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On Cn-Semistratifiable over α

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1. Preliminaries

In this section we now introduce some definitions and results which are used throughout this paper.

DEFINITION 1. 1. (Vaughan) An ordinal number α is called an initial ordinal provided that for every ordinal $\beta<\alpha$, there exists an injection from β to α, but there does not exist an injection from α to β. We assume that cardinal numbers and initial ordinal numbers are the same.

Let ω stand for the first infinite ordinal.

DEFINITION 1. 2. (Vaughan) Let $(X,\mathcal{J})$ be a T_1 *-space* and let α be an initial ordinal, $\alpha\geqq\omega$. The space $(X,\mathcal{J})$ is said to be stratifiable over α or linearly stratifiable provided that there exists a map $S: \alpha x\mathcal{J}\rightarrow\mathcal{J}$ (called an α-stratification) which satisfies the following (where we denote $S(\beta, U)$ by U_β)

LS1: $\bar{U}_\beta\subset U$ *for all* $\beta<\alpha$ *and all* $U\in\mathcal{J}$.

LS2: $\cup\{U_\beta: \beta<\alpha\}=U$ *for all* $U\in\mathcal{J}$.

LS3: If $U\subset W$, *then* $U_\beta\subset W_\beta$ *for all* $\beta<\alpha$

LS4: If $r<\beta<\alpha$, *then* $U_r\subset U_\beta$ *for all* $U\in\mathcal{J}$.

DEFINITION 1. 3. (Vaughan) A T_1-space X is called α-stratifiable provided that α is the smallest initial ordinal for which X is stratifiable over α. A space which is stratifiable over ω is called stratifiable, and the map S is called a stratification.

DEFINITION 1. 4. (Vaughan) A collection P of pairs $P=(p_1, p_2)$ of subsets of a topological space $(X,\mathcal{J})$ is said to be a linearly cushioned collection of pairs with respect to a linear order $\leqq$ provided that $\leqq$ is a linear order on P such that $(\cup\{P_1: P=(P_1,P_2)\in P^1\})\subset\cup\{P_2: p=(P_1,P_2)\in P^1\}$ for every subset P^1 of P which is majorized (i. e., has an upper bound) with respect to $\leqq$.

DEFINITION 1. 5. (Ceder) A collection P of pairs is called a pairbase for $(X,\mathcal{J})$ provided that (1) for each $P=(P_1,P_2)\in P$, P_1 is open and (2) for every x in x and every open set W containing x, there exists $P=(P_1,P_2)\in P$ such that $x\in P_1\subset P_2\subset W$.

THEOREM 1. 6. (Vaughan) *If* $(X,\mathcal{J})$ *is a* T_1*-topological space and* α *an infinite initial ordinal, then the following are equivalont*

(1) $(X, \mathcal{J})$ is stratifiable over α.

(2) $(X, \mathcal{J})$ has a linearly cushioned pair-base P and α is cofinal with P.

(3) There exists a family $\{g_\beta : \beta < \alpha\}$ of functions with domain X and range $\mathcal{J}$ such that the following hold.

(a) $x \in g_\beta(x)$ for all $\beta < \alpha$

(b) For every $F \subset X$, if $y \in [(U\{g_\beta(x) : x \in F\})]$ for all $\beta < \alpha$, then $y \in \bar{F}$.

(c) If $\beta < \gamma < \alpha$, then $g_\beta(x) \supset g_\gamma(x)$ for all x.

A linearly semistratifiable space is introduced by K. B. Lee [4]. The new class of spaces is an extension of semistratifiable spaces, definitions and main results of which are collected in the following.

DEFINITION 1.7. (K. B. Lee) Let $(X, \mathcal{J})$ be a topological space and α be an initial ordinal not less than ω. The space X is said to be sewistratifiable over α or linearly semistratifiable provided that there exists a map $S: \alpha \times \mathcal{J} \rightarrow \{$*closed subsets of* $X\}$(called an α-semistratification) which satisfies the following

LSS1: For every $U \in \mathcal{J}$, $U = U\{S(\beta, U) : \beta < \alpha\}$

LSS2: If $U, V \in \mathcal{J}$ and $U \subset V$, then $S(\beta, U) \subset S(\beta, V)$ for all $\beta < \alpha$

LSS3: If $\gamma < \beta < \alpha$, then $S(\gamma, U) \subset S(\beta, U)$ for all $U \in \mathcal{J}$.

DEFINITION 1.8. (K. B. Lee) A collection P of pairs $p = (p_1, p_2)$ of subsets of a space $(X, \mathcal{J})$ is called a pair-net provided that for every x in X and every open U containing x, there exists a $P = (p_1, p_2)$ such that $x \in P_1 \subset P_2 \subset U$.

THEOREM 1.9. (K. B. Lee) *If $(X, \mathcal{J})$ is a space and α an infinite initial ordinal, then the following are equivalent:*

(1) X is semistratifiable over α.

(2) X has a linearly cushioned pair-net P with which α is cofinal.

(3) There is a function g from $\alpha \times X$ into $\mathcal{J}$ such that

(a) for each $x \in X$, $x \in \cap\{g(\beta, x) : \beta < \alpha\}$

(b) if $x \in g(\beta, x_\beta)$ for each $\beta < \alpha$, then the net $\{x_\beta : \beta < \alpha\}$ accumulates at x

(c) if $\gamma < \beta < \alpha$, then $g(\gamma, x) \supset g(\beta, x)$ for every $x \in X$.

(4) There is a function g from $\alpha \times X$ into $\mathcal{J}$ such that (a) for each $x \in X$, $\cap\{g(\beta, x) : \beta < \alpha\} = cl\ \{x\}$: (b) if $x \in g(\beta, x_\beta)$ for each $\beta < \alpha$, then the net $\{x_\beta : \beta < \alpha\}$ converges to x: and (c) if $\gamma < \beta < \alpha$ then $g(\gamma, x) \supset g(\beta, x)$ for every $x \in X$.

DEFINITION 1.10. A pair-net is called a cn-pairnet if given any convergent net $\chi_\beta \rightarrow \chi$ and an open sudset U containing χ, there is a $P = (P_1, P_2) \in P$ such that $\chi \in P_1 \subset P_2 \subset U$ and $\{\chi_\beta\}$ is eventually in P_1

2. Definition of cn-semistratifiable over α and some characterizations

DEFINITION 2.1. Let $(X, \mathcal{J})$ be a topological space and α be an initial ordinal not less than ω. The space X is said to be cn-semistratifiable over α or linearly cn-semistratifiable provided that there exists a map $S: \alpha \times \mathcal{J} \rightarrow \{$*closed subsets of* $X\}$(*called* an-cn-semistratification) which satisfies the following.

a) For every $U \in \mathcal{J}$, $U = U\{S(\beta, U) = \beta < \alpha\}$

b) *If* $U, V\in \mathcal{J}$ *and* $U\subset V$, *then* $S(\beta, U)\subset S(\beta, V)$ *for all* $\beta<\alpha$

c) *If* $\gamma<\beta<\alpha$, *then* $S(\gamma, U)\subset S(\beta, U)$ *for all* $U\in \mathcal{J}$

d) *For each convergent net* $\chi_\beta\to\chi$ *and* $U\in\mathcal{J}$, *containing* χ, *there is a* $\beta<\alpha$ *such that* $\chi\in S(\beta, U)$ *and* $\{\chi_\beta|\beta<\alpha\}$ *is eventually in* $S(\beta, U)$.

DEFINITION 2. 2. A topological space X is called $\alpha-cn$-semistratifiable provided that α is the smallest initial ordinal for which X is cn—semistratifiable over α. A space which is cn-semistratifiable over ω is called cs-semistratifiable.

THEOREM 2. 3. *If* $(X, \mathcal{J})$ *is a space and a an infinite initial ordinal, then the following are equivalent:*

(1) X *is* cn—*semistratifiable over* α.

(2) X *has a linearly cushioned cn pairnet* P *with which* α *is cofinal.*

(3) *There is a function g from* $\alpha\times X$ *into* $\mathcal{J}$ *satisfying Theorem 1. 9(1) and an additional condition:*

(4) *Given a convergent net* $\{x_\beta|\beta<\alpha\}\to x$ *and an open subset U containing x, there is a* $\beta<\alpha$ *such that* $x\notin \bigcup_{y\in X-U} g(\beta, y)$ *and* $\{\beta<\alpha|x_\beta\in\bigcup_{y\in X-U} g(\beta, y)\}$ *such that* $J=\{\gamma_0|\gamma\leq\gamma_0<\alpha\}$ *is cofinal.*

Proof. For (1) ↔ (2), See the proof of Theorem 1.13

For (2) ↔ (3), Let P be a linearly cushioned cn pairnet for X, and α cofinal with P.

There is a subclass $P'=\{P_\beta: \beta<\alpha\}$ such that for every $P\in P$ there is a $\beta<\alpha$ such that $\boldsymbol{P}\leq\boldsymbol{P}_\beta$.

For each x in X and each $\beta<\alpha$, define $g(\beta, x)=X-cl(\bigcup\{P_1: x\notin P_2$ and $\boldsymbol{P}=(P_1, P_2)\leq\boldsymbol{P}_\beta\})$.

Lee. K. B proved g is a linearly semistratifiable function 4. To show g satisfies (d). Consider the following.

$$\bigcup_{y\notin V} g(\beta, y)=\bigcup_{y\notin V}\{X-cl(\bigcup\{P_1: y\notin P_2 \text{ and } \boldsymbol{P}=(P_1, P_2)\leq\boldsymbol{P}_\beta\})$$

$$=X-\bigcap_{y\notin V}(cl(\bigcup\{P_1: y\notin P_2 \text{ and } \boldsymbol{P}=(P_1, P_2)\leq\boldsymbol{P}_\beta\})$$

which is contained in $X-(cl(\bigcup\{P_1= y\notin P_2$ and $\boldsymbol{P}=(P_1, P_2)\leq\boldsymbol{P}_\beta\}))$

If $\{x_\beta|\beta<\alpha\}$ is eventually in $x\in P_1\subset P_2\subset U$ and $\boldsymbol{P}=(P_1, P_2)\leq\boldsymbol{P}_\beta$,

$J=\{\beta<\alpha|\chi_\beta\in\bigcup_{y\notin V} g(\beta, y)\}<\beta$. And consequently. J is cofinal.

(3) ⇨ (1) Let g be a map as is described in (3).

Define a map $S: \alpha\times\mathcal{J}\to\{$*closed subsets of* $X\}$

by $S(\beta, U)=X-\bigcup\{g(\beta, \chi): \chi\in X-U\}$.

$S(\beta, U)\subset X-(X-U)=U$. *Since* $\chi\in g(\beta, \chi)$ *for all* $\beta<\alpha$. Conversely, assume $\chi\notin\bigcup\{S(\beta, U): \beta<\alpha\}$

Then $\chi\in\bigcup\{g(\beta, y): y\in X-U\}$ for all $\beta<\alpha$. This implies there is an $y_\beta\in X-U$ such that $\chi\in g(\beta, y_\beta)$ for each $\beta<\alpha$.

Thus $\{y_\beta: \beta<\alpha\}$ satisfies the condition (b) of (4), and hence converges to χ.

Since $X-U$ is closed, we have $\chi\in cl(\{y_\beta: \beta<\alpha\})\subset X-U$. Finally the condition(d) of Definition 2.1 is satisfied by the property (d) of g. Thus the proof is completed.

3. Properties of Cn−semistratifiable over α

THEOREM 3.1. *Every subspace of a cn−semistratifiable over α is a cn−semistratifiable over α*

Proof. Let S be an α−cn−semistratification of X, and Y be a subspace of X. Define $S^1: \alpha\times \mathcal{J}_y\rightarrow\{closed\ subsets\ of\ Y\}$ by the restriction of S to $\mathcal{J}_y$−open subset of X. It is easily verified that S is an α−cn−semistratification for Y.

Now, we shall prove that a finite product of spaces cn−semistratifiable over the same α is again cn−semistratifiable over α.

LEMMA 3.2. *Let α be an infinite initial ordinal number, and Let $\{A_\lambda: \lambda\in\Lambda\}$ be a family of linearly ordered sets such that α has cardinality strictly greater than that of A, and α is cofinal with A for all $\lambda\in\Lambda$. If Λ is finite or if α is a regular ordinal, then $A=\cap\{A_\lambda: \lambda\in\Lambda\}$ can be well-ordered so that for every majorized $H\subset A$, we have $Pr(H)$ (i.e., the λ th projection) is majorized in A_λ for all $\lambda\in\Lambda_\lambda$ and α is cofinal in A. Further, if α is the smallest initial ordinal cofinal with each A_λ, then α is the smallest initial ordinal cofinal with A.*

Proof. See the proof of Lemma 5.1[9]

Theorem 3.3. *Let α be an initial ordinal number $\alpha\geqq\omega$. Let X_i be cn−semistratifiable over α for each $i<\omega$. Then $\Pi\{X_i: i\leqq n\}$ is cn−semistratifiable over α for all $n<\omega$.*

Proof. Each X_i has a linearly cushioned cn−pair−net P_i such that α is cofinal with P_i, For each $n<\omega$ and each $Q=(P^1,\cdots,P^n)$

$\Pi\{P_i: i\leqq n\}$ define $\prod_{i=1}^{n} P_1^i=\{\chi=(\chi_i): \chi_i\in P_1^i\ for\ i\angle n\}$, and Similarly define $\prod_{i=1}^{n}P_2^i$,

Set $B_{Q1}=\prod_{i=1}^{n}P_1^i$, $B_{Q2}=\prod_{i=1}^{n}P_2^i$, *and* $B_n=\{B_{Q1},B_{Q2}): Q\in\Pi\{P_i: i\leqq n\}\}$ and order the index set of B_n as Lemma 3.2 so that α is cofinal with B_n clearly B_n is a cn−pair−net for $\Pi\{X_i: i\leqq n\}$, and if we consider $(\chi_i)\in\Pi\{X_i: i<w\}$, then $B=U\{B_n: n<\omega\}$ is a cn −pair−net for $\Pi\{X_i\mid i<w\}$. We now show that each B_n is a linearly cushioned collection of pairs in $X=\Pi\{X_i: i\leqq n\}$.

Suppose H is a majorized subset of $\prod_{i=1}^{n}P_i$ *and* $\chi\notin U\{B_{Q2}: Q\in H\}$.

Let $N_i=X_i-(U\{P_1: P=(P_1,P_2)\in P_{\gamma i}(H)$ *and* $\chi_i\notin P_2\})$. Then N_i is an open neighborhood of χ_i in X_i because $P_{\gamma i}(H)$ is a majorized subset of P_i. Finally, $\prod_{i=1}^{n}N_i$ is a neighborhood of χ in X which misses $U\{B_{Q1}: Q\in H\}$. Thus $(U\{B_{Q1}: Q\in H\})^-\subset U\{B_{Q2}: Q\in H\}$, and this completes the proof.

Lemma 3.4. *Let X be cn−semistratifiable over α and Y be a closed subspace of X with an α−cn−semistratification S. Then there is an α−cn−semistratification T for X such that $S(\beta,V\cap Y)=T(\beta,V)\cap Y$ for every $\beta<\alpha$ and every open V in X.*

Proof. Let S' be any α−cn−semistratification for X. Detine an α−cn−semistratification T for X as follows :

$$T(\beta, V)=S(\beta, V\cap Y)\cup S'(\beta, V-Y)$$

It is clear that T is an α－cn－semistratification.

Now, we show that T satisfies (d).

Let $\{\chi_\beta\}$ be a net in X converging to χ. Given an open set U of X containing χ, if $\chi\in U\cap Y$. Since $U\cap Y$ is a relative open subset in Y there is $\gamma<\alpha$ such that $\{\chi_\beta\}$ is eventually in $S(\gamma, U\cap Y)$. Therefore $\{\chi_\beta\}$ is eventually in $T(\gamma, U)$

If $\chi\notin U\cap Y$ it is clear.

This completes the proof.

Lemma 3. 5. *The union of two closed (in the unión) subspaces which are cn－semistratifiable over* α *is also cn－semistratifiable over* α.

Proof. Apply Lemma 3. 4 with respect to the common subspace.

Theorem 3. 6. *If* X *is a locally finite union of closed cn－semistratifiable over* α, *then* X *is cn－semistratifiable over* α.

Proof. By Lemma 3. 5, the proof is verified easily.

4. Net－covering maps

Frank Siewiec introduced the concept of sequence－covering map in [8]. Now, we introduce the extnnded concept of sequence－covering map.

Definition 4. 1. A mapping $f: X\to Y$ is said to be net covering if given any convergent net $y_\beta\to y$ in Y, there exists a convergent net $\chi_\beta\to\chi$ in X such that $f(\chi_\beta)=y_\beta$, $\beta<\alpha$

Theorem 4. 2. *The image of a cn－semistratifiable over* α *under a closed continuous net－covering map is cn－semistratifiable over* α.

Proof. Let f be a closed continuous net－covering map from *cn－semistratifiable over* α X onto a space Y. Let S be a α－cn－semistratification for X. For each open V of Y and $\beta<\alpha$, Let $T(\beta, V)=f(S(\beta. f^{-1}(V))$ clearly T is a α－cn－semistratification. $y_\beta\to y$ be a convergent net in Y.

Then there is a convergent net $\chi_\beta\to\chi$ in X such that $f(\chi_\beta)=y_\beta$ for $\beta<\alpha$. Since X is cn－semistratifiable over α, there exists a $\gamma<\alpha$ such that $\{\chi_\beta|\beta<\alpha\}$ is eventually in $S(\gamma, f^{-1}(V))$ for any open V. Thus, y_β is eventually in $T(\gamma, V)=f[S(\gamma, f^{-1}(V)]$.

W. K. MIN proved that K－semistratifiable over α with α－fundamental system of neighborhoods $\{W_\beta(\chi): \beta<\alpha$ *and* $W_\beta(\chi)\subset W_\gamma(\chi)$ for $\gamma<\beta<\alpha\}$ for each $\chi\in X$ is stratifiable over α.

Theorem 4. 3. *A cn－semistratifiable over* α *with* α－*fundamental system of neighborhoods* $\{W_\beta(\chi): \beta<\alpha\}$ *for each* $\chi\in X$ *is stratifiable over* α.

Proof. Let S be an α－cn－semistratification for X. Suppose that $P\in V$, where V is open. Let $\{W_\beta(P): \beta<\alpha\}$ be α－fundamental system of neighborhoods for p such that $V\supset W_\gamma\supset W_\beta$ *for* $\gamma<\beta<\alpha$.

If $W_\beta\not\subset S(\beta, V)$ for each $\beta<\alpha$, choose points $y_\beta\in W_\beta-S(\beta, V)$ for each $\beta<\alpha$.

The net convergents to p, and so there is such that $\{y_\beta: \beta<\alpha\}$ is eventually in

$S(\gamma, V)$. Therefore, for some $\gamma < \alpha$, $W_\gamma(p) \subset S(\gamma, V)$,

By Lemma 3.4 [6] X is stratifiable over α.

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〈요 약〉

이 논문에서는 *CS-Semistratifiable* 공간보다 더 일반화된 공간 *Cn-Semistratifiable*을 정의하며 그에 따른 여러가지 성질들을 조사하였다.

위상 공간(X, τ)에 대하여 $\alpha \times \tau$에서 X의 폐집합족으로의 함수 S가 존재하여 다음 조건들을 만족할 때 공간X는 *Cn-Semistratifiable over* α라 정의한다.

a) 임의의 개집합 U에 대하여 $U = U\{S(\beta, U) : \beta < \alpha\}$

b) U, V가 X의 개집합이고 $U \subset V$이면 모든 $\beta < \alpha$에 대하여 $S(\beta, V) \subset S(\beta, V)$ 이다.

c) 만약 $\gamma < \beta < \alpha$ 이라면 임의의 개집합 U에 대하여 $S(\gamma, U) \subset S(\beta, U)$ 이다.

d) X의 수렴하는 *net* $\chi_\beta \rightarrow \chi$와 χ를 품는 임의의 개집합 U에 대하여 적당한 $\beta < \alpha$가 존재하여 $\chi \in S(\beta, U)$ 이고 $\{\chi_\beta\}$는 $S(\beta, U)$ 안에 *eventual*하게 들어간다.

위의 정의에 의하여 다음과 같은 성질들이 증명되었다.

1. *Strstifiable over* $\alpha \rightarrow$ *cn-semistratifiable over* $\rightarrow$ *semistratifiable over* α
2. 어떤 공간이 *cn-semistratifiable over* α이기 위한 필요충분 조건은 그것이 *linearly cushioned cn-pairnet*를 갖는 것이다.
3. *cn-semistratifiable over* α의 부분공간 역시 *cn-semistratifiable over* α 하다.
4. *cn-semistratifiable over* α의 유한개의 적공간 역시 *cn-semistratifiable over* α한다.
5. 폐 *cn-semistratifiable over a* 부분공간들의 합공간 역시 *cn-semistrbtifiable over* α 하다.
6. 폐연속 *net-cevering* 함수에 의하여 *cn-semistratifiable over* α 성질이 보존된다.

Introduction

In 1972, the concept of a linearly stratifiable space was introduced by J. E. Vaughan [9]

The class of linearly stratifiable space is composed of special sedclasses called α—stratifiable spaces (Where α is an infinite cardinal number) of which the class of stratifiable spaces is the subclass corresponding to the first infinite cardinal.

An analogous extension of the concept of a semistratifiable space [1] was introduced by K. B. Lee [4]

In this paper, a cn—semistratifiable over α is defined and some results will be given throughout this paper, all spaces will be T_1.