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COMPLEX SUBMANIFOLDS OF AN INDEFINITE KAEHLER-EINSTEIN MANIFOLD

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Introduction

Let M be an n-dimensional complex submanifold of a complex space form $M^{n+p}(c)$ of holomorphic sectional curvature c and $R^{\perp}$ denote the normal curvature of M. Chen and Ogiue [3] proved that M admits no parallel unit normal vector fields in the case where $c \neq 0$. They showed also that $R^{\perp}=0$ if and only if $c=0$ and M is totally geodesic.

Since Barros, Montiel and Romero [2], [5], [7], [8], [9], [10], [11] have recently made interesting results by a systematic study of indefinite Kaehlerian manifolds, in particular indefinite complex space forms, it seems to be natural to investigate whether some of the above results can be extended to indefinite complex submanifolds of an indefinite complex space form or not. From this point of view, Montiel and Romero [5] established the followings:

Let N be an $n(\geqq 3)$-dimensional complete and simply connected indefinite complex submanifold of an indefinite complex space form $M_{s+t}{}^{n+p}(c)$. Then N admits no parallel unit normal vector fields if $c \neq 0$. In particular, $R^{\perp}=0$ if and only if $c=0$ and $A^2=0$, where A denotes the shape operator in the direction of the unit normal vector field.

On the other hand, Ishihara [4] showed that a complex submanifold M of $M^{n+p}(c)$ satisfies the condition

$$(*) \qquad R^{\perp}(X, Y) = fg(X, JY)J$$

for any tangent vector fields X and Y, and a function f, then M is totally geodesic or M is locally isometric to a complex quadric Q_n, where g and J denote induced Kaehlerian metric tensor and the almost

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complex structure of M respectively.

The main purpose of the present paper is to prove that an indefinite complex submanifold of an indefinite Kaehler-Einstein manifold is also Einstein provided that the normal connection of the submanifold is flat. We also classified indefinite complex submanifolds satisfying the above condition (*) of an indefinite complex space form.

§1. Preliminaries

We begin by recalling fundamental properties on indefinite complex submanifolds of an indefinite Kaehlerian manifold. Let $\tilde{M}$ be a complex $(n+p)$-dimensional connected indefinite Kaehlerian manifold of index $2(s+t)$, $(0\leqq s\leqq n, 0\leqq t\leqq p)$. Then $\tilde{M}$ is equipped with a parallel almost complex structure F and an indefinite Riemannian metric tensor G which is F-Hermitian. Let M be an n-dimensional connected indefinite complex submanifold of index $2s$ of $\tilde{M}$. Then M is the indefinite Kaehlerian manifold endowed with the induced metric tensor g. We choose a local unitary frame field $\{E_1, \cdots, E_{n+p}\}$ on a neighborhood of $\tilde{M}$ in such a way that, restricted to $M, E_1, \cdots, E_n$ are tangent to M and the others are normal to M. Here and throughout this paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots=1, \cdots, n, \quad n+1, \cdots, n+p,$$
$$i, j, \cdots=1, \cdots, n,$$
$$\alpha, \beta, \cdots=n+1, \cdots, n+p.$$

Associated with the frame field $\{E_A\}$, let $\{w_A\}=\{w_i, w_\alpha\}$ be its dual frame field. Namely, it satisfies $w_A(E_B)=G(E_A, E_B)=\varepsilon_A\delta_{AB}$, where $\varepsilon_A=\pm 1$. Then the indefinite Kaehlerian metric G can be expressed locally as $G=2\Sigma\varepsilon_A w_A\otimes\bar{w}_A$. The canonical forms w_A and the connection forms w_{AB} of $\tilde{M}$ satisfy the following structure equations:

$$\begin{aligned} &dw_A+\Sigma\varepsilon_B w_{AB}\wedge w_B=0, \quad w_{AB}+\bar{w}_{BA}=0, \\ (1.1) \qquad &dw_{AB}+\Sigma\varepsilon_D w_{AD}\wedge w_{DB}=\Omega'_{AB}, \\ &\Omega'_{AB}=\Sigma\varepsilon_C\varepsilon_D K'_{\bar{A}BC\bar{D}}w_C\wedge\bar{w}_D, \end{aligned}$$

where Ω'_{AB} denotes the curvature form and $K'_{\bar{A}BC\bar{D}}$ are components of the Riemannian curvature tensor K'.

Restricting these forms to M, we have $w_\alpha=0$ and the induced indefinite Kaehlerian metric g of index $2s$ is given by $g=2\Sigma\ \varepsilon_j w_j\otimes\bar{w}_j$.

Then $\{E_j\}$ is a local unitary frame field with respect to g and $\{w_j\}$ is a local dual frame field relative to $\{E_j\}$, which consists of complex valued 1-forms of type (0, 1) on M. Moreover, the forms $w_1, \ldots, w_n, \bar{w}_1, \ldots, \bar{w}_n$ are linearly independent, and they are canonical forms on M. Using the Cartan lemma and fact that $w_\alpha=0$, it is seen that

$$w_{\alpha i}=\Sigma\, \varepsilon_j h_{ij}{}^{\alpha} w_j, \quad h_{ji}{}^{\alpha}=h_{ij}{}^{\alpha}. \tag{1.2}$$

The quadratic form $\Sigma_{i,j}\varepsilon_i\varepsilon_j\varepsilon_\alpha h_{ij}{}^{\alpha} w_i \otimes w_j \otimes E_\alpha$ with values in the normal bundle is called a *second fundamental form* of M.

It is easily, making use of (1.1), seen that the structure equations for M are obtained:

$$\begin{aligned} &dw_i+\Sigma\,\varepsilon_k w_{ik}\wedge w_k=0, \quad w_{ij}+\bar{w}_{ji}=0,\\ &dw_{ij}+\Sigma\,\varepsilon_k w_{ik}\wedge w_{kj}=\Omega_{ij},\\ &\Omega_{ij}=\Sigma\,\varepsilon_k\varepsilon_m K_{\bar{i}jk\bar{m}} w_k\wedge \bar{w}_m, \end{aligned} \tag{1.3}$$

where Ω_{ij} (resp. $K_{\bar{i}jk\bar{m}}$) denotes the curvature form (resp. components of the Riemannian curvature tensor K) of M. Furthermore the following equations are defined:

$$\begin{aligned} &dw_{\alpha\beta}+\Sigma\,\varepsilon_\gamma w_{\alpha\gamma}\wedge w_{\gamma\beta}=\Omega_{\alpha\beta},\\ &\Omega_{\alpha\beta}=\Sigma\,\varepsilon_k\varepsilon_m K_{\bar{\alpha}\beta k\bar{m}} w_k\wedge \bar{w}_m, \end{aligned} \tag{1.4}$$

where $\Omega_{\alpha\beta}$ is called the *normal curvature form* of M. For the Riemannian curvature tensor K and K' of M and $\tilde{M}$ respectively, it follows from (1.1), (1.2) and (1.3) that we have the Gauss equations

$$K_{\bar{i}jk\bar{m}}=K'_{\bar{i}jk\bar{m}}-\Sigma\,\varepsilon_\alpha h_{jk}{}^{\alpha}\bar{h}_{im}{}^{\alpha}, \tag{1.5}$$

and by means of (1.2) and (1.4) we have

$$K_{\bar{\alpha}\beta k\bar{m}}=K'_{\bar{\alpha}\beta k\bar{m}}+\Sigma\,\varepsilon_j h_{kj}{}^{\alpha}\bar{h}_{jm}{}^{\beta}. \tag{1.6}$$

Thus, components of the Ricci tensor S and the scalar curvature r of M are respectively given by

$$S_{\bar{i}j}=\Sigma\,\varepsilon_k K'_{\bar{i}jk\bar{k}}-\Sigma\,\varepsilon_r\varepsilon_\alpha \bar{h}_{ir}{}^{\alpha} h_{rj}{}^{\alpha}, \tag{1.7}$$

$$r=2\,\Sigma\,\varepsilon_j\varepsilon_k K'_{\bar{j}jk\bar{k}}-2h_2, \tag{1.8}$$

where $h_2=\Sigma\,\varepsilon_i\varepsilon_j\varepsilon_\alpha h_{ij}{}^{\alpha}\bar{h}_{ij}{}^{\alpha}$ is a real number. The indefinite Kaehlerian manifold M is said to be *Einstein* if the Ricci tensor S has the form $S_{i\bar{j}}=r\varepsilon_i\delta_{ij}/2n$.

§ 2. Normal connections of Kaehlerian submanifolds

Let $(\tilde{M}, G)$ be a complex $(n+p)$-dimensional indefinite Kaehlerian manifold of index $2(s+t)$ with almost complex structure F.

A holomorphic plane spanned by u and Fu which is a tangent vector at any point in $\tilde{M}$ is non-degenerate if and only if it contains some vector v such that $G(v, v)\neq 0$. The manifold $\tilde{M}$ is said to be *of constant holomorphic sectional curvature* c' if all non-degenerate holomorphic planes have the same constant sectional curvature c'. A comlete, simply connected and connected indefinite Kaehlerian manifold $\tilde{M}$ is called an *indefinite complex space form,* which is denoted by $M_{s+t}{}^{n+p}(c')$, provided that it is of constant holomorphic sectional curvature c' and of index $2(s+t)$. There are three kinds of types about indefinite complex space forms [2], [13]: an indefinite complex projective space $P_{s+t}{}^{n+p}C$, an indefinite complex Euclidean space $C_{s+t}{}^{n+p}$ or an indefinite hyperbolic space $H_{s+t}{}^{n+p}C$, according as c' is positive, zero or negative. The components $K'_{\bar{D}CB\bar{A}}$ of the Riemann curvature tensor of $M_{s+t}{}^{n+p}(c')$ are of the form:

$$k'_{\bar{D}CB\bar{A}}=\frac{1}{2}c'\varepsilon_C\varepsilon_D(\delta_{DC}\delta_{BA}+\delta_{DB}\delta_{CA}).$$

Let M be an n-dimensional connected indefinite complex submanifold of index $2s$ of $M_{s+t}{}^{n+p}(c')$. Then equations (1.5) and (1.6) for M is reduced respectively to

$$K_{\bar{i}jk\bar{m}}=\frac{c'}{2}\varepsilon_j\varepsilon_k(\delta_{ij}\delta_{km}+\delta_{ik}\delta_{jm})-\Sigma\,\varepsilon_a h_{jk}{}^{a}\bar{h}_{im}{}^{a}, \tag{3.1}$$

$$K_{\bar{\alpha}\beta k\bar{m}}=\frac{c'}{2}\varepsilon_\beta\varepsilon_k\delta_{\alpha\beta}\delta_{km}+\Sigma\,\varepsilon_r h_{kr}{}^{\alpha}\bar{h}_{rm}{}^{\beta}. \tag{3.2}$$

Thus, components of the Ricci tensor S and the scalar curvature r of M are respectively given by

$$S_{i\bar{j}}=\frac{(n+1)}{2}c'\varepsilon_i\delta_{ij}-h_{i\bar{j}}{}^{2}, \tag{3.3}$$

$$r=n(n+1)c'-2h_2, \tag{3.4}$$

where $h_{i\bar{j}}{}^{2}=\Sigma\,\varepsilon_r\varepsilon_\alpha h_{ir}{}^{\alpha}\bar{h}_{rj}{}^{\alpha}$.

We give here some examples, which is introduced by Montiel and Romero [5] and Romero [9], of indefinite complex Einstein hypersurfaces of an indefinite complex space form.

Example 1. The indefinite Euclidean space $C_s{}^n$ is a totally geodesic complex hypersurface of $C_s{}^{n+1}$ and $C_{s+1}{}^{n+1}$ in a natural way.

Example 2. For an indefinite complex projective space $P_s{}^{n+1}(c')$, if $(z_1, \dots, z_{s+1}, \dots, z_{n+2})$ is the usual homogeneous coordinate system of

$P_s^{n+1}(c')$, then for each fixed j, the equation $z_j=0$ defines a totally geodesic complex hypersurface identifiable with $P_s^n(c')$ or $P_{s-1}^n(c')$ according as $s+1\leqq j\leqq n+2$ or $1\leqq j\leqq s$. Taking account of the fact that $H_s^n(-c')$ is obtained from $P_{n-s}^n(c')$ by reversing the sign of its indefinite Kaehlerian metric, the above discussion shows that $H_s^n(-c')$ is a totally geodesic complex hypersurface of both $H_s^{n+1}(-c')$ and $H_{s+1}^{n+1}(-c')$.

Example 3. Let Q_s^n be an indefinite complex hypersurface of $P_s^{n+1}(c')$ defined by the equation

$$-\sum_{i=1}^{s} z_i^2+\sum_{j=s+1}^{n+2} z_j^2=0.$$

Then Q_s^n is an complete complex Einstein hypersurface of index $2s$ and then the Ricci tensor S satisfies $S_{i\bar{j}}=\frac{n}{2}c'\varepsilon_i\delta_{ij}$. Q_s^n can be also considered as an indefinite complex Einstein hypersurface of $H_{s+1}^{n+1}(-c')$. Q_s^n is called an *indefinite complex quadric* [5].

Remark 1. Smyth [12] showed that a complex Einstein hypersurface M of a complex space form $M^{n+1}(c')$ is totally geodesic or $c'>0$ and M is locally the complex quadric Q^n. Example 3 means that the situation of indefinite complex Einstein hypersurfaces is quitely different from those of definite cases.

Example 4. Let us consider an indefinite complex hypersurface of $P_{n+1}^{2n+1}(c')$ defined by the equation

$$\sum_{j=1}^{n+1} z_j z_{n+j}=0$$

in the homogeneous coordimate system of $P_{n+1}^{2n+1}(c')$. It is a complete complex hypersurface of index $2n$, which is denoted by Q_*^n. It is easily seen that components of the Ricci tensor S satisfies

$$S_{A\bar{B}}=(n+1)c'\varepsilon_A\delta_{AB},\ h_{A\bar{B}}^2=-c'\varepsilon_A\delta_{AB}/2.$$

Thus, Q_*^n is a complete Einstein hypersurface of index $2n$. Q_*^n also an indefinite Einstein hypersurface of $H_n^{2n+1}(c')$.

Remark 2. We can not always diagonalize the shape operator A in the indefinite Einstein hypersurface M of $M_{s+1}^{n+1}(c')$, and hence the classification for self-adjoint endomorphisms of a scalar product space is used. Then we have the followings:

(1) A is diagonalizable;

or

(2) it is not diagonalizable, but either $\varepsilon h_2<0$ or $h_2=0$ and not totally geodesic, where $\varepsilon=\varepsilon_{n+1}$.

A complex Einstein hypersurface is said to be *proper* if the shape operator is diagonalizable.

For indefinite complex Einstein hypersurfaces of an indefinite complex space form Montiel and Romero proved the followings:

THEOREM M.-R.-1. [5]. *If* $n\geqq 3$, *we have*:

(1) *The proper complete and simply connected indefinite complex Einstein hypersurfaces of* $P_{s+t}{}^{n+1}(c')$ *are only* $P_s{}^n(c')$ *with* $t=0, 1$ *and* $Q_s{}^n$ *with* $t=0$.

(2) *In* $H_{s+t}{}^{n+1}(-c')$ *they are only* $H_s{}^n(-c')$ *with* $t=0, 1$ *and* $Q_s{}^n$ *with* $t=1$.

(3) *In* $C_{s+t}{}^{n+1}$ *they are only* $C_s{}^n$ *with* $t=0, 1$.

THEOREM M.-R.-2. [5], [9]. *Let* M *be an indefinite Einstein hypersurface of* $M_{s+t}{}^{n+1}(c')$.

(1) If $h_2\neq 0$, *then* M *is locally symmetric.*

(2) *If* $h_2=0$ *but not totally geodesic, then* $c'=0$ *provided that* M *is locally symmetric.*

THEOREM M.-R.-3. [5]. $Q_*{}^n$ *is the unique indefinite complex Einstein hypersurface of* $P_{n+1}{}^{2n+1}(c')$ *and of* $H_n{}^{2n+1}(-c')$ *which is complete and simply connected, and satisfies* $\varepsilon h_2<0$.

Now, suppose that the normal curvature tensor in the normal bundle of M satisfies

$$K_{\bar{\alpha}\beta k\bar{m}}=f\varepsilon_\beta\varepsilon_k\delta_{km}\delta_{\alpha\beta} \tag{3.5}$$

for some function f on M. In particular, the normal connection of M is said to be *semi-flat* if $f=1$ [4]. The justification of this definition is given in [14].

REMARK 3. By using the first Bianchi identity, we easily from (3.5) see that f is constant everywhere if $n\geqq 2$ (for detail, see [14]). It is, utilizing (3.2) and (3.5), clear that

$$h_{i\bar{j}}{}^2=p\left(f-\frac{c'}{2}\right)\varepsilon_i\delta_{ij},$$

which together with (3.3) yield

$$S_{i\bar{j}}=b\varepsilon_i\delta_{ij},$$

where $b=\frac{n+1}{2}c'-p\left(f-\frac{c'}{2}\right)$ is constant. Thus, the complex submanifold M of $M_{s+t}^{n+p}(c')$ is Einstein if $n\geqq 2$. From (3.2) and (3.5), it is easy to see that

$$\text{(3.6)}\qquad \Sigma\,\varepsilon_r h_{ir}{}^{\alpha}\bar{h}_{rj}{}^{\beta}=\left(f-\frac{c'}{2}\right)\varepsilon_\beta\varepsilon_i\delta_{ij}\delta_{\alpha\beta},$$

which implies

$$\text{(3.7)}\qquad h_{i\bar{j}}{}^2=c\varepsilon_i\delta_{ij},$$

where $c=\left(f-\frac{c'}{2}\right)p$ is constant.

The relationships (3.6) and (3.7) tell us that $c=0$ i.e., $f=\frac{c'}{2}$ if and only if $\Sigma\,\varepsilon_r h_{ir}{}^{\alpha}\bar{h}_{rj}{}^{\beta}=0$ for any index α, β, i and j, and hence $h_{i\bar{j}}{}^2=0$.

From now on the case where $c\neq 0$ and $n\geqq 3$ is considered: As in the proof of Theorem 6.3 of [5], we know, using (3.8), that there exists a complex submanifold $M_{s+a}^{n+1}(c')$ (where $a=0, 1$ according as $c<0, c>0$) embedded in a totally geodesic way in $M_{s+t}^{n+p}(c')$ where M is immersed as an indefinite complex hypersurface. We, making use of (3.7) and the Gauss equation (3.1), see that M is Einstein manifold for this last immersion. The relationship (3.7) means that $h_2=nc$.

The case $\varepsilon c<0$ is considered, where $\varepsilon=\varepsilon_{n+1}$. Now, components $h_{ijk}{}^{\alpha}$ and $h_{ij\bar{k}}{}^{\alpha}$ of the covariant derivative of $h_{ij}{}^{\alpha}$ can be defined by

$$\text{(3.8)}\qquad \Sigma\,\varepsilon_k(h_{ijk}{}^{\alpha}w_k+h_{ij\bar{k}}{}^{\alpha}\bar{w}_k)=dh_{ij}{}^{\alpha}-\Sigma\,\varepsilon_k(h_{kj}{}^{\alpha}w_{ki}+h_{ik}{}^{\alpha}w_{kj}) +\Sigma\,\varepsilon_\beta h_{ij}{}^{\beta}w_{\alpha\beta}.$$

Then, substituting $dh_{ij}{}^{\alpha}$ of (3.8) into the exterior derivative of (1.2), we have

$$\text{(3.9)}\qquad h_{ijk}{}^{\alpha}=h_{jik}{}^{\alpha}=h_{ikj}{}^{\alpha},\ h_{ij\bar{k}}{}^{\alpha}=0$$

because the ambient space is a complex space form.

Similarly, components $h_{ijkl}{}^{\alpha}$ and $h_{ijk\bar{l}}{}^{\alpha}$ of the covariant derivative of $h_{ijk}{}^{\alpha}$ can be also defined by

$$\text{(3.10)}\qquad \Sigma\,\varepsilon_l(h_{ijkl}{}^{\alpha}w_l+h_{ijk\bar{l}}{}^{\alpha}\bar{w}_l)=dh_{ijk}{}^{\alpha}-\Sigma\,\varepsilon_l(h_{ljk}{}^{\alpha}w_{li}+h_{ilk}{}^{\alpha}w_{lj} +h_{ijl}{}^{\alpha}w_{lk})+\Sigma\,\varepsilon_\beta h_{ijk}{}^{\beta}w_{\alpha\beta},$$

and the simple calculation gives rise to

$$\text{(3.11)}\qquad h_{ijk\bar{l}}{}^{\alpha}=\frac{c'}{2}(\varepsilon_k h_{ij}{}^{\alpha}\delta_{kl}+\varepsilon_i h_{jk}{}^{\alpha}\delta_{il}+\varepsilon_j h_{ki}{}^{\alpha}\delta_{jl})$$

$$-\Sigma\varepsilon_r\varepsilon_\beta(h_{ri}{}^\alpha h_{jk}{}^\beta+h_{rj}{}^\alpha h_{ki}{}^\beta+h_{rk}{}^\alpha h_{ij}{}^\beta)\bar{h}_{rl}{}^\beta$$

by virtue of (3.1), (3.2) and (3.9).

Differentiating $h_{i\bar{j}}{}^2$ exteriorly twice and using (3.8) and (3.10), we get

$$\Sigma\,\varepsilon\varepsilon_r(h_{irk\bar{l}}h_{rj}+h_{irk}\bar{h}_{rjl})=0,$$

which implies $h_2h_{ijk\bar{l}}=0$. Since $\varepsilon h_2<0$, it follows from this equation, (3.1) and (3.11) that

$$(nc'-2h_2)(\varepsilon_k h_{ij}\delta_{kl}+\varepsilon_i h_{jk}\delta_{il}+\varepsilon_j h_{ki}\delta_{jl})=0$$

and hence $nc'=2h_2$ because of (3.7) and $\varepsilon c<0$.
Consequently (3.4) is reduced to $r=n^2c'$ and hence

$$S_{i\bar{j}}=\frac{n}{2}c'\varepsilon_i\delta_{ij}.$$

According to above three theorems, we conclude that

THEOREM 1. *Let M be an indefinite complex submanifold of complex dimension $n(n\geqq 2)$ and of index $2s$ in $M_{s+t}{}^{n+p}(c')$. If M is complete and if the normal connection of M satisfies the condition* (3.5), *then $h_{i\bar{j}}{}^2=0$ or M is an indefinite complex quadric $Q_s{}^n$ in $M_{s+t}{}^{n+1}(c')$, $c'\neq 0$.*

REMARK 4. As in the proof of Theorem 1, it follows that the normal connection of M is not flat if $c\neq 0$.

REMARK 5. (1) If M is a space-like complex submanifold with $f=\frac{c'}{2}$ in Theorem 1, then M is totally geodesic. (2) There exists an indefinite complex submanifold M with $f=\frac{c'}{2}$ and $\Sigma\,\varepsilon_r h_{ir}{}^\alpha\bar{h}_{rj}{}^\beta=0$. But M is not totally geodesic: Let $M_s{}^{2n}$ be an indefinite complex submanifold with Ricci flat, not flat in $C_s{}^{2n+1}(p\geqq 2)$ [1]. Then $M_s{}^{2n}$ satisfies the condition (3.5), that is, $K_{\bar{\alpha}\beta k\bar{m}}=0$, $f=0$ and $h_{k\bar{m}}{}^2=0$, but it is not totally geodesic.

§ 3. Einstein submanifolds of an indefinite Kaehler-Einstein manifold

Let $(\tilde{M}, G)$ be a complex $(n+p)$-dimensional indefinite Kaehler-Einstein manifold of index $2(s+t)$ and M be an n-dimensional indefinite complex submanifold of index $2s$ of $\tilde{M}$. Then we have (1.5) and (1.6).

Thus, it follows that

$$S_{k\bar{m}}=\Sigma\,\varepsilon_i K'_{\bar{i}ik\bar{m}}-h_{k\bar{m}}{}^2,$$

where $h_{k\bar{m}}{}^2=\Sigma\,\varepsilon_r\varepsilon_\alpha h_{kr}{}^\alpha \bar{h}_{rm}{}^\alpha$, which is equivalent to

$$S_{k\bar{m}}=S'_{k\bar{m}}-\sum_\alpha \varepsilon_\alpha K_{\bar{\alpha}\alpha k\bar{m}}-h_{k\bar{m}}{}^2.$$

If we take account of (1.6), then the equation turns out to be

$$S_{k\bar{m}}=S'_{k\bar{m}}-\sum_\alpha \varepsilon_\alpha K_{\bar{\alpha}\alpha k\bar{m}}.$$

Because $\tilde{M}$ was assumed to be Einstein, it follows that

$$S_{i\bar{j}}=\frac{r'}{2(n+p)}\varepsilon_i\delta_{ij}-\sum_\alpha \varepsilon_\alpha K_{\bar{\alpha}\alpha i\bar{j}},$$

where r' is the scalar curvature of $\tilde{M}$. Thus, we have

THEOREM 2. *Let M be an indefinite complex submanifold of an indefinite Kaehler-Einstein manifold $\tilde{M}$. If the normal connection of M is flat, then M is Einstein.*

REMARK 6. The converse assertion of Theorem 2 is not always true. For example, let $Q_s{}^n$ be an indefinite complex quadric in an indefinite complex projective space $P_s{}^{n+1}(c')$. Then $Q_s{}^n$ and $P_s{}^{n+1}(c')$ are Einstein. But, the normal connection of $Q_s{}^n$ is not flat.

REMARK 7. If the normal connection of M is semi-flat in Theorem 2, then M is Einstein.

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