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The Spherical Derivative Near An Essential Singularity

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1. Introduction

In this paper we investigate the behavior of a meromorphic function in a neighborhood of an essential singularity.

In our discussion we need the concept of spherical distance. For a presentation of this see [3]. Let $P=C\cup\{\infty\}$ denote the extended complex plane or Riemann sphere. For geometric purposes we view P as the sphere in $\mathbb{R}^3$ with center $(0,0,0)$ and radius 1. The identification is given by explicitly by stereographic projection. A circle on P is called a great circle if its image under stereographic projection is a great circle. Similarly, the open unit disk D in the complex plane can be regarded as a hemisphere.

The spherical metric is the Riemannian metric

$$\lambda_P(z)\,|dz|=\frac{|dz|}{1+|z|^2}$$

on P which is half the pull-back via stereographic projection of the restriction of the euclidean metric to the sphere in R^3. The spherical metric has constant Gaussian curvature 4. The spherical distance between z and w in P is defined by

$$d_P(z,w)=\inf\int_\delta \lambda_P(\zeta)\,|d\zeta|,$$

where the infimum is taken over all paths δ on P joining z and w. In fact, this infimum is a minimum. The minimum value is attained for the shorter arc γ of any great circle through z and w. The arc γ is unique unless z and w are antipodal points; when z and w are antipodal then either of the subarcs of any great circle through z and w is a possible choice for γ. In general, any path γ that satisfies

$$d_P(z,w)=\int_\gamma \lambda_P(\zeta)\,|d\zeta|$$

is called a spherical geodesic. Explicitly,

$$d_P(z,w)=\begin{cases} \arctan(|z-w|/|1+\bar{w}z|) & \text{if } z,w\in C \\ \arctan(1/|z|) & \text{if } z\in C,\ w=\infty, \end{cases}$$

and $d_P(z,w)$ is half the angle at the center of the sphere that is subtended by any geodesic.

2. The spherical derivative

Let f be a meromorphic function in a plane region, and let a be pole of f order m. Then

$$f(z)=g(z)+\frac{A_m}{(z-a)^m}+\cdots+\frac{A_1}{(z-a)}$$

for z in some disk about a and g holomorphic in that disk. This yields that

$$\lim_{z\to a}\frac{|f'(z)|}{1+|f(z)|^2}=\begin{cases}0 & \text{if } m\geqq 2\\ \dfrac{1}{|A_1|} & \text{if } m=1.\end{cases}$$

The spherical derivative $f^{\#}(z)$ of $f(z)$ is defined by

$$f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2}$$

whenever z is not a pole of f, and

$$f^{\#}(a)=\lim_{z\to a}\frac{|f'(z)|}{1+|f(z)|^2}$$

if a is a pole of f. We note that $f^{\#}$ is a complex-valued continuous function.

Theorem 1. *Let $f(z)$ be a meromorphic function in $D^*=\{z: 0<|z|<1\}$ and have an essential singularity at the origin. Then there exists a meromorphic function g in D^* and a real number θ in $[0, 2\pi)$ such that $g^{\#}=f^{\#}$, and $G_\theta(z)=g(z)\bar{g}(\bar{z}e^{i\theta})$ has an essential singularity at the origin.*

Proof First, suppose there is a sequence (z_n) in D^* with $z_n\to 0$ and $f(z_n)=0$ for all n. Because f has just countably many poles in D^*, it is possible to select θ in $[0, 2\pi)$ so that $\bar{z}_n e^{i\theta}$ is not a pole of f for all n. Fix such a value of θ. Then the function $F_\theta(z)=f(z)\bar{f}(\bar{z}e^{i\theta})$ is meromorphic in D^*. Since $F_\theta(z_n)=0$ for all n, it follows that $F_\theta(z)$ has an essential singularity at the origin.

Now, suppose such a sequence (z_n) does not exist. The big Picard Theorem implies that for any a in P, with at most two exceptions, there is a sequence (z_n) in D^* with $z_n\to 0$ and $f(z_n)=a$ for all n. Fix such a value a. Then $R(w)=(w-a)/(1+\bar{a}w)$ is a rotation of P and $g=R\circ f$ is meromorphic in D^*. Since the spherical derivative is invariant under rotations of the sphere, we have $g^{\#}(z)=f^{\#}(z)$ for all z in D^*. Clearly, $g(z_n)=0$ for all n, so the first part of the proof shows that there is a real number θ in $[0, 2\pi)$ such that $G_\theta(z)=g(z)\bar{g}(\bar{z}e^{i\theta})$ has an essential singularity at the origin.

Remark. *Set*

$$f(z)=\prod_{n=1}^{\infty}\left[1-\left(\frac{1}{nz}\right)^{3^n}\right]\Big/\prod_{n=1}^{\infty}\left[1+\left(\frac{1}{nz}\right)^3\right]^n.$$

Then f is meromorphic on $P-\{0\}$ and has an essential singularity at the origin. If $\theta=\pi j/3^m$, for j an odd integer and m a positive integer, then $F_\theta(z)=f(z)\bar{f}(\bar{z}e^{i\theta})$ is a rational function. Thus, $F_\theta(z)$ does not have an essential singularity at the origin for a countable, dense set of value in $[0, 2\pi)$. For $\theta\neq j/3^m$ the function $F_\theta(z)$ does have an *essentia singularity at the origin.*

Theorem 2. *Let $f(z)$ be a meromorphic function in D^* and have an essential singularity at the origin. Then*

$$\limsup_{z\to 0} |z| f^{\#}(z) \geqq \frac{1}{2}.$$

Proof. By Theorem 1, we can choose a meromorphic function g in D^* and a real number θ in $[0, 2\pi)$ such that $g^{\#} = f^{\#}$ and $G_\theta(z) = g(z)\bar{g}(\bar{z}e^{i\theta})$ has an essential singularity at the origin. The Casorati-Weierstrass Theorem implies that for every $\varepsilon > 0$ there is a sequence (z_n) in D^* with $z_n \to 0$ such that $|G_\theta(z_n) + 1| < \varepsilon$. The points $g(z_n)$ and $g(\bar{z}_n e^{i\theta})$ lie almost diametrically opposite on the Riemann sphere, and hence the spherical length L of the image of $|z| = |z_n|$ by $g(z)$ is greater than $\pi - \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let γ be the image of $|z| = |z_n|$ by $g(z)$. Then

$$L \leqq \int_\gamma \lambda_P(w)|dw| = \int_{|z|=|z_n|} \frac{|g'(z)||dz|}{1+|g(z)|^2}$$
$$\leqq 2\pi |z_n| \max g^{\#}(z).$$

Combining these two inequalities, we obtain

$$\limsup_{z\to 0} |z| f^{\#}(z) = \limsup_{z\to 0} |z| g^{\#}(z) \geqq \frac{1}{2}.$$

Now we give a brief introduction to the hyperbolic metric. For a general discussion of the hyperbolic metric we refer the reader to [1] and [2].

Let G be a hyperbolic region in the complex plane; that is, the complement of G in C contains at least two points. Then there is a holomorphic universal covering projection f of the open unit disk D onto G. If G is simply connected, then f is just a one-to-one conformal mapping of D onto G. The hyperbolic metric $\lambda_G(z)|dz|$ on G is defined as follows: if $a \in G$ and $b \in f^{-1}(a)$, then

$$\lambda_G(a) = 1/|f'(b)|(1-|b|^2).$$

The value of $\lambda_G(a)$ is independent of both the choice of $b \in f^{-1}(a)$ and the selection of the covering f. It follows from the definition of the hyperbolic metric that

$$\lambda_G(f(z))|f'(z)| = \frac{1}{1-|z|^2}$$

whenever f is a holomorphic universal covering projection of D onto G.

Example. Let $G = \{z: 0 < |z| < R\}$. The function

$$w = f(z) = R\exp\left(\frac{z+1}{z-1}\right): D \to G$$

is a holomorphic universal covering projection. We have

$$\lambda_G(f(z))|f'(z)| = \lambda_D(z) = \frac{1}{1-|z|^2},$$

$$\lambda_G(w)|w| \frac{2}{|z-1|^2} = \frac{1}{1-|z|^2},$$

$$\lambda_G(w) = \frac{1}{2|w|} \cdot \frac{|1-z|^2}{1-|z|^2}.$$

Since $w = R\, exp(\frac{z+1}{z-1})$, it follows that

$$\frac{|w|}{R} = |exp(\frac{z+1}{z-1})| = exp(Re\frac{z+1}{z-1})$$

$$= exp(Re\ \frac{|z|^2 - 1^2(z-\bar{z})}{|z-1|^2})$$

$$= exp\frac{|z|^2-1}{|z-1|^2};$$

hence $log\frac{R}{|w|} = \frac{1-|z|^2}{|z-1|^2}$. Therefore, the hyperbolic metric $\lambda_G(w)|dw|$ on G is

$$\lambda_G(w)|dw| = \frac{1}{2|w|log(R/|w|)}.$$

A meromorphic function f on a hyperbolic region G is called a normal function if

$$sup\{\frac{f^{\#}(z)}{\lambda_G(z)} : z \in G\} < \infty.$$

Theorem 3. *Let f be a meromorphic function in D^*. If f has an essential singularity at the origin, then f can not be normal in D^*.*

Proof. If f is normal in D^*, then there exists a positive number M such that

$$f^{\#}(z) \leqq M\lambda_{D^*}(z) = \frac{1}{2|z|log(1/|z|)}$$

for all z in D^*. This yields

$$\lim_{z\to 0} sup\ |z|f^{\#}(z) = 0.$$

But $\lim_{z\to 0} sup\ |z|f^{\#}(z) \geqq \frac{1}{2}$, since f has an essential singularity at the origin. This contradiction establishes the theorem.

References

1. L. V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory,* McGraw-Hill, New York, 1973.
2. G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable.* Translations of Mathematical Monographs, 26, *Amer. Math. Soc.,* Providence, 1969.
3. D. Minda, *The hyperbolic metric and Bloch constants for spherically convex regions,* Complex Variables Theory Appl., 5(1986), 127-140.