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ON COUNTABLY APPROXIMATING LATTICES

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1. Introduction

In 1972, Scott has [16] introduced a concept of continuous lattices and has shown the equivalence between continuous lattices and injective T_0-spaces. We recall that an algebraic lattice is just a sublattice of a power of the two point chain with respect to the arbitrary meets and directed joins. Lawson has [14] shown that continuous lattices are precisely sublattices of a power of the unit interval under the same operations as above.

There have been many efforts made to generalize continuous lattices and extend corresponding properties to them, notably in [7 ; 11 ; 12 ; 15].

The purpose to write this paper is to introduce another class of lattices, namely countably approximating lattices, generalizing continuous lattices and study its basic properties. To do so, we introduce the countably way below relation and show that it satisfies the strong interpolation property on a countably approximating lattice. We show that a regular space X is a locally Lindelöf space iff its open set lattice $\Omega(X)$ is a countably approximating lattice and that a complete lattice L is countably approximating iff the join map on the lattice of σ-ideals of L preserves arbitrary meets and joins. Using these, it is shown that countably approximating lattices are productive, hereditary and closed under homomorphic images.

Finally we show that a complete lattice is countably approximating iff it is a homomorphic image of a countably algebraic lattice.

For the terminology not introduced in the paper, we refer to [6].

2. Countably Approximating Lattices

Throughout this paper, a partially ordered set is called a poset.

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DEFINITION 2.1. Let L be a poset and D a subset of L. Then we say:

(1) D is *countably directed* if every countable subset of D has an upper bound in D.

(2) D is a *σ-ideal* if it is a countably directed lower set.

EXAMPLE 2.2. Let 2^X denote the power set lattice of a set X, i.e., the power set endowed with the inclusion relation. Then the set of all finite subsets of X, denoted by Fin X, is an ideal of 2^X and the set of all countable subsets of X, denoted by Count X, is a σ-ideal of 2^X. We note that Fin X is not countably directed if X is infinite.

NOTATION 2.3. For a poset L, the set of all ideals of L will be denoted by IdL and the set of all σ-ideals by σIdL.

REMARK 2.4. For any poset L with the smallest element, σIdL is closed under countably directed unions and arbitrary intersections.

DEFINITION 2.5. Let x and y be elements of a complete lattice L. We say that x is *countably way below* y, in symbols $x \ll_c y$, if for any countably directed subset D of L with $y \leq \bigvee D$, there is an element $d \in D$ with $x \leq d$. If $x \ll_c x$, then x is said to be a *Lindelöf element*

PROPOSITION 2.6. *Let L be a complete lattice. For $x, y \in L$, the following are equivalent*:

(1) $x \ll_c y$.

(2) *If $y \leq \bigvee X$ $(X \subseteq L)$, then there is a countable subset K of X with $x \leq \bigvee K$.*

(3) *If I is a σ-ideal of L and $y \leq \bigvee I$. then $x \in I$.*

Proof. (1) $\Longrightarrow$ (2). Let $D = \{\bigvee K \mid K \in \text{Count } X\}$, then D is countably directed and $\bigvee D = \bigvee X$. Since $x \ll_c y$, there is an element $d \in D$ with $x \leq d$, which is of the form $d = \bigvee K$ for a countable subset K of X.

(2) $\Longrightarrow$ (3). Suppose I is a σ-ideal of L with $y \leq \bigvee I$, then by the assumption, there is a countable subset K of I with $x \leq \bigvee K$. Since $\bigvee K \in I = \downarrow I$, $x \in I$.

(3) $\Longrightarrow$ (1). Let D be a countably directed subset of L with $y \leq \bigvee D$. Define $I = \downarrow D$, then I is a σ-ideal and $\bigvee I = \bigvee D$. By the assumption, $x \in I$, so that there is an element $d \in D$ with $x \leq d$.

EXAMPLE 2.7. (1) Let X be a set and L the power set lattice 2^X of X. Then for $A, B \in L$, $A \ll_c B$ iff there is a countable subset K of X with $A \subseteq K \subseteq B$.

(2) Let L be the lattice IdA of ideals of a ring A. For any $I, J \in L$, $I \ll_c J$ iff $I \subseteq K \subseteq J$ for some ideal K of A generated by a countable subset.

(3) If L is the unit interval $[0, 1]$ with the usual order $\leq$, then $x \ll_c x$ for all $x \in [0, 1]$.

Definition 2.8. A topological space X is said to be a *locally Lindelöf space* if each point of X has a local base consisting of Lindelöf subsets of X.

Clearly, every locally compact space is locally Lindelöf. But the rational line with the usual topology is locally Lindelöf but not locally compact.

Proposition 2.9. *In the open set lattice $\Omega(X)$ of a topological space X, $A \ll_c B$ if there is a Lindelöf set U on X with $A \subseteq U \subseteq B$.*

Furthermore, if X is a locally Lindelöf space, then the converse also holds.

Proof. Suppose that there is a Lindelöf subspace U of X with $A \subseteq U \subseteq B$. Take any open cover $\mathcal{Q}$ of B, then $\mathcal{Q}$ also covers U. Since U is Lindelöf, there is a countable subset $\mathcal{Q}'$ of $\mathcal{Q}$ which covers U. For $A \subseteq U$, $\mathcal{Q}'$ is a countable subset of $\mathcal{Q}$ which covers A. Thus $A \ll_c B$.

Now let X be a locally Lindelöf space. If $A \ll_c B$, then for any $b \in B$, there is a Lindelöf neighborhood Q_b of b with $Q_b \subseteq B$. It follows that $B = \cup \{\mathring{Q}_b \mid b \in B\}$ where $\mathring{Q}_b$ is the interior of Q_b. Since $A \ll_c B$, there is a countable subset $\{b_n \mid n \in N\}$ of B such that $A \subseteq \cup \{\mathring{Q}_{b_n} \mid n \in N\}$. Let $U = \cup \{Q_{b_n} \mid n \in N\}$, then U is a Lindelöf subspace of X and $A \subseteq U \subseteq B$. This completes the proof.

Using the exactly same arguments as those in the proof of Proposition I-1.2 in [6], one has,

Proposition 2.10. *In a complete lattice L, one has the following:*

(1) *If $x \ll_c y$, then $x \leq y$.* $(x, y \in L)$

(2) *If $u \leq x \ll_c y \leq v$, then $u \ll_c v$.* $(x, y, u, v \in L)$

(3) *For any sequence (x_n) in L such that $x_n \ll_c y$ for all $n \in N$, $\bigvee \{x_n \mid n \in N\} \ll_c y$.*

(4) $0 \ll_c x$. $(x \in L)$

(5) *If $x \ll y$, then $x \ll_c y$.* $(x, y \in L)$

Remark 2.11. (1) The converse of (5) in the above proposition need not be true. Indeed, let L be the power set lattice 2^X of an infinite set X. For an infinitely countable subset Y of X, $Y \ll_c Y$ but not $Y \ll Y$.

(2) For a complete lattice L and $x \in L$, let $\Downarrow_c x = \{y \in L \mid y \ll_c x\}$. Then by (2), (3) and (4) of the above proposition, $\Downarrow_c x$ is a σ-ideal of L, which is clearly contained in $\downarrow x$. Moreover if $x \leq y$, then $\Downarrow_c x \subseteq \Downarrow_c y$.

(3) Let $(L_i)_{i \in I}$ be a family of complete lattices and $\Pi_{i \in I} L_i$ the product of the family. Then $x \ll_c y$ in $\Pi_{i \in I} L_i$ iff $x_i \ll_c y_i$ for all $i \in I$ and $x_i = 0$ for all but a countable number of indices i, where $x = (x_i)$ and $y = (y_i)$.

Using the countably way below relation, we now introduce a new concept of countably approximating lattices as a generalization of continuous lattices.

Definition 2.12. A complete lattice L is said to be a *countably approximating lattice* if

$$x = \bigvee \{u \in L \mid u \ll_c x\}$$

for all $x \in L$, equivalently $x = \bigvee \Downarrow_c x$.

Proposition 2.13. *For a complete lattice L, the following are equivalent:*

(1) *L is countably approximating.*

(2) *If $x \not\leq y$ in L, then there is an element $u \in L$ with $u \ll_c x$ but $u \not\leq y$.*

The above proposition amounts to saying that every element in a countably approximating lattice can be sufficiently well approximated by elements countably way below it, and the order structure in that is completely determined by the countably way below relation.

Remark 2.14. In a countably approximating lattice L, we have:

$$x \leq y \text{ in } L \text{ iff } \Downarrow_c x \subseteq \Downarrow_c y.$$

Proof. If $x \leq y$, then by (2) of Proposition 2.10, $\Downarrow_c x \subseteq \Downarrow_c y$. Conversely, assume $\Downarrow_c x \subseteq \Downarrow_c y$, then

$$x = \bigvee \Downarrow_c x \leq \bigvee \Downarrow_c y = y,$$

for L is a countably approximating lattice.

Example 2.15. (1) Every continuous lattice is countably approximating.

(2) Every countable complete lattice is countably approximating.

(3) By Proposition 2.9, the open set lattice $\Omega(X)$ of a locally

Lindelöf space X is countably approximating.

(4) countably approximating lattice need not be a continuous lattice. Indeed, the open set lattice $\Omega(Q)$ of the rational line Q with the usual topology is countably approximating but it is not continuous for Q is regular but not locally compact.

THEOREM 2.16. *If a topological space X is regular and the open set lattice $\Omega(X)$ is countably approximating, then X is locally Lindelöf.*

Proof. Take any $x \in X$ and let V be any open neighborhood of x. Since $\Omega(X)$ is countably approximating, there is an open subset U of X with $x \in U \ll_c V$. Since X is regular, there is an open neighborhood W of x with $\overline{W} \subseteq U$. Now let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $\overline{W}$, then $\mathcal{G} \cup \{X - \overline{W}\}$ is an open cover of V. Since $U \ll_c V$, a countable subset of $\mathcal{G} \cup \{X - \overline{W}\}$ covers U, and hence there is a countable subfamily of $\mathcal{G}$ which covers $\overline{W}$. Thus $\overline{W}$ is a Lindelöf neighborhood of x contained in V.

EXAMPLE 2.17. Let S denote the space of real numbers endowed with the topology generated by $\{[a, b] \mid a < b, a, b \in R\}$. Then it is known that S is a Lindelöf space. Let X be the product space $S \times S$, then clearly X is regular and for any $(x_0, y_0) \in X$, $\{[x_0, x_0+\varepsilon] \times [y_0, y_0+\varepsilon] \mid \varepsilon > 0\}$ is a local base at (x_0, y_0). Noting that for any $\varepsilon > 0$, $[x_0, x_0+\varepsilon] \times [y_0, y_0+\varepsilon]$ is a closed and open neighborhood of (x_0, y_0) and contains uncountable discrete closed subspace, we conclude that X is not a locally Lindelöf space. Thus by the above theorem, the open set lattice $\Omega(X)$ is not countably approximating but a complete Heyting algebra.

The concept of an auxiliary relation has been introduced by [17]. Similarly, we introduce a concept of countably auxiliary relation and using that, we characterize countably approximating lattices.

DEFINITION 2.18. (1) A binary relation $\prec$ on a complete lattice L is said to be a *conutably auxiliary relation* on L it satisfies the following:

(i) If $x \prec y$, then $x \leq y$.

(ii) If $u \leq x \prec y \leq z$, then $u \prec z$.

(iii) If $x_n \prec z$ for all $n \in N$, then $\bigvee \{x_n \mid n \in N\} \prec z$.

(iv) $0 \prec x$ for all $x \in L$.

(2) A countably auxiliary relation $\prec$ on a complete lattice L is said to be *approximating* if we have $x = \bigvee \{u \in L \mid u \prec x\}$ for all $x \in L$.

Clearly every countably auxiliary relation is antisymmetric and transitive by (1) and (2), and the countably way below relation is a countably auxiliary relation. Moreover, every countably auxiliary relation is an auxiliary relation, but the converse need not be true. Indeed, in the chain $[0, 1]$, if we define $x \prec y$ iff $x=0$ or $x<y$, then the relation $\prec$ is not a countably auxiliary relation but an auxiliary relation.

The relation $\leq$ is trivially approximating, and a complete lattice is a countably approximating lattice iff the relation $\ll_c$ is approximating.

REMARK 2.19. (1) For any countably auxiliary relation $\prec$ on a complete lattice L, it is immediate from the definition that for any $x \in L$, $\{y \in L \mid y \prec x\}$ is a σ-ideal of L.

(2) In a complete lattice L and $x \in L$, we have: $\Downarrow_c x = \cap \{I \in \sigma \mathrm{IdL} \mid x \leq \bigvee I\}$, for $y \in \Downarrow_c x$ iff $y \ll_c x$ iff $y \in I$ for all $I \in \sigma\mathrm{IdL}$ with $x \leq \bigvee I$.

(3) In a complete lattice L, the relation $\prec$ is approximating iff whenever $x \not\leq y$ in L, there is an element $u \in L$ with $u \prec x$ but $u \not\leq y$.

THEOREM 2.20. *Let L be a complete lattice. Then the following are equivalent*:

(1) *L is a countably approximating lattice.*

(2) *$\ll_c$ is the smallest approximating countably auxiliary relation on L.*

Proof. Suppose L is countably approximating, then clearly $\ll_c$ is an approximating relation on L. Take any approximating relation $\prec$ on L. Then by Remark 2.19, for any $x \in L$, $\{y \in L \mid y \prec x\}$ is a σ-ideal and since $\prec$ is approximating, $x = \bigwedge \{y \in L \mid y \prec x\}$. Hence by Remark 2.19, $\Downarrow_c x \subseteq \{y \in L \mid y \prec x\}$. Therefore $y \ll_c x$ implies $y \prec x$. Thus the relation $\ll_c$ is contained in the relation $\prec$. Hence $\ll_c$ is the smallest approximating countably auxiliary relation. The converse is trivial.

By Theorem I-1.18 in [6], the way below relation $\ll$ on a continuous lattice satisfies the strong interpolation property.

THEOREM 2.21. *Let L be a countably approximating lattice. Then one has*:

If $x \ll_c z$ and $x \neq z$ in L, then there is an element $y \in L$ with $x \ll_c y \ll_c z$ but $x \neq y$.

Proof. Let $I=\{u\in L \mid u\ll_c y\ll_c z$ some $y\in L\}$. First we claim that I is a σ-ideal. Clearly I is a lower set. If $u_n\in I$ for all $n\in N$, then there are elements $y_n\in L$ with $u_n\ll_c y_n\ll_c z$ for all $n\in N$, and hence $\bigvee u_n\ll_c\bigvee y_n\ll_c z$. Thus $\bigvee u_n\in I$.

Next we claim that $\bigvee I=z$.
Let $\bigvee I=z^*$ and suppose that $z^*\neq z$. Since $\ll_c$ is approximating, $z=\bigvee\{y\in L \mid y\ll_c z\}$. Thus there is an element $y\in L$ with $y\ll_c z$ but $y\not\leq z^*$, for $z\not\leq z^*$. Since $y=\bigvee\{u\in L \mid u\ll_c y\}$, there is an element $u\in L$ with $u\ll_c y$ but $u\not\leq z^*$. But because $u\ll_c y\ll_c z$, $u\in I$ and hence $u\leq\bigvee I=z^*$. This leads to a contradiction. Therefore $z^*=z$.

Since $x\ll_c z=\bigvee I$ and I is a σ-ideal, $x\in I$. Thus there is an element $y^*\in L$ with $x\ll_c y^*\ll_c z$. Since $z=\bigvee\{y\in L \mid y\ll_c z\}$ and $z\not\leq x$, there is an element $y^{**}\in L$ with $y^{**}\ll_c z$ but $y^{**}\not\leq x$. Now let $y=y^*\vee y^{**}$, then clearly one has $x\ll_c y\ll_c z$ and $x\neq y$. This completes the proof.

The following is immediate from the above theorem.

COROLLARY 2.22. *In a countably approximating lattice L, if $x\ll_c z$, then there is an element $y\in L$ with $x\ll_c y\ll_c z$.*

3. Permanence Properties of Countably Approximating Lattices

In this section, we show that countably approximating lattices can be described by equations and that countably approximating lattices are productive, hereditary and closed under maps preserving arbitrary meets and countably directed joins, which are called homomorphisms. Furthermore, countably approximating lattices are precisely homomorphic images of countably algebraic lattices.

Let L be a complete lattice, $r:\sigma\mathrm{Id}L\to L$ the map defined by $r(I)=\bigvee I$, and $p:L\to\sigma\mathrm{Id}L$ the map defined by $p(x)=\downarrow x$. Then one has immediately the following:

LEMMA 3.1. *The map $r:\sigma\ IdL\to L$ is a left adjoint of the map $p: L\to\sigma\ IdL$. In particular, r preserves joins.*

Using the above lemma, we characterize countably approximating lattices via σ-ideals.

THEOREM 3.2. *Let L be a complete lattice. Then the following are equivalent*:

(1) *L is a countably approximating lattice.*

(2) *For each* $x \in L$, *the set* $\Downarrow_c x$ *is the smallest* σ*-ideal* I *with* $x \leq \bigvee I$.

(3) *For each* $x \in L$, *there is a smallest* σ*-ideal* I *with* $x \leq \bigvee I$.

(4) *The map* $r : \sigma\, IdL \to L$ *defined by* $r(I) = \bigvee I$ $(I \in \sigma\, IdL)$, *has a left adjoint.*

(5) *The map* $r : \sigma IdL \to L$ *preserves arbitrary meets and joins.*

Proof. We recall that for any $x \in L$, $\Downarrow_c x = \cap J(x)$ where $J(x) = \{I \in \sigma\mathrm{IdL} \mid x \leq \bigvee I\}$.

$(1) \Longrightarrow (2)$. Since $x = \bigvee \Downarrow_c x$, $\Downarrow_c x$ belongs to $J(x)$, and hence $\Downarrow_c x$ is the smallest σ-ideal with $x \leq \bigvee I$.

$(2) \Longrightarrow (3)$. Trivial.

$(3) \Longrightarrow (1)$. Take any $x \in L$ and let I_0 be the smallest element of $J(x)$, then $x \leq \bigvee I_0 \leq \bigvee(\cap J(x))$.
Thus $\Downarrow_c x = \cap J(x) \in J(x)$. Therefore $x \leq \bigvee \Downarrow_c x \leq x$, so that $x = \bigvee \Downarrow_c x$.

$(3) \Longleftrightarrow (4)$. The map r has a left adjoint iff $\min r^{-1}(\uparrow x)$ exists for all $x \in L$.
Since $r^{-1}(\uparrow x) = \{I \in \sigma\mathrm{IdL} \mid r(I) = \bigvee I \geq x\} = J(x)$, $\min r^{-1}(\uparrow x) = \min J(x)$, and therefore, one has the equivalence between (3) and (4).

$(4) \Longrightarrow (5)$. Since the map r has a left adjoint, r preserves arbitrary meets. By the above lemma, r has a right adjoint, and hence r preserves arbitrary joins.

$(5) \Longrightarrow (4)$. Because σIdL is a complete lattice and r preserves arbitrary meets, r has a left adjoint.

By (5) of the above theorem, every countably approximating lattice L is the image of the lattice σIdL under a map preserving meets and joins. But by Remark 2.4, σIdL is closed under intersections and countably directed unions, and hence any equations that these operations, i.e., arbitrary meets and countably directed joins satisfy, will transfer to L. Using these observations and the same arguments as those in the proof of Theorem I-2.3 in [6], we have the following.

THEOREM 3.3. *For a complete lattice* L, *the following are equivalent*:

(1) L *is countably approximating.*

(2) *Let* $\{x_{j,k} \mid j \in J, k \in K(j)\}$ *be a family of elements in* L *such that for all* $j \in J$, $\{x_{j,k} \mid k \in K(j)\}$ *is countably directed, and* $M = \Pi_{j \in J} K(j)$. *Then the following holds*:

$$\begin{aligned} &\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} \\ &= \bigvee_{f \in M} \bigwedge_{j \in J} x_{j, f(j)} \end{aligned} \qquad \cdots\cdots\cdots (a)$$

(3) *For any family* $\{x_{j,k} \mid (j, k) \in J \times K\}$ *of elements in* L *such that* $\{x_{j,k} \mid k \in K\}$ *is countably directed for all* $j \in J$, *the following holds*:

$$\bigwedge_{j\in J}\bigvee_{k\in K} x_{j,k} = \bigvee_{f\in K^J}\bigwedge_{j\in J} x_{j,f(j)} \quad \cdots\cdots\cdots (b)$$

(4) *Let* $\{x_{j,j} \mid (j, k) \in J \times K\}$ *be a family of elements in* L *and* $F = (Countk)^J$. *Then the following holds*:

$$\bigwedge_{j\in J}\bigvee_{k\in K} x_{j,k} = \bigvee_{f\in F}\bigwedge_{j\in J}\bigvee_{k\in f(j)} x_{j,k} \quad \cdots\cdots\cdots (c)$$

Note that all the joins in (a) and (b) are countably directed joins. Strictly speaking, (a) and (b) are not equations because its validity requires the assumption that certain sets are countably directed. On the other hand, (c) is a pure lattice equation in meets and joins.

The above theorem motivates the following definition of structure preserving maps between countably approximating lattices.

Definition 3.4. If S and T are countably approximating lattices, then a map $g: S \to T$ is said to be a *homomorphism*, if it preserves arbitrary meets and countably directed joins.

It is clear that the identity map of a countably approximating lattice is a homomorphism and that the composition of two homomorphisms is again a homomorphism. Thus the class of all countably approximating lattices and homomorphisms between them forms a category which will be denoted by <u>*CALat*</u>.

Proposition 3.5. *The category* <u>*CALat*</u> *has products.*

Proof. For any family $(A_i)_{i\in I}$ in <u>*CALat*</u>, let $(\Pi A_i, (\pi_i)_{i\in J})$ be the usual product of complete lattices.

Take any family $\{x_{j,k} \mid (j, k) \in J \times K\}$ of elements of ΠA_i such that for each $j \in J$, $\{x_{j,k} \mid k \in K\}$ is countably directed. For any $i \in I$,

$$\pi_i(\bigwedge_{j\in J}\bigvee_{k\in K} x_{j,k}) = \bigwedge_{j\in J}\bigvee_{k\in K}\pi_i(x_{j,k})$$

Since $\{\pi_i(x_{j,k}) \mid k \in K\}$ is countably directed and A_i is countably approximating, by the above theorem, one has,

$$\bigwedge_{j\in J}\bigvee_{k\in K}\pi_i(x_{j,k}) = \bigvee_{f\in K^J}\bigwedge_{j\in J}\pi_i(x_{j,f(j)}) = \pi_i(\bigvee_{f\in K^J}\bigwedge_{j\in J} x_{j,f(j)}).$$

Using the above two equations together with the fact that $(\pi_i)_{i\in I}$

separates points, one has

$$\bigwedge_{j\in J}\bigvee_{k\in K}x_{j,k}$$
$$=\bigvee_{f\in K^J}\bigwedge_{j\in J}x_{j,f(j)}$$

Once again by the above theorem, ΠA_i is also countably approximating, so that $(\Pi A_i, (\pi_i)_{i\in I})$ is a product of the family in CALat.

DEFINITION 3.6. Let T be a countably approximating lattice and S a subset of T. If S is closed under arbitrary meets and countably directed joins, then S is said to be a *subalgebra* of T.

REMARK 3.7. By Theorem 3.3, a subalgebra S of a countably approximating lattice T is again countably approximating. Indeed, since S has an arbitrary meets, it is a complete lattice. Furthermore, for any family $\{x_{j,k}\mid (j,k)\in J\times K\}$ in S satisfying the conditions of (3) in Theorem 3.3, both sides of the equation (a) are contained in S since S is closed under arbitrary meets and countably directed joins in T. If the equation holds in T, then it holds in S, and thus S is a countably approximating lattice.

PROPOSITION 3.8. *The category CALat has equalizers.*

Proof. For homomorphisms $f,\ g: A\rightarrow B$ in CALat, let $E=\{x\in A\mid f(x)=g(x)\}$. Since f and g preserve meets and countably directed joins in A, E is a subalgebra of A and hence E is countably approximating. Moreover, the inclusion map $e: E\rightarrow A$ also preserves meets and countably directed joins, i.e., e is a homomorphism in CALat. It is now clear that (E, e) is the equalizer of (f, g) in CALat.

Combining Proposition 3.5 and 3.8, the following is immediate from Theorem 23.8 in [10].

THEOREM 3.9. *The category CALat is complete.*

Using Theorem 3.3 and the same arguments as those in [6], one has

THEOREM 3.10. *Let S be a countably approximating lattice and T a complete lattice. If there is an onto map $g: S\rightarrow T$ which preserves arbitrary meets and countably directed joins, then T is again countably approximating.*

Interchanging directed joins by countably directed joins in the proof of Proposition 0-3.14 in [6], we have the following:

LEMMA 3.11. *For a complete lattice L, there is a one–to–one correspondence between the set of closure operators on L preserving countably directed joins and the set of closure systems on L which are closed under countably directed joins.*

Using the above lemma, we have the following:

THEOREM 3.12. *Let L be a countably approximating lattice. Then a subset S of L is a subalgebra of L iff there is a closure operator c on L such that c preserves countably directed joins and $S=c(L)$.*

NOTATION 3.13. The set of all Lindelöf elements of a complete lattice A is denoted by $L(A)$.

Clearly every compact element of A is a Lindelöf element and the converse need not be true, for 0 is the only compact element in the chain $[0, 1]$ but every element of $[0, 1]$ is a Lindelöf element by Example 2.7.

The following definition is due to Grätzer [8, 9].

DEFINITION 3.14. A complete lattice A is said to be countably algebraic if for any $x\in A$, $x=\bigvee(\downarrow x\cap L(A))$.

REMARK 3.15. (1) Countably algebraic lattices are called $\aleph_1$–algebraic in [8, 9].

(2) Every algebraic lattice is also countably algebraic.

The open set lattice $\Omega(Q)$ with the usual topology on the rational line is countably algebraic but not algebraic. Also the unit interval $[0, 1]$ with the usual order is countably algebraic but not algebraic.

(3) A complete lattice A is countably algebraic iff A is isomorphic to σIdS for some posets with countable joins.

(4) A product of countably algebraic lattices is again countably aigebraic and a subalgebra of a countably algebraic lattice is again countably algebraic.

PROPOSITION 3.16. *Let A be a complete lattice. Then the following are equivalent:*

(1) *A is countably algebraic.*

(2) *A is countably approximating and for all $x\ll_c y$, there is an element $t\in L(A)$ with $x\leq t\leq y$.*

In particular, every countably algebraic lattice is countably approxi-

mating.

Proof. (1) $\Longrightarrow$ (2). Since $\downarrow x\cap L(A)\subseteq \Downarrow_c x$; we have $x=\bigvee(\downarrow x\cap L(A))\leq\bigvee\Downarrow_c x\leq x$. Thus A is countably approximating. Now let $x\ll_c y$. Since $\downarrow y\cap L(A)$ is countably directed and $y=\bigvee(\downarrow y\cap L(A))$, there is an element $t\in\downarrow y\cap L(A)$ with $x\leq t$. Thus $x\leq t\leq y$ and $t\in L(A)$.

(2) $\Longrightarrow$ (1). Take any $y\in A$, then $y=\bigvee\Downarrow_c y$. Since $\downarrow y\cap L(A)$ is cofinal in $\Downarrow_c y$,

$$\bigvee(\downarrow y\cap L(A))=\bigvee\Downarrow_c y=y.$$

Thus A is countably algebraic.

Using countably algebraic lattices, we now characterize countably approximating lattices.

THEOREM 3.17. *A complete lattice is countably approximating iff it is a homomorphic image of a countably algebraic lattice.*

Proof. Let A be a countably approximating lattice and $r:\sigma\mathrm{IdA}\to A$ a map defined by $r(I)=\bigvee I$ for each $I\in\sigma$ IdA, then r is an onto homomorphism because for any $x\in A$, $x=\bigvee\Downarrow_c x$ and $\Downarrow_c x\in\sigma$ IdA. Thus A is a homomorphic image of a countably algebraic lattice σIdA.

The converse is immediate from the fact that every countably algebraic lattice is countably approximating and the fact that a homomorphic image of a countably approximating lattice is again countably approximating.

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