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A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction.

Let $S_p(p\geqslant 1)$ denote the class of functions of the form $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ which are analytic and p-valent in the unit disc $U=\{z : |z|<1\}$. A function f is said to be subordinate to a function $F(f<F)$ if there exists an analytic function $\phi(z)$ with $|\phi(z)|\leqslant|z|, z\in U,$ such that $f=F\circ\phi$. For A, B fixed, $-1\leqslant A<B\leqslant 1$, $0<B\leqslant 1$, and $0\leqslant\alpha<p$, we say that $f\in S_p^*(A, B, \alpha)$ if and only if

$$\frac{zf'(z)}{f(z)}<\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz},\quad z\in U,$$

or equivalently $f\in S_p^*(A, B, \alpha)$ if and only if

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{B\frac{zf'(z)}{f(z)}-[pB+(A-B)(p-\alpha)]}\right|<1,\quad z\in U.$$

Further f is said to belong to the class $K_p(A, B, \alpha)$ if and only if $\frac{zf'(z)}{p}\in S_p^*(A, B, \alpha)$.

Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$, and we set

$$T_p^*(A, B, \alpha)=S_p^*(A, B, \alpha)\wedge T_p \text{ and } C_p(A, B, \alpha)=K_p(A, B, \alpha)\wedge T_p.$$

Silverman [4], Gupta and Jain [2] and Silverman and Silvia [5, 6] have studied certain subclasses of univalent functions with negative coefficients. Also Goel and Sohi [1] have studied certain subclases of multivalent functions with negative coefficients. In this paper we obtain

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coefficient estimates, distortion and covering theorems for the classes $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$. We also determine the radius of convexity for the class $T_p^*(A, B, \alpha)$. It is further shown that the classes $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$ are closed under arithmetic mean and convex linear combinations. Also in this paper we obtain for these classes some distortion theorems for the fractional calculus.

2. Coefficient inequalities.

THEOREM 1. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in* $T_p^*(A, B, \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}|\leqslant(B-A)(p-\alpha). \quad (2.1)$$

The result is sharp.

Proof. Let $|z|=1$, then

$$\begin{aligned}&|zf'(z)-pf(z)|-|Bzf'(z)-[pB+(A-B)(p-\alpha)]f(z)|\\&=\left|\sum_{n=1}^{\infty}-n|a_{p+n}|z^{p+n}\right|-|(B-A)(p-\alpha)z^p\\&\quad-\sum_{n=1}^{\infty}[nB+(B-A)(p-\alpha)]|a_{p+n}|z^{p+n}|\\&\leqslant\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}|-\\&\qquad(B-A)(p-\alpha)\leqslant 0.\end{aligned}$$

Hence by the principle of maximum modulus $f(z)\in T_p^*(A, B, \alpha)$.

Conversely, suppose that

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-p}{B\cdot\dfrac{zf'(z)}{f(z)}-[pB+(A-B)(p-\alpha)]}\right|$$

$$=\left|\frac{\sum_{n=1}^{\infty}n|a_{p+n}|z^{p+n}}{(B-A)(p-\alpha)z^p-\sum_{n=1}^{\infty}[nB+(B-A)(p-\alpha)]|a_{p+n}|z^{p+n}}\right|<1,\ z\in U.$$

Since $|\mathrm{Re}\, z|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}n|a_{p+n}|z^{p+n}}{(B-A)(p-\alpha)z^p-\sum_{n=1}^{\infty}[nB+(B-A)(p-\alpha)]|a_{p+n}|z^{p+n}}\right\}<1. \quad (2.2)$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2. 2) and letting $z \longrightarrow 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} n|a_{p+n}| \leqslant \left\{(B-A)(p-\alpha)-\sum_{n=1}^{\infty}[nB+(B-A)(p-\alpha)]|a_{p+n}|\right\}$$

which implies that

$$\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leqslant (B-A)(p-\alpha).$$

The function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}z^{p+n}$$

is an extremal function.

Corollary 1. *If* $f \in T_p^*(A, B, \alpha)$ *then* $|a_{p+n}| \leqslant \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}$, *with equality only for functions of the form*

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}z^{p+n}.$$

Corollary 2. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in* $C_p(A, B, \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leqslant (B-A)(p-\alpha).$$

Proof. It is well known that $f \in C_p(A, B, \alpha)$ if and only if $\frac{zf'(z)}{p} \in T_p^*(A, B, \alpha)$. Since

$$\frac{zf'(z)}{p}=z^p-\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)|a_{p+n}|z^{p+n}$$

we may replace $|a_{p+n}|$ with $\left(\frac{n+p}{p}\right)|a_{p+n}|$ in Theorem 1.

3. Representation formula.

Theorem 2. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in* $T_p^*(A, B, \alpha)$ *if and only if*

$$f(z)=z^p\exp\left\{(B-A)(p-\alpha)\int_0^z\frac{\phi(t)}{1-Bt\phi(t)}dt\right\},\tag{3.1}$$

where $\phi(z)$ is analytic in U and satisfies $|\phi(z)|<1$, $z\in U$.

Proof. Let $f(z)\in T_p^*(A,B,\alpha)$, then

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{B\cdot\frac{zf'(z)}{f(z)}-[pB+(A-B)(p-\alpha)]}\right|<1,\quad z\in U.$$

Since the absolute value vanishes for $z=0$, we have

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{B\cdot\frac{zf'(z)}{f(z)}-[pB+(A-B)(p-\alpha)]}\right|=h(z)\tag{3.2}$$

where $h(z)$ is analytic in U and $|h(z)|<1$ for $z\in U$. Integrating (3.2) with $h(z)=z\phi(z)$ we find that

$$f(z)=z^p\cdot\exp\left\{(B-A)(p-\alpha)\int_0^z\frac{\phi(t)}{1-Bt\phi(t)}dt\right\}.$$

The converse is obtained by differentiating (3.1).

4. Distortion and covering theorems for $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$.

THEOREM 3. *If $f(z)\in T_p^*(A,B,\alpha)$, then*

$$r^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^{p+1}\leqslant|f(z)|$$
$$\leqslant r^p+\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^{p+1}\quad(|z|=r),\tag{4.1}$$

with equality for $f(z)=z^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}z^{p+1}$ $(z=\pm r)$.

Proof. From Theorem 1, we have

$$[1+B+(B-A)(p-\alpha)]\sum_{n=1}^{\infty}|a_{p+n}|$$
$$\leqslant\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}|\leqslant(B-A)(p-\alpha).$$

This implies that

$$\sum_{n=1}^{\infty}|a_{p+n}|\leqslant\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}.\tag{4.2}$$

Thus

$$\begin{aligned}|f(z)| &\leqslant |z|^p+\sum_{n=1}^{\infty}|a_{p+n}|\,|z|^{p+n}\\ &\leqslant r^p\left(1+r\sum_{n=1}^{\infty}|a_{p+n}|\right)\\ &\leqslant r^p+\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^{p+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}|f(z)| &\geqslant |z|^p-\sum_{n=1}^{\infty}|a_{p+n}|\,|z|^{p+n}\\ &\geqslant r^p\left(1-r\sum_{n=1}^{\infty}|a_{p+n}|\right)\\ &\geqslant r^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^{p+1}.\end{aligned}$$

Corollary 3. *If* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}\in C_p(A,B,\alpha)$, *then*

$$\begin{aligned}r^p-\frac{(B-A)(p-\alpha)p}{(p+1)[1+B+(B-A)(p-\alpha)]}r^{p+1}&\leqslant|f(z)|\\ \leqslant r^p+\frac{(B-A)(p-\alpha)p}{(p+1)[1+B+(B-A)(p-\alpha)]}r^{p+1}&\quad(|z|=r),\end{aligned}$$

with equality for

$$f(z)=z^p-\frac{(B-A)(p-\alpha)p}{(p+1)[1+B+(B-A)(p-\alpha)]}z^{p+1}\quad(z=\pm r).$$

Theorem 4. *The disc* $|z|<1$ *is mapped onto a domain that contains the disc* $|w|<\dfrac{1+B}{1+B+(B-A)(p-\alpha)}$ *by any* $f\in T_p^*(A,B,\alpha)$, *and onto a domain that contains the disc*

$$|w|<\frac{(p+1+B)+[pB+(A-B)(p-\alpha)]}{(p+1)[1+B+(B-A)(p-\alpha)]}\ \text{by any } f\in C_p(A,B,\alpha).$$

The theorem is sharp, with extremal functions

$$\begin{aligned}&z^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}z^{p+1}\in T_p^*(A,B,\alpha)\ \text{and}\\ &z^p-\frac{(B-A)(p-\alpha)p}{(p+1)[1+B+(B-A)(p-\alpha)]}z^{p+1}\in C_p(A,B,\alpha).\end{aligned}$$

Proof. The results follow upon letting $r\longrightarrow 1$ in Theorem 3 and Corollary 3.

Theorem 5. *If* $f\in T_p^*(A,B,\alpha)$, *then*

$$pr^{p-1}-\frac{(p+1)(B-A)(p-\alpha)}{1++(B-A)(p-\alpha)}r^p\leqslant|f'(z)|$$

$$\leqslant pr^{p-1}+\frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^p\quad(|z|=r).$$

Equality holds for

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}z^{p+1}\quad(z=\pm r).$$

Proof. We have

$$|f'(z)|\leqslant pr^{p-1}+\sum_{n=1}^{\infty}(p+n)|a_{p+n}|r^{p+n-1}$$

$$\leqslant pr^{p-1}+r^p\sum_{n=1}^{\infty}(p+n)|a_{p+n}|$$

$$=r^{p-1}\left[p+r\sum_{n=1}^{\infty}(p+n)|a_{p+n}|\right].\tag{4.3}$$

In view of Theorem 1,

$$\sum_{n=1}^{\infty}(1+B)\left[n+p-\frac{p(1+B)+(A-B)(p-\alpha)}{1+B}\right]|a_{p+n}|\leqslant(B-A)(p-\alpha)$$

or

$$\sum_{n=1}^{\infty}(1+B)(n+p)|a_{p+n}|\leqslant(B-A)(p-\alpha)+$$

$$[p(1+B)+(A-B)(p-\alpha)]\sum_{n=1}^{\infty}|a_{p+n}|\tag{4.4}$$

(4.4) with the help of (4.2) impies that

$$\sum_{n=1}^{\infty}(n+p)|a_{p+n}|\leqslant\frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}.\tag{4.5}$$

A substitution of (4.5) into (4.3) yields the right-hand inequality.

On the other-hand,

$$|f'(z)|\geqslant r^{p-1}\left[p-r\sum_{n=1}^{\infty}(p+n)|a_{p+n}|\right]$$

$$\geqslant pr^{p-1}-\frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}r^p.$$

This completes the proof.

Corollary 4. *If* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}\in C_p(A,B,\alpha)$, *then*

$$pr^{p-1}-\frac{(B-A)(p-\alpha)p}{[1+B+(B-A)(p-\alpha)]}r^{p}\leqslant|f'(z)|$$
$$\leqslant pr^{p-1}+\frac{(B-A)(p-\alpha)p}{[1+B+(B-A)(p-\alpha)]}r^{p}\quad(|z|=r).$$

Equality holds for $f(z)=z^{p}-\dfrac{(B-A)(p-\alpha)P}{(p+1)[1+B+(B-A)(p-\alpha)]}z^{p+1}$, $(z=\pm r)$.

5. Radius of convexity for the class $T_p^*(A,B,\alpha)$.

THEOREM 6. *If* $f(z)\in T_p^*(A,B,\alpha)$, *then* $f(z)$ *is p-valently convex in the disc*

$$|z|<R_p=\inf_n\left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}\cdot\left(\frac{p}{n+p}\right)^2\right]^{1/n}\quad(n=1,2,\ldots).\tag{5.1}$$

The result is sharp, with the extremal function

$$f(z)=z^{p}-\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}z^{p+n}.$$

Proof. It is sufficient to show that $\left|\left[1+\dfrac{zf''(z)}{f'(z)}\right]-p\right|\leqslant p$ for $|z|<R_p$. We have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|=\left|\frac{-\sum_{n=1}^{\infty}n(n+p)|a_{p+n}|z^{n}}{p-\sum_{n=1}^{\infty}(n+p)|a_{p+n}|z^{n}}\right|$$
$$\leqslant\frac{\sum_{n=1}^{\infty}n(n+p)|a_{p+n}||z|^{n}}{p-\sum_{n=1}^{\infty}(n+p)|a_{p+n}||z|^{n}}.$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\leqslant p\ \ if\ \sum_{n=1}^{\infty}(n+p)^2|a_{p+n}||z|^{n}\leqslant p^2$$

or

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^2|a_{p+n}||z|^{n}\leqslant 1.\tag{5.2}$$

According to Theorem 1, $\sum_{n=1}^{\infty}\dfrac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}|a_{p+n}|\leqslant 1.$

Hence (5.2) will be true if

$$\left(\frac{n+p}{p}\right)^2|z|^n \leqslant \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}$$

or if

$$|z| \leqslant \left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \cdot \left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}} \quad (n=1, 2, \ldots). \quad (5.3)$$

The theorem follows easily from (5.3).

6. Closure theorems.

In this section we shall prove that the classes $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$ are closed under convex linear combinations.

THEOREM 7. *If* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *and* $g(z)=z^p-\sum_{n=1}^{\infty}|b_{p+n}|z^{p+n}$ *are in* $T_p^*(A, B, \alpha)$, *then* $h(z)=z^p-\frac{1}{2}\sum_{n=1}^{\infty}|a_{p+n}+b_{p+n}|z^{p+n}$ *is also in* $T_p^*(A, B, \alpha)$.

Proof. Since $f(z)$ and $g(z)$ are in $T_p^*(A, B, \alpha)$, we have

$$\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leqslant (B-A)(p-\alpha) \quad (6.1)$$

and

$$\sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|b_{p+n}| \leqslant (B-A)(p-\alpha). \quad (6.2)$$

From (6.1) and (6.2) we get

$$1/2 \sum_{n=1}^{\infty}[(1+B)n+(B-A)(p-\alpha)]|a_{p+n}+b_{p+n}| \leqslant (B-A)(p-\alpha)$$

which implies that $h(z) \in T_p^*(A, B, \alpha)$.

The following theorem can be proved similarly.

THEOREM 8. *If* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *and* $g(z)=z^p-\sum_{n=1}^{\infty}|b_{p+n}|z^{p+n}$ *are in* $C_p(A, B, \alpha)$, *then* $h(z)=z^p-1/2\sum_{n=1}^{\infty}|a_{p+n}+b_{p+n}|z^{p+n}$ *is also in* $C_p(A, B, \alpha)$.

THEOREM 9. *Let* $f_p(z)=z^p$, $f_{p+n}(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}z^{p+n}$ $(n=1, 2, 3, \cdots)$. *Then* $f\in T_p^*(A, B, \alpha)$ *if and only if it can be* *expressed in the form*

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}\cdot f_{p+n}(z) \quad \textit{where } \lambda_{p+n}\geqslant 0 \textit{ and } \sum_{n=0}^{\infty}\lambda_{p+n}=1.$$

Proof. Suppose $f(z)=\sum_{n=0}^{\infty}\lambda_{p+n} f_{p+n}(z)$

$$=z^p-\sum_{n=1}^{\infty}\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}\lambda_{p+n}\, z^{p+n}.$$

Then

$$\sum_{n=1}^{\infty}\left[\lambda_{p+n}\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}\cdot\left(\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}\right)\right]$$
$$=\sum_{n=1}^{\infty}\lambda_{p+n}\leqslant 1-\lambda_p\leqslant 1.$$

So by Theorem 1, $f(z)\in T_p^*(A, B, \alpha)$.

Conversely suppose $f(z)\in T_p^*(A, B, \alpha)$. Then

$$|a_{p+n}|\leqslant\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}.$$

Setting $\lambda_{p+n}=\dfrac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}|a_{p+n}|$ $(n=1, 2, \ldots)$, and

$$\lambda_p=1-\sum_{n=1}^{\infty}\lambda_{p+n},$$

we have

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}\cdot f_{p+n}(z).$$

This completes the proof of the theorem.

REMARKS. (1) Putting $\alpha=0$ in the above theorems we get the results obtained by R. M. Goel and N. S. Sohi [1].

(2) Putting $p=1$ and taking $A=-\beta$, $B=\beta$, where $0<\beta\leqslant 1$, in the above theorems we get the results obtained by Gupta and Jain [2].

(3) Putting $p=1$ and taking $A=-1$, $B=1$ in the above theorems we get the results obtained by Silverman [4].

7. Fractional calculus.

There are many definitions of the fractional culculus, that is, the fractional derivative and the fractional integral. In 1978, Owa [3] gave the following delfinitions for the fractional calculus.

DEFINITION 1. The fractional integral of order k is defined by

$$D_z^{-k}f(z)=\frac{1}{\Gamma(k)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-k}},$$

where $k>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order k is defined by

$$D_z^kf(z)=\frac{1}{\Gamma(1-k)}\frac{d}{dz}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{k}},$$

where $0\leqslant k<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-k}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypothese of Definition 2, the fractional derivative of order $(n+k)$ is defined by

$$D_z^{n+k}f(z)=\frac{d^n}{dz^n}D_z^kf(z),$$

where $0\leqslant k<1$ and $n\in NU\{0\}$.

THEOREM 10. *Let a function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *be in the class* $T_p^*(A, B, \alpha)$. *Then we have*

$$|D_z^{-k}f(z)|\geqslant\frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k}\cdot\left\{1-\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{\{1+B+(B-A)(p-\alpha)\}}|z|\right\}$$

and

$$|D_z^{-k}f(z)|\leqslant\frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k}\cdot\left\{1+\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}|z|\right\}$$

for $0<k<1$ *and* $z\in U$. *The result is sharp.*

Proof. Let

$$\begin{aligned}F(z)&=\frac{\Gamma(p+1+k)}{\Gamma(p+1)}z^{-k}D_z^{-k}f(z)\\&=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1+k)}{\Gamma(p+n+1+k)\Gamma(p+1)}|a_{p+n}|z^{p+n}\end{aligned}$$

$$=z^p-\sum_{n=1}^{\infty}A(n)\,|a_{p+n}|\,z^{p+n},$$

where

$$A(n)=\frac{\Gamma(p+n+1)\Gamma(p+1+k)}{\Gamma(p+n+1+k)\Gamma(p+1)}\quad(n\geqslant 1).$$

Since

$$0<A(n)\leqslant A(1)=\frac{p+1}{p+1+k},$$

we have, with the help of Theorem 1,

$$|F(z)|\geqslant|z|^p-A(1)\,|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$
$$\geqslant|z|^p-\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{\{1+B+(B-A)(p-\alpha)\}}|z|^{p+1}$$

and

$$|F(z)|\leqslant|z|^p+A(1)\,|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$
$$\leqslant|z|^p+\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{\{1+B+(B-A)(p-\alpha)\}}|z|^{p+1}$$

which prove the inequalities of Theorem 10. Further, equalities are attained for the function

$$D_z^{-k}f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1+k)}z^{p+k}\cdot\left\{1-\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{\{1+B+(B-A)(p-\alpha)\}}z\right\},$$

or

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}z^{p+1}.$$

COROLLARY 5. *Under the hypotheses of Theorem* 10, $D_z^{-k}f(z)$ *is incl uded in the disc with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)}\left\{1+\frac{p+1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)}{\{1+B+(B-A)(p-\alpha)\}}\right\}.$$

Using Corollary 2, we have

THEOREM 11. *Let a function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *be in the class* $C_p(A, B, \alpha)$. *Then we have*

$$|D_z^{-k}f(z)| \geqslant \frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k} \cdot \left\{1-\frac{1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|\right\}$$

and

$$|D_z^{-k}f(z)| \leqslant \frac{\Gamma(p+1)}{\Gamma(p+1+k)}|z|^{p+k} \cdot \left\{1+\frac{1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|\right\}$$

for $0<k<1$ and $z\in U$. The result is sharp for the function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)p}{(p+1)\{1+B+(B-A)(p-\alpha)\}}z^{p+1}.$$

COROLLARY 6. *Under the conditions of Theorem* 11, $D_z^{-k}f(z)$ *is included in the disc with radiu*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)}\left\{1+\frac{1}{p+1+k}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}\right\}.$$

Finally, we derive

THEOREM 12. *Let a function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *be in the class* $C_p(A,B,\alpha)$. *Then we have*

$$|D_z^{k}f(z)| \geqslant \frac{\Gamma(p+1)}{\Gamma(p+1-k)}|z|^{p-k} \cdot \left\{1-\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|\right\}$$

and

$$|D_z^{k}f(z)| \leqslant \frac{\Gamma(p+1)}{\Gamma(p+1-k)}|z|^{p-k} \cdot \left\{1+\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|\right\}$$

for $0\leqslant k<1$ *and* $z\in U$. *The result is sharp.*

Proof. Let

$$G(z)=\frac{\Gamma(p+1-k)}{\Gamma(p+1)}z^k\ D_z^kf(z)$$
$$=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+n+1-k)\Gamma(p+1)}|a_{p+n}|z^{p+n}$$

$$=z^p-\sum_{n=1}^{\infty}(p+n)B(n)\,|a_{p+n}|\,z^{p+n},$$

where

$$B(n)=\frac{\Gamma(p+n)\Gamma(p+1-k)}{\Gamma(p+n+1-k)\Gamma(p+1)}\quad(n\geqslant 1).$$

Noting

$$0<B(n)\leqslant B(1)=\frac{1}{(p+1-k)},$$

with Corollary 2, we have

$$|G(z)|\geqslant|z|^p-B(1)\,|z|^{p+1}\sum_{n=1}^{\infty}(p+n)\,|a_{p+n}|$$
$$\geqslant|z|^p-\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|^{p+1}$$

and

$$|G(z)|\leqslant|z|^p+B(1)\,|z|^{p+1}\sum_{n=1}^{\infty}(p+n)\,|a_{p+n}|$$
$$\leqslant|z|^p+\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}|z|^{p+1}$$

which give the inequalities of Theorem 12. Since equalities are attained for the function $f(z)$ defined by

$$D_z^k f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1-k)}z^{p-k}\cdot\left\{1-\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}z\right\}$$

that is, by

$$f(z)=z^p-\frac{(B-A)(p-\alpha)p}{(p+1)\{1+B+(B-A)(p-\alpha)\}}z^{p+1},$$

we complete the assertion of theorem 12.

COROLLARY 7. *Under the conditions of Theorem* 12, $D_z^k f(z)$ *is included in the disc with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1-k)}\left\{1+\frac{1}{(p+1-k)}\cdot\frac{(B-A)(p-\alpha)p}{\{1+B+(B-A)(p-\alpha)\}}\right\}.$$

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