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ARITHMETIC OF INTEGRAL EVEN UNIMODULAR QUADRATIC FORMS OF 24 VARIABLES

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§1. Introduction

Let $A=(a_{ij})$ be a symmetric, positive definite and integral $n\times n$ matrix for which $a_{ii}\equiv 0 \bmod 2$ and $\det A=1$. We associate to A a quadratic form $A[X]={}^tXAX$, $X=(x_1, x_2, \ldots, x_n)$ which we call a positive definite integral even unimodular quadratic form in n variables. Then it turns out that $n\equiv 0 \bmod 8$ (Ch. V [9]). Let $Q(n,1)$ be the set of all such quadratic forms. Two forms $A[X]$ and $B[X]$ are equivalent (write $A[X]\sim B[X]$) if $B={}^tCAC$ for some C in $GL_n(\boldsymbol{Z})$. Set $\tilde{Q}(n,1)=Q(n,1)/\sim$. The cardinality $|\tilde{Q}(n,1)|$ is finite (Ch. II, §5[10]), so we speak of the class number $h(Q(n,1))=|\tilde{Q}(n,1)|$. Historically $h(Q(8,1))=1$([6]), $h(Q(16,1))=2$([12]) and $h(Q(24,1))=24$([7]). The class number $h(Q(n,1))$ has not been determined yet for $n\geqq 32$. Instead, we see that the class number grows remarkably fast (Ch. V [9]).

Among 24 classes, following Niemeier, the Leech lattice G_0 created by J. Leech ([4]) (also see Appendix 5 [5]) attracts particular interest. Let $A_{G_0}[X]$ be the quadratic form whose quadratic matrix A_{G_0} is constructed from G_0(see §4). Since $A_{G_0}[X]$ is even, let $r_{A_{G_0}}(m)$ be the number of integral solutions X in $\boldsymbol{Z}^{24}$ to the equation $A_{G_0}[X]=2m$, $m=1,2,3,\cdots$. As is well known (Theorem 7 [1]), $r_{A_{G_0}}(1)=0$ and $r_{A_{G_0}}(m)=0$ for some $m\geqq 1$. The aim of this paper is to improve Conway's result and extend it to the other lattices heavily due to Niemeier, which asserts:

THEOREM 1.1. *Let $\bar{A}_{G_0}[X]$ be the equivalent class of the quadratic form $A_{G_0}[X]$. If $A[X]\in Q(24,1)-\bar{A}_{G_0}[X]$,*

$$r_{A_{G_0}}(m)>0 \text{ for all } m\geqq 1.$$

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If $A[X]\in\bar{A}_{G_0}[X]$,

$$r_A(m)\begin{cases}>0 \text{ for all } m\geqq 2\\ =0 \text{ for } m=1.\end{cases}$$

To prove it in §6 we shall need Niemeier's classification (§4) and theta functions as modular forms (§5).

As byproduct we present in §7 perhaps another conjecture on Ramanujan numbers $\tau(m)$ (p. 98 [9]).

§2. Notations

H is upper half plane $\{z\in\boldsymbol{C}\mid \mathrm{Im}(z)>0\}$. $\Gamma(1)$ denotes the group $SL_2(\boldsymbol{Z})$. For an even integer $k(\geqq 2)$, $M_k(\Gamma(1))$ is the vector space over $\boldsymbol{C}$ of modular forms of weight k for $\Gamma(1)$ and $M_k^\circ(\Gamma(1))$ is the subspace of $M_k(\Gamma(1))$ consisting of cusp forms of weight k for $\Gamma(1)$. $\zeta(s)$ is the Riemann zeta function. $\sigma_k(n)$ is the divisor function defined by $\sum_{\substack{d\mid n\\ d>0}} d^k$. $[x]$ is the largest integer not exceeding x.

§3. Preliminaries

For $z\in H$ and $A[X]\in Q(n,1)$, we define a theta function θ_A associated with a quadratic form $A[X]$ by

$$\theta_A(z)=\sum_{X\in Z^n} e^{\pi i z A[X]}.$$

Then θ_A is analytic in H and $\theta_A\in M_{n/2}(\Gamma(1))$. On the other hand,

$$\theta_A=\theta_B \text{ for } A[X]\sim B[X]. \tag{1}$$

We recall that

$$M_k(\Gamma(1))=M_k^\circ(\Gamma(1))\oplus\boldsymbol{C}E_k, \text{ where}$$
$$E_k(z)=1+\frac{(-1)^{k/2}(2\pi)^k}{(k-1)!H(k)}\sum_{m=1}^{\infty}\sigma_{k-1}(m)e^{2\pi i m z} \tag{2}$$

and

$$\dim_{\boldsymbol{C}} M_k(\Gamma(1))=\begin{cases}[k/12], & k\equiv 2 \bmod 12\\ [k/12]+I, & \text{otherwise}\end{cases} \tag{3}$$
$$\dim_{\boldsymbol{C}} M_k^\circ(\Gamma(1))=\dim_{\boldsymbol{C}} M_k(\Gamma(1))-1,\quad k\geqq 4$$
$$\dim_{\boldsymbol{C}} M_2^\circ(\Gamma(1))=0.$$

§ 4. Niemeir's Classification

In this section we briefly sketch the idea of classifying positive definite integral even unimodular quadratic forms of 24 variables, and the main result.

On an n-dimensional vector space V over $\boldsymbol{Q}$, let a positive definite scalar prodct (x, y) be given. A $\boldsymbol{Z}$-submodule L of V of rank n is called an integral lattice if $(x, y) \in \boldsymbol{Z}$. Let $b_1, b_2, \ldots\ldots, b_n$ be a basis for an integral lattice L and let $q: L \to \boldsymbol{Z}$ be the quadratic map with respect to a basis $\{b_i\}$ defined by

$$q(x) = (x, x) = \sum_{i,j} (b_j, b_j) x_i x_j \qquad (4)$$

for $x = \sum_{i=1}^{n} x_i b_i$. An integral lattice L is called even if $q(L) \subset 2\boldsymbol{Z}$. The discriminant $d(L)$ of L is the determinant of the quadratic matrix $A_L = ((b_i, b_j))$. We call a lattice L unimodular if $d(L) = 1$. From now on by a lattice we shall mean an even unimodular lattice of dimension n endowed with a positive definite scalar product (x, y). Since the isomrophic lattices correspond to the equivalent quadratic forms, there is no harm to identify lattices with forms in what follows.

Let L be a lattice and x be a vector in L. We call (x, x) the (squared) length of x. A vector x of length 2 is called a minimal vector. The lattice M generated by all minimal vectors of L is a sublattice of L. We say that L is of type M when it exists. It is known ([2]) that every definite lattice can be uniquely written as an orthgonal sum of indecomposable sublattices. Therefore we search for the system of indecomposable sublattices M_i of M, from which we can get every system of minimal vectors of L. In this case we simply write $\bigoplus_i M_i$ instead of L. Up to isomorphism there are exactly the following lattices generated by minimal vectors ([11]):

Let $e_1, e_2, \ldots\ldots, e_k$ be the orthogonal unit vectors in a k-dimensional space.

$$A_n = \left\{ \sum_{i=1}^{n+1} x_i e_i \,|\, x_i \in \boldsymbol{Z},\ \sum_{i=1}^{n+1} x_i = 0 \right\},$$

$$D_n = \left\{ \sum_{i=1}^{n} x_i e_i \,|\, x_i \in \boldsymbol{Z},\ \sum_{i=1}^{n} x_i \equiv 0 \bmod 2 \right\} \quad (n \geq 4),$$

$$E_8 = \left\{ \sum_{i=1}^{8} x_i e_i \,|\, 2x_i \in \boldsymbol{Z},\ x_i - x_j \in \boldsymbol{Z},\ \sum_{i=1}^{8} x_i \equiv 0 \bmod 2 \right\},$$

$$E_7=\left\{\sum_{i=1}^{8} x_i e_i \in E_8 \mid \sum_{i=1}^{8} x_i=0\right\} \text{ and}$$
$$E_6=\left\{\sum_{i=1}^{8} x_i e_i \in E_8 \mid \sum_{i=1}^{8} x_i=x_7+x_8=0\right\}.$$

Due to Niemeier (Theorem 8. 3 [7]) we have

THEOREM 4. 1. *Up to isomorphism there are 24 classes of even unimodular lattices of dimension 24. They are characterized by the following configurations of minimal vectors*: $3\times E_8$, $E_8\oplus D_{16}$, $E_7\oplus E_7\oplus D_{10}$, $E_7\oplus A_{17}$, D_{24}, $D_{12}\oplus D_{12}$, $3\times D_8$, $D_9\oplus A_{15}$, $4\times E_6$, $E_6\oplus D_7\oplus A_{11}$, $4\times D_6$, $D_6\oplus A_9\oplus A_9$, $D_5\oplus D_5\oplus A_7\oplus A_7$, $3\times A_8$, A_{24}, $A_{12}\oplus A_{12}$, $6\times D_4$, $D_4\oplus (4\times A_5)$, $4\times A_6$, $6\times A_4$, $8\times A_3$, $12\times A_2$, $24\times A_1$ *and* G_0 *without minimal vectors* ($\oplus$ *orthogonal sum*, $n\times L=\underbrace{L\oplus L\oplus\cdots\cdots\oplus L}_{n \text{ copies}}$).

§ 5. Theta Functions in Case $n=24$

Unless otherwise specified we fix the dimension of a lattice to be 24 in what follows. If a lattice L is of type $\bigoplus_{i=1}^{r} M_i$, it is understood that

$$\theta_L(z)\ (=\theta_{\bigoplus_{i=1}^{r} M_i}(z)=\theta_{A_L}(z))=\sum_{X\in Z^{24}} e^{\pi i z A_L[X]}$$

where A_L is the quadratic matrix with respect to a basis $\{b_i\}$ of L.

By (1) and Theorem 4. 1 we have at most 24 theta functions (, in fact 19) as modular forms of weight 12 for $\Gamma(1)$. Observe that

$$\theta_L(z)=1+\sum_{m=1}^{\infty} r_L(m)\, e^{2\pi i m z} \tag{5}$$

where $r_L(m)=r_{A_L}(m)$ as defined in § 1. Let $F(z)=(2\pi)^{-12}\Delta(z)$, where $\Delta(z)$ is the modular discriminant. Then $F(z)=\sum_{m=1}^{\infty}\tau(m)e^{2\pi i m z}$ and the coefficients $\tau(m)$ are the Ramanujan numbers. If follows from (2) and (3) that

$$\theta_L(z)=E_{12}(z)+c_L F(z). \tag{6}$$

By (2), (5) and (6), for all $m\geqq 1$

$$r_L(m)=\frac{65520}{691}\sigma_{11}(m)+c_L\tau(m) \tag{7}$$

Putting $m=1$

$$c_L=r_L(1)-\frac{65520}{691}. \tag{8}$$

In view of (6) and (8) we should know the number of integral solutions $X \in \boldsymbol{Z}^{24}$ to the equation $A_L[X] = {}^tXA_L\,X = 2$ in order to completely determine the theta function θ_L associated with a lattice L. It is obvious, however, by (4) that $r_L(1) = m(L)$ where $m(L)$ is the number of minimal vectors of L. Let a lattice L be of type $\bigoplus_{i=1}^{k} M_i$.

LEMMA 5.1. $m(L) = \sum_{i=1}^{k} m(M_i)$.

Proof. It is immediate because any minimal vector from M_i can not be combined with another one from M_j $(i \neq j)$ to be a minimal vector in L.

Since $m(A_n) = n(n+1)$, $m(D_n) = 2n(n-1)$ $(n \geqq 4)$, $m(E_8) = 240$, $m(E_7) = 126$ and $m(E_6) = 72$, we obtain the following list for $r_L(1)$ with the aid of Theorem 4.1 and Lemma 5.1: Putting $r_L(1) = r_{\bigoplus_{i=1}^{k} M_i}(1)$

$$
\begin{aligned}
&r_{3\times E_8}(1) = 720, \quad r_{E_8\oplus D_{16}}(1) = 720,\\
&r_{E_7\oplus E_7\oplus D_{10}}(1) = 432, \quad r_{E_7\oplus A_{17}}(1) = 432,\\
&r_{D_{24}}(1) = 1104, \quad r_{D_{12}\oplus D_{12}}(1) = 528,\\
&r_{3\times D_8}(1) = 336, \quad r_{D_9\oplus A_{15}}(1) = 384,\\
&r_{4\times E_6}(1) = 288, \quad r_{E_6\oplus D_7\oplus A_{11}}(1) = 288,\\
&r_{4\times D_6}(1) = 240, \quad r_{D_6\oplus A_9\oplus A_9}(1) = 240,\\
&r_{D_5\oplus D_5\oplus A_7\oplus A_7}(1) = 192, \quad r_{3\times A_8}(1) = 216,\\
&r_{A_{24}}(1) = 600, \quad r_{A_{12}\oplus A_{12}}(1) = 312,\\
&r_{6\times D_4}(1) = 144, \quad r_{D_4\oplus(4\times A_5)}(1) = 144,\\
&r_{4\times A_6}(1) = 168, \quad r_{6\times A_4}(1) = 120,\\
&r_{8\times A_3}(1) = 96, \quad r_{12\times A_2}(1) = 72,\\
&r_{24\times A_1}(1) = 48, \quad r_{G_0}(1) = 0.
\end{aligned}
\tag{9}
$$

We easely get by (7) and (8)

LEMMA 5.2. *Let L and L' be two lattices in $\tilde{Q}(24, 1)$. Then $r_L(m) = r_{L'}(m)$ for all $m \geqq 1$ if and only if $r_L(1) = r_{L'}(1)$.*

Therefore by (5), (9) and Lemm 5.2 we come up with

THEOREM 5.3. *$\theta_{3\times E_8} = \theta_{E_8\oplus D_{16}}$, $\theta_{E_7\oplus E_7\oplus D_{10}} = \theta_{E_7\oplus A_{17}}$, $\theta_{4\times E_6} = \theta_{E_6\oplus D_7\oplus A_{11}}$, $\theta_{4\times D_6} = \theta_{D_6\oplus A_9\oplus A_9}$ and $\theta_{6\times D_4} = \theta_{D_4\oplus(4\times A_5)}$.*

COROLLARY 5.4. *For all $m \geqq 1$, $r_{3\times E_8}(m) = r_{E_8\oplus D_{16}}(m)$, $r_{E_7\oplus E_7\oplus D_{10}}(m) = r_{E_7\oplus A_{17}}(m)$, $r_{4\times E_6}(m) = r_{E_6\oplus D_7\oplus A_{11}}(m)$, $r_{4\times D_6}(m) = r_{D_6\oplus A_9\oplus A_9}(m)$ and $r_{6\times D_4}(m) = r_{D_4\oplus(4\times A_5)}(m)$.*

THEOREM 5.5. *If* $\theta_L \neq \theta_{L'}$ *for* $L, L' \in \tilde{Q}(24, 1)$ *then* θ_L *and* $\theta_{L'}$ *are linearly independent.*

proof. If they were linearly dependent, we would have

$$\alpha\theta_L + \beta\theta_{L'} = 0 \tag{10}$$

for $\alpha, \beta \in \boldsymbol{C}$, not both zero. It follows by (5) that

$$\alpha = -\beta \tag{11}$$

$$\alpha r_L(1) = -\beta r_{L'}(1). \tag{12}$$

If $r_{L'}(1)=0$, then $r_L(1)=0$ as well, whence $r_L(1)=r_{L'}(1)$.
By Lemma 5.2 $\theta_L = \theta_{L'}$, which is impossible.
If $r_{L'}(1) \neq 0$, by (11) and (12)

$$1 = -\beta/\alpha = r_L(1)/r_{L'}(1),$$

and so $r_L(1) = r_{L'}(1)$. Again $\theta_L = \theta_{L'}$, impossible.

PROPOSITION 5.6. $\tau(m)=0$ *for some integer* m $(>10^{15})$ *if and only if* $r_L(m) = r_{L'}(m)$ *for any two lattices* L, L' *such that* $\theta_L \neq \theta_{L'}$.

Proof. If $\tau(m)=0$, the statement follows from (7). Conversely, assume that there is an integer $m > 10^{15}$ such that $r_L(m) = r_{L'}(m)$ for any two lattices L, L' with $\theta_L \neq \theta_{L'}$. Then

$$\begin{aligned} 0 &= r_L(m) - r_{L'}(m) \\ &= \tau(m)(c_L - c_{L'}). \end{aligned}$$

However $c_L \neq c_{L'}$ by (6), whence $\tau(m)=0$.

§6. Proof of Theorem 1.1

Let $A[X] \in Q(24, 1) - \bar{A}_{G_0}[X]$. Then $r_A(1) > 0$ because of (1) and the list (9). In like manner, $r_A(1)=0$ for $A[X] \in \bar{A}_{G_0}[X]$. To complete the proof it remains to show that $r_A(m) > 0$ for all $m \geqq 2$ and for any $A[X] \in Q(24, 1)$, or equivalently $r_L(m) > 0$ for all $m \geqq 2$ and for any lattice L in $\tilde{Q}(24, 1)$. To this end we need two elementary lemmas.

LEMMA 6.1. *Let* p *be a prime number. Then* $e+1 \leqq p^e$ *for all integers* $e \geqq 1$.

Proof. It is obvious by induction on e.

LEMMA 6.2. *If* m *is a positive integer, then* $\sigma_0(m) \leqq m$.

Proof. Let $m=p_1^{e_1}p_2^{e_2}\ldots\ldots p_k^{e_k}$ with $e_j\geqq 1$. Since $\sigma_0(m)=\prod_{j=1}^{k}(e_j+1)$, it follows from Lemma 6. 1.

Suppose $r_L(m)=0$ for some $m\geqq 2$. Since the corresponding theta function θ_L is a modular form of weight 12 for $\Gamma(1)$,

$$r_L(m)=\frac{65520}{691}\sigma_{11}(m)+c_L\tau(m)$$

with $c_L\in\boldsymbol{Q}$. By the assumption we obtain that

$$|c_L|\,|\tau(m)|=\frac{65520}{691}\sigma_{11}(m). \tag{14}$$

As is well known (p. 104[9]) the function $F(z)=\sum_{m=1}^{\infty}\tau(m)e^{2\pi imz}$, as a cusp form of weight 12 for $\Gamma(1)$, is a normalizd eigen function of all the Hecke operators $T(n)$. Thus ([8], [9])

$$|\tau(m)|\leqq m^{11/2}\sigma_0(m). \tag{15}$$

It then follows from (14) and (15) that

$$\frac{65520}{691}\sigma_{11}(m)\leqq|c_L|m^{11/2}\sigma_0(m).$$

By appendix A, we get

$$\frac{65520}{691}\sigma_{11}(m)\leqq\frac{697344}{691}m^{11/2}\sigma_0(m),\quad\text{or}$$
$$\sigma_{11}(m)<11m^{11/2}\sigma_0(m).$$

Since $\sigma_{11}(m)>m^{11}$, $m^{11}<11\sigma_0(m)m^{11/2}$, or $m^{11/2}<11\sigma_0(m)$.
Hence $m^5<11\sigma_0(m)$. By Lemma 6. 2, $\sigma_0(m)^5<11\sigma_0(m)$ so that $\sigma_0(m)^4<11$. Since $\sigma_0(m)\geqq 2$, $16<11$ impossible.

§ 7. Remarks on Ramanujan Numbers

In this section we shall express $r_L(m)$ in terms of the divisor functions $\sigma_k(m)$ only, which might be useful in the study of Ramanujan numbers $\tau(m)$

Recall from (7) that

$$r_L(m)=\frac{65520}{691}\sigma_{11}(m)+c_L\tau(m).$$

It is known (p. 55[3]) that

$$\tau(m)=\frac{65}{756}\sigma_{11}(m)+\frac{691}{756}\sigma_5(m)-\frac{691}{3}\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k),$$

from which we have

$$r_L(m)=\left(\frac{65520}{691}+c_L\frac{65}{756}\right)\sigma_{11}(m)+c_L\frac{691}{756}\sigma_5(m)$$
$$-c_L\frac{691}{3}\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k). \tag{16}$$

See appendix B for the list of $r_L(m)$. We close this section with the following open question: Can we find two lattices L and L' in $\tilde{Q}(24,1)$ such that $r_L(m)\neq r_{L'}(m)$ for all $m\geqq 1$?

If it is true, we can readily prove D. H. Lehmer's conjecture on $\tau(m)$:

$$\tau(m)\neq 0 \text{ for all } m\geqq 1.$$

The reason is as follows: Suppose $\tau(m)=0$ for some integer m. We may asume $m>10^{15}$ because it is known that $\tau(m)\neq 0$ for all $m\leqq 10^{15}$. Then by Proposition 5. 6 $r_L(m)=r_{L'}(m)$ for any two lattices L and L' in $\tilde{Q}(24,1)$ such that $\theta_L\neq\theta_{L'}$. It contradicts the above statement. Theorem 5. 3 asserts that there are exactly 19 distinct theta functions, and so we may work with concrete examples to approach the problem. For instance choose L and L' to be of types $12\times A_2$ and $24\times A_1$, respectively. Then we must show that for all integers $m>10^{15}$

$$\frac{650}{7}\sigma_{11}(m)-\frac{146}{7}\sigma_5(m)+5256\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)\neq$$
$$\frac{5720}{63}\sigma_{11}(m)-\frac{2696}{63}\sigma_5(m)+10784\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k), \text{ or}$$
$$\frac{130}{63}\sigma_{11}(m)+\frac{1382}{63}\sigma_5(m)-5528\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(m)\neq 0.$$

It seems that for large integer m the orders of magnitudes of $\sigma_{11}(m)$ and $\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$ are the same, and hece the middle term can be neglected. Therefore if it is the right choice of ${}_{19}C_2=171$ ones, the problem reduces to the question: Can we find mutually disjoint narrow boundaries of $\frac{130}{63}\sigma_{11}(m)$ and $5528\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$ for every large integer $m(>10^{15})$?

Appendix A

By (8) and (9) we have the following list for c_L: putting $c_L=c_{\overset{k}{\underset{i=1}{\oplus}}M_i}$

$$c_{3\times E_8}=c_{E_8\oplus D_{16}}=\frac{432060}{691}$$

$$c_{E_7\oplus E_7\oplus D_{10}}=c_{E_7\oplus A_{17}}=\frac{232992}{691}$$

$$c_{D_{24}}=\frac{697344}{691}$$

$$c_{D_{12}\oplus D_{12}}=\frac{299328}{691}$$

$$c_{3\times D_8}=\frac{166656}{691}$$

$$c_{D_9\oplus A_{15}}=\frac{199824}{691}$$

$$c_{4\times E_6}=c_{E_6\oplus D_7\oplus A_{11}}=\frac{133488}{691}$$

$$c_{4\times D_6}=c_{D_6\oplus A_9\oplus A_9}=\frac{100320}{691}$$

$$c_{D_5\oplus D_5\oplus A_7\oplus A_7}=\frac{67152}{691}$$

$$c_{3\times A_8}=\frac{83736}{691}$$

$$c_{A_{24}}=\frac{349080}{691}$$

$$c_{A_{12}\oplus A_{12}}=\frac{150072}{691}$$

$$c_{6\times D_4}=c_{D_4\oplus(4\times A_5)}=\frac{33984}{691}$$

$$c_{4\times A_6}=\frac{17400}{691}$$

$$c_{8\times A_3}=\frac{816}{691}$$

$$c_{12\times A_2}=-\frac{15768}{691}$$

$$c_{24\times A_1}=-\frac{32352}{691}$$

$$c_{G_0}=-\frac{65520}{691}$$

Appendix B

By (16) and appendix A we obtain the following list for $r_L(m)$: Putting

$r_L(m)=r_{\overset{k}{\underset{i=1}{\oplus}}M_i}$,

$$r_{3\times E_8}(m)=r_{E_8\oplus D_{16}}(m)=\frac{1040}{7}\sigma_{11}(m)+\frac{4000}{7}\sigma_5(m)-144000\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{E_7\oplus E_7\oplus D_{10}}(m)=r_{E_7\oplus A_{17}}(m)=\frac{2600}{21}\sigma_{11}(w)+\frac{6472}{21}\sigma_5(m)-77664\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{D24}(m)=\frac{11440}{63}\sigma_{11}(m)+\frac{58112}{63}\sigma_5(m)-232448\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{D_{12}\oplus D_{12}}(m)=\frac{8320}{63}\sigma_{11}(m)+\frac{24944}{63}\sigma_5(m)-99776\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{3\times D_8}(m)=\frac{1040}{9}\sigma_{11}(m)+\frac{1984}{9}\sigma_5(m)-55552\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{D_9\oplus A_{15}}(m)=\frac{7540}{63}\sigma_{11}(m)+\frac{16652}{63}\sigma_5(m)-66608\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{4\times E_6}(m)=r_{E_6\oplus D_7\oplus A_{11}}(m)=\frac{780}{7}\sigma_{11}(m)+\frac{1236}{7}\sigma_5(m)-44496\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{4\times D_6}(m)=r_{D_6\oplus A_9\oplus A_9}(m)=\frac{6760}{63}\sigma_{11}(m)+\frac{8360}{63}\sigma_5(m)-33440\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{D_5\oplus D_5\oplus A_7\oplus A_7}(m)=\frac{5500}{63}\sigma_{11}(m)+\frac{5596}{63}\sigma_5(m)-22384\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{3\times A_8}(m)=\frac{2210}{21}\sigma_{11}(m)+\frac{2326}{21}\sigma_5(m)-27912\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{A_{24}}(m)=\frac{8710}{63}\sigma_{11}(m)+\frac{29090}{63}\sigma_5(m)-116360\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{A_{12}\oplus A_{12}}(m)=\frac{7150}{63}\sigma_{11}(m)+\frac{12506}{63}\sigma_5(m)-50024\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{6\times D_4}(m)=r_{D_4\oplus(4\times A_5)}(m)=\frac{2080}{21}\sigma_{11}(m)+\frac{944}{21}\sigma_5(m)-11328\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{4\times A_6}(m)=\frac{910}{9}\sigma_{11}(m)+\frac{602}{9}\sigma_5(m)-16856\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{6\times A_4}(m)=\frac{6110}{63}\sigma_{11}(m)+\frac{1450}{63}\sigma_5(m)-5800\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{8\times A_3}(m)=\frac{5980}{63}\sigma_{11}(m)+\frac{68}{63}\sigma_5(m)-272\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{12\times A_2}(m)=\frac{650}{7}\sigma_{11}(m)-\frac{146}{7}\sigma_5(m)+5256\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{24\times A_1}(m)=\frac{5720}{63}\sigma_{11}(m)-\frac{2696}{63}\sigma_5(m)+10784\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

$$r_{G_0}(m)=\frac{260}{3}\sigma_{11}(m)-\frac{260}{3}\sigma_5(m)+21840\sum_{k=1}^{m-1}\sigma_5(m-k)\sigma_5(k)$$

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