that there exists a c-firrt countable and locally compact space which is not a metric space.

EXAMPLE 2.2. For each irrational x, we choose a sequence (x_n) of rationals converging to it in the Euclidean topology. The rational sequence topology $\mathscr{T}$ on $\boldsymbol{R}$ is then defined by declaring each rational open, and selecting the sets $U_n(x)=\{x_i : i=n, n+1, n+2, \ldots\} \cup \{x\}$ as a basis for the irrational point x. The space $(\boldsymbol{R}, \mathscr{T})$ is Hausdorff, locally compact and not metrizable [2]. We will show that $(\boldsymbol{R}, \mathscr{T})$ is c–first countable. Let K be a compact subset of $\boldsymbol{R}$. If K–Q is infinite, where Q is the set of rationals, then the open cover $\{U_1(x) : x \in K\text{–}Q\} \cup \{Q\}$ of K has no finite subcover, this is a contradiction. Thus K–Q is finite, say $K\text{–}Q=\{x^1, x^2, \ldots, x^m\}$. Let U be aneighborhood of K. For each $i=1, 2, \ldots, m$, since $x^i \in K\text{–}Q \subset U-Q$, there is an n_i such that $U_{n_i}(x^i) \subset U$. Let $N=\max n_i$. Then $\bigcup_{i=1}^{m} U_N(x^i) \cup (K \cap Q) \subset U$. Thus $\left\{\bigcup_{i=1}^{m} U_n(x^i) \cup (K \cap Q) : n=1, 2, \ldots\right\}$ is a countable neighborhood base of K. Hence $(\boldsymbol{R}, \mathscr{T})$ is c–first countable.

Lemma 2.1. *Let X be a c–first countable and locally compact space, and let K be a compact subset of X. For each neighborhood U of K, there exists a countable neighborhood base $\{U(r) : r \in D\}$ of K such that*

(1) *$U(1)=U$, and that*

(2) *if $r_1<r_2$, then $\overline{U(r_1)} \subset U(r_2)$*

where D is the set of all rationals of form $\frac{k}{2^n}$, $0<\frac{k}{2^n}\leq 1$.

Proof. Let us show that for each $r \in D$ we can associate a neighborhood $U(r)$ of K satisfying the above conditions (1) and (2). We proceed by induction on exponent of dyadic fractions, letting $\mathscr{U}_n=\left\{U\left(\frac{k}{2^n}\right) : k=1, 2, \ldots, 2^n\right\}$. There exists a countable neighborhood base $\{V_m : m=1, 2, \ldots\}$ of K. We may assume that $V_1 \supset V_2 \supset \ldots$ and $\overline{V}_1$ compact. There is an m_1 such that $\overline{V}_{m_1} \subset U$. $\mathscr{U}_1$ consists of $U\left(\frac{1}{2}\right)=V_{m_1}$ and $U(1)=U$. Assume $\mathscr{U}_{n-1}$ constructed. Note that only $U\left(\frac{k}{2^n}\right)$ for odd k requires definition. There is an $m_n>m_{n-1}$ such that $\overline{V}_{m_n} \subset U\left(\frac{1}{2^{n-1}}\right)$. We define $U\left(\frac{1}{2^n}\right)=V_{m_n}$. For each odd $k \neq 1$, we have from $\mathscr{U}_{n-1}$ that $\overline{U\left(\frac{k-1}{2^n}\right)} \subset U\left(\frac{k+1}{2^n}\right)$, so we define $U\left(\frac{k}{2^n}\right)$ to be an open set V satisfying

$$\overline{U\left(\frac{k-1}{2^n}\right)}\subset V\subset\overline{V}\subset U\left(\frac{k+1}{2^n}\right)$$

and $\overline{V}$ compact. This completes inductive step. Given any neighborhood W of K, there is an n such that $V_{m_n}=U\left(\frac{1}{2^n}\right)\subset W$. Thus the family $\{U(r) : r\in D\}$ is a countable neighborhood base of K.

THEOREM 2.3. *Let X be a locally compact space. Then X is c-first countable if and only if for each compact subset K of X there exists a continuous nonnegative real valued function f defined on X such that f vanishes exactly on K.*

Proof. ($\Rightarrow$) By Lemma 2.1, there exists a countable neighborhood base $\{U(r) : r\in D\}$ such that $U(1)=X$, and that if $r_1<r_2$ then $\overline{U(r_1)}\subset U(r_2)$. Define a function $f : X\to\boldsymbol{R}^+$ by $f(x)=\inf\{r\in D : x\in U(r)\}$. Clearly, $0\leq f\leq 1$. It is easy to show that f vanishes exactly on K. Given any $\varepsilon>0$, we can choose an $r\in D$ such that $r<\varepsilon$. Since $f(U(r))\subset(-\varepsilon,\varepsilon)$, f is continuous on K. We will show that f is continuous at $x\in X-K$. There are two possibilities;

Case 1. $f(x)<1$; Given any $\varepsilon>0$, we can choose r_1 and r_2 in D such that $f(x)-\varepsilon<r_1<f(x)<r_2<f(x)+\varepsilon$. Then $U(r_2)-\overline{U(r_1)}$ is a neighborhood of x and $f(U(r_2)-\overline{U(r_1)})\subset(f(x)-\varepsilon, f(x)+\varepsilon)$.

Case 2. $f(x)=1$; Given any $\varepsilon>0$, there exists a number $r\in D$ such that $1-\varepsilon<r<1$. Then $X-\overline{U(r)}$ is a neighborhood of x and $f(X-\overline{U(x)})\subset(1-\varepsilon, 1+\varepsilon)$. Thus f is continuous.

($\Leftarrow$) There exists a neighborhood U of K such that $\bar{U}$ is compact. For each positive integer n, the set $U_n=f^{-1}\left[0,\frac{1}{n}\right)\cap U$ is a neighborhood of K. Given any neighborhood V of K, suppose that $U_n\not\subset V$ for all n. For each n, we can choose an $x_n\in U_n-V$. Since $\bar{U}$ is compact, the sequence (x_n) in $\bar{U}$ has a convergent subsequence. Let $x_n\to x$. It is clear that $x\in X-V$ and $f(x_n)\to f(x)$. Since $f(x_n)<\frac{1}{n}$ for all $n, f(x_n)\to 0$. Thus $f(x)=0$ and so $x\in K$. This is a contradiction. So $U_n\subset V$ for some n. Hence the family $\{U_n : n=1, 2, \ldots\}$ is a countable neighborhood base of K.

3. Asymptotic stability

Throughout this section (X, φ) is a flow whose phase space X is c-first countable and locally compact.

For a point x of X, the *positive* (*negative*) *limit set* $L^+(x)$ $(L^-(x))$ *of* x defined by

$$L^+(x) = \bigcap_{t \in R^+} \overline{x[t, \infty)} \quad (L^-(x) = \bigcap_{t \in R^-} \overline{x(-\infty, t]})$$

where $\boldsymbol{R}^+$ $(\boldsymbol{R}^-)$ denotes the set of nonnegative (nonpositive) real numbers. It is easy to show that $y \in L^+(x)$ $(L^-(x))$ if and only if there is a sequence (t_n) in $\boldsymbol{R}^+$ $(\boldsymbol{R}^-)$ such that $t_n \to \infty$ $(-\infty)$ and $xt_n \to y$. Obviously, the set $L^+(x)$ $(L^-(x))$ is invariant. Furthermore, the set $L^+(x)$ $(L^-(x))$ is nonempty whenever $\overline{x\boldsymbol{R}^+}$ $(\overline{x\boldsymbol{R}^-})$ is compact. A subset M of X is said to be *stable* if for each neighborhood U of M, there exists a neighborhood V of M such that $V\boldsymbol{R}^+ \subset U$. It is clear that a stable set is positively invariant. For a subset M of X, the *region of attraction* $A(M)$ is defined by $A(M) = \{x \in X : L^+(x) \neq \phi \subset M\}$. Note that $A(M)$ is invariant. A subset M of X is called an *attractor* if the set $A(M)$ is a neighborhood of M. When a subset M of X is stable and an attractor, the set M is said to be *asymptotically stable.*

Lemma 3.1 *Let M be a compact subset of X. Then $x \in A(M)$ if and only if for each neighborhood U of M there exists a $t \in \boldsymbol{R}^+$ such that $x[t, \infty) \subset U$.*

Proof. ($\Rightarrow$) Let $x \in A(M)$ and U a neighborhood of M. We can choose a neigborhood V of M such that $\bar{V} \subset U$ and $\bar{V}$ compact. Suppose that for each $t \in \boldsymbol{R}^+$ there is an $s \geq t$ such that $xs \notin U$. Then there is an $r_1 \geq 1$ such that $xr_1 \in X - U \subset X - \bar{V}$. Since $x \in A(M)$, there exists a $t_1 > r_1$ such that $xt_1 \in V$. We can choose an s_1 such that $r_1 < s_1 < t_1$ and $xs_1 \in \partial V$ where ∂V is the boundary of V. By the same way we can choose r_2, t_2 and s_2 such that

$$r_2 \geq \max(2, t_1), \quad xr_2 \in X - \bar{V}, \quad xt_2 \in V, \quad r_2 < s_2 < t_2 \text{ and } xs_2 \in \partial V,$$

and so on. Thus we obtain a sequence (s_n) in $\boldsymbol{R}^+$ such that $s_n \to \infty$ and $xs_n \in \partial V$ for all n. Since ∂V is compact, the sequence (xs_n) has a convergent subsequence. Let $xs_n \to z \in \partial V$. Since $z \in L^+(x) \subset M \subset V$, we have a contradiction, Thus there is a $t \in \boldsymbol{R}^+$ such that $x[t, \infty) \subset U$.

($\Leftarrow$) There exists a neighborhood U of M such that $\bar{U}$ is compact. We can choose a $t \in \boldsymbol{R}^+$ such that $x[t, \infty) \subset U$. Since

$$\overline{xR^+} = x[0, t] \cup \overline{x[t, \infty)} \subset x[0, t] \cup \bar{U},$$

$\overline{xR^+}$ is compact. Thus $L^+(x) \neq \phi$. To show $L^+(x) \subset M$, suppose that there exists an $y \in L^+(x) - M$. There are neighborhoods V of M and W of y such $V \cap W = \phi$. We can choose a $t \in \boldsymbol{R}^+$ such that $x[t, \infty) \subset V$. Since $W \cap x[t, \infty) = \phi$, $y \notin \overline{x[t, \infty)}$ and so $y \notin L^+(x)$. This is a contradiction. Thus $L^+(x) \subset M$. Hence $x \in A(M)$.

LEMMA 3.2 *Let a compact subset M of X be asymptotically stable and U a neighborhood of M. For any point x of $A(M)$, if $x\boldsymbol{R}^+ \subset U$, then there exists a neighborhood V of x such that $V\boldsymbol{R}^+ \subset U$.*

Proof. Since M is stable, there is a neighborhood U_1 of M such that $U_1\boldsymbol{R}^+ \subset U$. By Lemma 3.1, there is an $s \in \boldsymbol{R}^+$ such that $x[s, \infty) \subset U_1$, we can choose a neighborhood W_1 of x such that $W_1 s \subset U_1$. For each $t \in [0, s]$, since $xt \in U$, there exist neighborhoods V_t of x and I_t of t such that $V_t I_t \subset U$. There are finitely many $0 \leq t_1, t_2, \ldots, t_n \leq s$ such that $[0, s] \subset \bigcup_{i=1}^{n} I_{t_i}$. Let $W_2 = \bigcap_{i=1}^{n} V_{t_i}$. Then W_2 is a neighborhood of x. Given any $y \in W_2$ and $t \in [0, s]$, since $t \in I_{t_i}$ for some i, $yt \in V_{t_i} I_{t_i} \subset U$. Thus $W_2[0, s] \subset U$. Let $V = W_1 \cap W_2$. Then V is a neighborhood of x. From the fact that

$$V[0, s] \subset W_2[0, s] \subset U \text{ and } V[s, \infty) \subset W_1[s, \infty) = (W_1 s)\boldsymbol{R}^+ \subset U_1\boldsymbol{R}^+ \subset U$$

we have $V\boldsymbol{R}^+ = V[0, s] \cup V[s, \infty) \subset U$.

LEMMA 3.3 *Let U be a neighborhood of a point x of X. If y is a point of X and $y\boldsymbol{R}^+ \not\subset \bar{U}$, then there is a neighborhood V of y such that $z\boldsymbol{R}^+ \not\subset \bar{U}$ for all points z of V.*

Proof. There is a $t \in \boldsymbol{R}^+$ such that $yt \notin \bar{U}$. Since $X - \bar{U}$ is a neighborhood of yt, there exists a neighborhood V of y such that $Vt \subset X - \bar{U}$. Then V is a desired neighborhood.

THEOREM 3.1 *Let M be an asymptotically stable compact subset of X. Then there exists a continuous nonnegative real valued function f defined on $A(M)$ such that f vanishes exactly on M, and that $f(xt) < f(x)$ for all points x of $A(M) - M$ and all positive real numbers t.*

Proof. Let D be the set of all rationals r of form $\frac{k}{2^n}$, $0 < \frac{k}{2^n} \leq 1$.

By Lemma 2.1, there exists a countable neighborhood base $\{U(r) : r \in D\}$ of M satisfying

(1) $U(1) = A(M)$ and

(2) if $r_1 < r_2$ then $\overline{U(r_1)} \subset U(r_2)$.

Define a function $g : A(M) \to \boldsymbol{R}^+$ by $g(x) = \inf\{r \in D : x\boldsymbol{R}^+ \subset U(r)\}$. Clearly, $0 \leq g \leq 1$. Let $x \in M$. For any $r \in D$, since $x\boldsymbol{R}^+ \subset M \subset U(r)$, $g(x) \leq r$. Thus $g(x) = 0$. Let $x \in A(M) - M$. We can choose an $r \in D$ such that $x \notin U(r)$. Then $g(x) \geq r > 0$. Thus g vanishes exactly on M. Let us show that g is continuous on M. Given any $\varepsilon > 0$, there exists a number $r \in D$ such that $r < \varepsilon$. Since M is stable, there exists a neighborhood V of M such that $V\boldsymbol{R}^+ \subset U(r)$. Since $g(V) \subset (-\varepsilon, \varepsilon)$, g is continuous on M. We further show that g is continuous at each point x in $A(M) - M$. There are two possibilities;

(1) In case $g(x) = 1$, given any $\varepsilon > 0$, we can choose an $r \in D$ such that $1 - \varepsilon < r < 1$. Since $x\boldsymbol{R}^+ \not\subset \overline{U(r)}$, by Lemma 3.3, there is a neighborhood V of x such that $yR^+ \not\subset \overline{U(r)}$ for all $y \in V$. Then $g(V) \subset (1 - \varepsilon, 1 + \varepsilon)$.

(2) In case $g(x) < 1$, given any $\varepsilon > 0$, we choose $r_1, r_2 \in D$ such that $g(x) - \varepsilon < r_1 < g(x) < r_2 < g(x) + \varepsilon$. Since $x\boldsymbol{R}^+ \subset U(r_2)$, there is a neighborhood V_1 of x such that $V_1\boldsymbol{R}^+ \subset U(r_2)$ by Lemma 3.2. By Lemma 3.3, there exists a neighborhood V_2 of x such that $y\boldsymbol{R}^+ \not\subset \overline{U(r_1)}$ for all $y \in V_2$ since $x\boldsymbol{R}^+ \not\subset \overline{U(r_1)}$. Let $V = V_1 \cap V_2$. Then V is a neighborhood of x and $g(V) \subset (g(x) - \varepsilon, g(x) + \varepsilon)$. Thus g is continuous. We claim that $g(xt) \leq g(x)$ for all $x \in A(M)$ and $t \in \boldsymbol{R}^+$. Suppose that $g(xt) > g(x)$ for some $x \in A(M)$ and $t \in \boldsymbol{R}^+$. We can choose an $r \in D$ such that $g(x) < r < g(xt)$. Since $(xt)\boldsymbol{R}^+ = x[t, \infty) \subset x\boldsymbol{R}^+ \subset U(r)$, $g(xt) \leq r$. This is a contradiction. Thus $g(xt) \leq g(x)$ for all $x \in A(M)$ and $t \in \boldsymbol{R}^+$.

Define a function $f : A(M) \to \boldsymbol{R}^+$ by

$$f(x) = \int_0^\infty e^{-s} g(xs)\, ds.$$

Clearly, f is continuous and vanishes exactly on M. It remains to prove that $f(xt) < f(x)$ for all $x \in A(M) - M$ and $t > 0$. Since $g((xt)s) = g((xs)t) \leq g(xs)$ for all $s \in \boldsymbol{R}^+$, $f(xt) \leq f(x)$. To rule out $f(xt) = f(x)$, observe that in this case we must $g(x(t+s)) = g((xt)s) = g(xs)$ for all $s \in \boldsymbol{R}^+$. In particular, letting $s = 0, t, 2t, \ldots$, we get $g(x(nt)) = g(x)$, $n = 1, 2, \ldots$ Given any $r \in D$, since $x \in A(M)$, by Lemma 3.1, there exists an $s \in \boldsymbol{R}^+$ such that $x[s, \infty) \subset U(r)$. Since $nt \to \infty$ as $n \to \infty$, mt

$\geq s$ for some m. Since

$$(x(mt))\boldsymbol{R}^+ = x[mt, \infty) \subset x[s, \infty) \subset U(r),$$

$g(x) = g(x(mt)) \leq r$. Thus $g(x) = 0$. But as $x \in A(M) - M$, we must $g(x) > 0$, a contradiction. We have thus proved that $f(xt) < f(x)$. The theorem is proved.

THEOREM 3.2 *Let M be an asymptotically stable compact invariant subset of X. Then there exists a continuous function $f : A(M) \to \boldsymbol{R}^+$ such that f vanishes exactly on M, and that $f(xt) = e^{-t}f(x)$ for all $x \in A(M)$ and all $t \in \boldsymbol{R}$.*

Proof. By Theorem 3.1, there exists a continuous function $g : A(M) \to \boldsymbol{R}^+$ such that g vanishes exactly on M, and that $g(xt) < g(x)$ for all $x \in A(M) - M$ and all $t > 0$. Since $A(M)$ is a neighborhood of M, we can choose a neighborhood U of M such that $\bar{U} \subset A(M)$ and $\bar{U}$ is compact. Set $a = \min g(\partial U)$. Clearly, $a > 0$. Let $V = g^{-1}[0, a)$. Then V is a neighborhood of M. Suppose that $\bar{V} \not\subset \bar{U}$ and choose a point $x \in \bar{V} - \bar{U}$. Since $x \in \bar{V} \subset g^{-1}[0, a] \subset A(M)$, there exists a number $s > 0$ such that $x[s, \infty) \subset U$ by Lemma 3.1. Thus we can choose a $t > 0$ such that $xt \in \partial U$. Since $a \leq g(xt) < g(x) \leq a$, we have a contradiction. This shows that $\bar{V} \subset \bar{U}$. We claim that $\partial V \cap (\partial V)t = \varnothing$ for all $t > 0$. Suppose that $\partial V \cap (\partial V)t \neq \varnothing$ for some $t > 0$. Then there exists an $x \in \partial V$ such that $xt \in \partial V$. Since $\partial V \subset g^{-1}(a)$, $a = g(xt) < g(x) = a$. This is a contradiction. Thus $\partial V \cap (\partial V)t = \varnothing$ for all $t > 0$. We will show that for every $x \in A(M) - M$, there is unique $t \in \boldsymbol{R}$ such that $xt \in \partial V$. There are three possibilities;

(1) In case $x \notin \bar{V}$, by Lemma 3.1, there is an $s > 0$ such that $x[s, \infty) \subset V$. Thus we can choose a $t > 0$ such that $xt \in \partial V$.

(2) In case $x \in \partial V$, $x0 = x \in \partial V$.

(3) In case $x \in V$, assume that $x\boldsymbol{R} \subset V$. Since $\overline{x\boldsymbol{R}} \subset \bar{V}$ is compact, $L^-(x) \neq \varnothing$. If $L^-(x) \cap M \neq \varnothing$, then we can choose an $y \in L^-(x) \cap M$. There exists a sequence (t_n) in $\boldsymbol{R}^-$ such that $t_n \to -\infty$ and $xt_n \to y$. Since $g(x) \leq g(xt_n)$ for all n, $g(x) \leq g(y) = 0$, this is a contradiction. Thus $L^-(x) \cap M = \varnothing$. Choose a point $z \in L^-(x)$. Since $L^+(z) \subset \overline{z\boldsymbol{R}} \subset L^-(x)$, $L^+(z) \cap M = \varnothing$. But $L^+(z)$ is nonempty and contained in M because of $z \in L^-(x) \subset \overline{x\boldsymbol{R}} \subset \bar{V} \subset A(M)$. This is a contradiction. Thus $x\boldsymbol{R} \not\subset V$. Hence we can choose a $t \in \boldsymbol{R}$ such that $xt \in \partial V$. The uniqueness of such t can be obtained from the fact that $\partial V \cap (\partial V)t = \varnothing$ for all $t > 0$.

Define a function $m: A(M)-M \to \boldsymbol{R}$ by $xm(x) \in \partial V$. Let $x \in A(M)-M$. Given any $t \in \boldsymbol{R}$, since $(xt)(m(x)-t)=xm(x) \in \partial V$, $m(xt)=m(x)-t$. Thus $m(xt) \to \pm\infty$ as $t \to \mp\infty$. We will show that m is continuous. Given any $x \in A(M)-M$ and $\varepsilon > 0$, since $x(m(x)+\varepsilon) \in V$, $W_1 = V(-m(x)-\varepsilon)$ is a neighborhood of x. For all $y \in W_1$, $y(m(x)+\varepsilon) \in V$ implies $m(y) < m(x)+\varepsilon$. Since $x(m(x)-\varepsilon) \in X-\overline{V}$, $W_2=(X-\overline{V})(-m(x)+\varepsilon)$ is a neighborhood of x. For all $y \in W_2$, $y(m(x)-\varepsilon) \in X-\overline{V}$ implies $m(x)-\varepsilon < m(y)$. Let $W=W_1 \cap W_2$. Then W is a neighborhood of x and $m(x)-\varepsilon < m(y) < m(x)+\varepsilon$ for all $y \in W$. Thus m is continuous.

Define a function $f: A(M) \to \boldsymbol{R}^+$ by

$$f(x)=\begin{cases} e^{m(x)} & \text{if } x \in A(M)-M \\ 0 & \text{if } x \in M. \end{cases}$$

We will show that f is continuous. It is sufficient to show that f is continuous on M. Suppose that there exists an $\varepsilon > 0$ such that $f(U) \not\subset [0, \varepsilon)$ for all neighborhoods U of M. There is a $T \in \boldsymbol{R}^-$ such that $e^T < \varepsilon$. For each neighborhood U of M, $f(U) \not\subset [0, e^T)$ and so $m(U-M) \not\subset (-\infty, T)$. Since X is c-first countable, there is a countable neighborhood base $\{V_n : n=1, 2, \dots\}$ of M. We may assume that $V \supset V_1 \supset V_2 \supset \dots$ For each n, since $m(V_n - M) \not\subset (-\infty, T)$, there is an $x_n \in V_n - M$ such that $T \leq m(x_n) \leq 0$. There is a $y \in M$ such that $x_n \to y$. $(m(x_n))$ is a sequence in $[T, 0]$. Since $[T, 0]$ is compact, $(m(x_n))$ has a convergent subsequence. Let $m(x_n) \to t \in [T, 0]$. Then $x_n m(x_n) \to yt \in M$ and $yt \in \partial V$. This is a contradiction. Thus for each $\varepsilon > 0$, there is a neighborhood U of M such that $f(U) \subset [0, \varepsilon)$. Hence f is continuous on M. Clearly, f vanishes exactly on M. For any $x \in A(M)$ and $t \in \boldsymbol{R}$,

$$f(xt)=e^{m(xt)}=e^{m(x)-t}=e^{-t}e^{m(x)}=e^{-t}f(x).$$

Thus the theorem is proved.

LEMMA 3.4 *Let M be a compact subset of X, U an invariant neighborhood of M and $f: U \to \boldsymbol{R}^+$ a continuous function such that f vanishes exactly on M and $f(xt)=e^{-t}f(x)$ for all $x \in U$ and $t \in \boldsymbol{R}$. If K is a compact positively invariant subset of U then K is contained in $A(M)$.*

Proof. Let $x \in K$. Since $\overline{x\boldsymbol{R}^+} \subset K$ is compact, $L^+(x) \neq \emptyset$. Let $y \in L^+(x)$. Take a $t > 0$. Since $yt \in L^+(x)$, there are sequence (t_n), (s_n) in $\boldsymbol{R}^+$ such that $t_n \to \infty$, $s_n \to \infty$, $xt_n \to y$ and $xs_n \to yt$. We may assume that $t_n \geq s_n$ for all n. Since $f(xt_n) \leq f(xs_n)$, $f(y) \leq f(yt)$. Since $f(yt) \leq f(y)$,

$f(y)=f(yt)=e^{-t}f(y)$. Thus $f(y)=0$ and so $y\in M$. Hence $L^+(x)\subset M$. Therefore $x\in A(M)$.

THEOREM 3.3 *Let M be a compact invariant subset of X. If there exists a continuous nonnegative real valued function f defined on an invariant neighborhood U of M such that f vanishes exactly on M, and that $f(xt)=e^{-t}f(x)$ for all points x of U and all real numbers t, then M is asymptotically stable and $U=A(M)$.*

Proof. Given any neighborhood V of M, we can choose a neighborhood W_1 of M such that $\overline{W}_1\subset U\cap V$ and $\overline{W}_1$ is compact. Let $a=\min f(\partial W_1)$. Then $a>0$. Let $W=f^{-1}[0,a)$. Then $W\subset W_1$ and W is a positively invariant neighborhood of M. Thus M is stable. We can choose a neighborhood V of M such that $\overline{V}\subset U$ and $\overline{V}$ is compact. Let $a=\min f(\partial V)$. Then $a>0$. Take a number r such that $0<r<a$, and let $W=f^{-1}[0,r]$. Then $W\subset V$ and W is compact positively invariant. By Lemma 3.4, $W\subset A(M)$. Given any $x\in U$, if $x\in W$, then $x\in A(M)$, and if $x\notin W$, then $f(x)>r$. There is a $t>0$ such that $f(xt)=e^{-t}f(x)=r$. Since $K=x\boldsymbol{R}^+\cup W=x[0,t]\cup W$ is a compact positively invariant subset of U, by Lemma 3.4, $K\subset A(M)$ and so $x\in A(M)$. Thus $U\subset A(M)$. Given any $x\in A(M)$, since U is a neighborhood of M, by Lemma 3.1, there is a $t\in\boldsymbol{R}^+$ such that $xt\in U$. Since U is invariant, $x=(xt)(-t)\in U$. Hence $A(M)=U$ and so $A(M)$ is a neighborhood of M. Therefore M is asymptotically stable.

References

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