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A characterization of Banach spaces with the weak Radon-Nikodym property

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1. Introduction

In 1974 Huff, Davis, Phelps [2] proved the following theorem.

Theorem(Huff-Davis-Phelps) : A Banach space containing a bounded nondentable set is a Banach space without the Radon-Nikodym property.

In this paper, we will extend this result and obtain the following theorem.

Theorem 3.4 : A Banach space containing a bounded non-weak-norm-one dentable set is a Banach space without the weak Radon-Nikodym property.

2. Preliminaries

Throughout this paper $\mathcal{X}$ is a Banach space with dual $\mathcal{X}^*$. The closed unit ball in $\mathcal{X}$ will be denoted by B_X. $([0, 1], \Sigma, \mu)$ 나denote the Lebesgue measure space on $[0, 1]$.

Definition 2.1 : Let $T : L_1(\mu) \rightarrow \mathcal{X}$ be a bounded linear operator. The operator T is said to be a Dunford-Pettis operator if T takes weakly compact sets into norm compact sets.

Definition 2.2 : Let (Ω, Σ, μ) be a finite measure space and $T : L_1(\mu) \rightarrow \mathcal{X}$ be a bounded linear operator. T is said to be Pettis representable if there exists a Pettis integrable function $g : \Omega \rightarrow \mathcal{X}$ such that

$$x^*T(f) = \int_\Omega f x^* g \, d\mu$$

for every $f \in L_1(\mu)$ and for all $x^* \in X^*$. g is called the Pettis kernel of T. The following theorem is due to [5].

Theorem 2.3*(Stegall) : Let $T : L_1[0, 1] \rightarrow X$ be Pettis representable. Then T is a Dunford-Pettis operator.*

Definition 2.4 : Let Σ_0 be a sub σ-algebre of Σ and $f \in P(\mu, X)$. Suppose $g \in P(\mu, X)$ is such that g is weakly measurable with respect to Σ_0 and $(P)-\int_A g d\mu = (P)-\int_A f d\mu$ for all $A \in \Sigma_0$, then g is the Pettis conditional expectation of f with respect to Σ_0, denoted $g = (P)-E(f| \Sigma_0)$. Also if f, $g \in L_1(\mu, X)$, with g strongly measurable with respect to Σ_0 and $\int_A g d\mu = \int_A f d\mu$ for all $A \in \Sigma_0$, then g is called the(Bochner) conditional expectation of f with respect to Σ_0, usually denoted $g = E(f|\Sigma_0)$.

Definition 2.5 : For $g \in P(\mu, X)$ we define its Pettis norm

$|||\, g\, ||| = \sup\{ \int |x^*g| d\mu : x^* \in X^*, \| x^* \| \leq 1\}$.

A martingale (g_n) in $P(\mu, X)$ is called Pettis Cauchy if (g_n) is a Cauchy sequence for the Pettis norm.

The following theorem due to several authors allows the pontential to use martingales to study Dunford-pettis operators on $L_1[0, 1]$ in much the same way that martingales are used to study representable operators on $L_1[0, 1]$.

Theorem 2.6 : *An operator $T : L_1[0, 1] \to X$ is a Dunford-Pettis operator if and only if the martingale (ξ_n, Σ_n) associated with T is Cauchy in the Pettis norm ; i. e.*

$$\lim_{n, m} \lim_{\|x^*\| \leq 1} \int |x^* \xi_n - x^*_m| \; d\mu = 0, \; x^* \in X^*.$$

Definition 2.7 : A subset D of a Banach space is not dentable if there exists an $\varepsilon > 0$ such that, for each $x \in D$, $x \in \overline{co}(D \backslash B_\varepsilon(x))$ where $B_\varepsilon(x) = \{y : \| y - x \| < \varepsilon\}$ and $\overline{co}(D \backslash B_\varepsilon(x))$ is the closed convex hull of $(D \backslash B^*\varepsilon(x))$.

3. Weak-norm-one dentable sets and the weak Radon-Nikodym property

In this section, we get a characterization of Banach spaces with the weak Radon-Nikodym property.

Definition 3.1 : A subset D of a Banach space X is not weak-norm-one dentable if there is an $\varepsilon > 0$ such that for each finite subset F of D, there is $x^*_F \in X^*$ with $\|x^*_F\| = 1$ with the property that $x \in F \Rightarrow x \in \overline{co}(D \cap \{y \in X : |x^*_F(x - y)| \geq \varepsilon\})$. Naturally a non-weak-norm-one dentable set is not dentable, i.e., dentability implies weak-norm-one dentability.

The following Lemma is needed to our main theorem.

Lemma 3.2 : *Let X be a Banach space containing a bounded non-weak-norm-one dentable set D. Choose $\varepsilon > 0$ to satisfy the criterion of Definition 3.1. Then there exists a sequence of finite partitions $\sigma\pi_n$ of $[0, 1)$ into half-open intervals and a sequence of (f_n) of finitely valued functions on $[0, 1)$ such that*

(a) *Each f_n has the form $f_n = \sum_{E \in \pi_n} x_E \chi_E$ where $x_E \in D$ for all $E \in \pi_n$.*

(b) *$|||\, f_n - f_{n+1} ||| \geq \varepsilon$ for all n.*

(c) *$\| \int_E (f_m - f_n) d\mu \| < \mu(E)/2^n$ for all $E \in \pi_n$ and all $m \geq n$.*

Proof : To construct (f_n) and (π_n), choose $\bar{x}$ arbitrarily in D and set $f_1 = \overline{X}\, X_{[0, 1)}$ and $\pi_1 = \{[0, 1]\}$. Suppose π_n and $f_n = \sum_{E \in \pi_n} x_E \chi_E$ have been defined with $x_E \in D$ for all $E \in \pi_n$ and with each $E \in \pi_n$ a half-open interval. Choose k and $0 = c_0 < c_1 < \cdots < c_k = 1$ (depending on n) so that $\pi_n = \{[c_{j-1}, c_j)] : 1 \leq j \leq k\}$. Fix j, set $E(=E_j) = [c_{j-1}, c_j)$ and let x_E be the value of f_n on E. Now by assumption we may choose elements $x_1, \cdots, x_m$ in D and positive numbers $\lambda_1, \cdots, \lambda_m$ with $|x^*(x_i) - x^*(x_E)| \geq \varepsilon$ for all $i = 1, \cdots, m$ and for some $x^* \in X^*$ with $\|x^*\| =$

1 and $\| x_E - \sum_{i=1}^{m} \lambda_i x_i \| < \frac{1}{2^{n+1}}$. Set $d_0 = c_{j-1}$ and define $d_1, \cdots, d_m$ successively by $d_i - d_{i-1} = \lambda_i (c_j - c_{j-1})$. It follows that $c_{j-1} = d_0 < d_1 < \cdots < d_m = c_j$. Now $\pi_{n+1} = \{[d_{i-1}, d_i) : 1 \leq i \leq m\}$. Again fixing j, define f_{n+1} on E by $f_{n+1}\chi_E = \sum_{i=1}^{m} x_i \chi[d_{i-1}, d_i]$. Do this for every $E \in \pi_n$ and note that $||| f_n - f_{n+1} ||| \geq \varepsilon$. Indeed, for each $E \in \pi_n$ and for all $x^* \in X^*$ with $\| x^* \| \leq 1$, $\int_E | x^* f_n(t) - x^* f_{n+1}(t) | d\mu(t) = \sum_{i=1}^{m} \int_E | x^*(x_E) - x^*(x_i) | \chi[d_{i-1}, d_i)(t)\, d\mu(t)$
$= \sum_{i=1}^{m} | x^*(x_E) - x^*(x_i) | \mu(E_j \cap [d_{i-1}, d_i))$.

Thus

$$||| f_n - f_{n+1} ||| = \sup_{\| x^* \| \leq 1} \int_{[0,1]} | x^* f_n - x^* f_{n+1} | d\mu = \sup_{\| x^* \| \leq 1} \sum_{j=1}^{k} \int_{E^j} | x^* f_n - x^* f_{n+1} | d\mu$$

$$= \sup_{\| x^* \| \leq 1} \sum_{j=1}^{k} \sum_{i=1}^{m} | x^*(x_E) - x^*(x_i) | \mu(E_j \cap [d_{i-1}, d_i))$$

$$\geq \sum_{j=1}^{k} \sum_{i=1}^{m} | x^*_F(x_E) - x^*_F(x_i) | \mu(E_j \cap [d_{j-1}, d_j)) \text{ where } F = \{x_1, x_1, \cdots, x_m\} \geq \sum_{j=1}^{k} \sum_{i=1}^{m} \varepsilon \mu(E_j [d_{i-1}, d_i)) = \varepsilon.$$

This establishes statement (b). Furthermore note that

$$\left\| \int_E (f_n - f_{n+1}) d\mu \right\| = \left\| x_E - \sum_{i=1}^{m} \lambda_i x_i \right\| \mu(E) < \frac{\mu(E)}{2^{n+1}}$$

This establishes statement (c) and completes the proof.

We need the following Theorem to our main theorem.

Theorem 3.3 : *Let X be a Banach space containing a bounded non-weak-norm-one dentable set D. Then there is a non-Pettis Cauchy martingale with values in D.*

Proof : By the above Lemma(c), we see that $F(E) = \lim_n \int_E f_n d\mu$ exists for all $E \in \bigcup_n \pi_n$. Write $g_n = \sum_{E \in \pi_n} \frac{F(E)}{\mu(E)} \chi_E (0/0)$.

If $\sum_n$ be the σ-algebra generated by π_n, then $(g_n, \sum_n)$ is a martingale in $L_1([0, 1), X)$. Moreover since D is bounded $(g_n, \sum_n)$ is a uniformly bounded martingale. Now for each $x^* \in X^*$ with $\| x^* \| \leq 1$, $\int_{[0,1)} | x^* f_n - x^* g_n | d\mu$

$$= \sum_{E \in \pi_n} | x^*(x_E)\mu(E) - x^* F(E) | \leq \sum_{E \in \pi_n} \| x_E \mu(E) - F(E) \|$$

$$= \lim_m \sum_{E \in \pi_n} \left\| \int (f_n - f_m) d\mu \right\| \leq \sum_{E \in \pi_n} \frac{\mu(E)}{2^n} = \frac{1}{2^n}.$$

Thus $||| f_n - g_n ||| \to 0$ as $n \to \infty$ i.e. $(f_n - g_n)$ is a Pettis Cauchy sequence in $L_1([0, 1], X)$. Glancing at (b), we see that (f_n) is not Pettis Cauchy.

Thus(g_n) is not Pettis Cauchy and completes the proof.

The following theorem is the main theorem of this section.

Theorem 3.4 : *A Banach space X containing a bounded non-weak-norm-one dentable set is a Banach space without the weak Radon-Nikodym property.*

Proof : Let (g_n) be the martingale in the above theorem, and let $T : L_1[0, 1) \to X$ be the associated bounded linear operator. Then by the theorem 2.6. T is not Dunford-Pettis and therefore T is not Pettis representable by the theorem 2.3. Hence X does not have the weak Radon-Nikodym property. This completes the proof.

References

1. J. Bourgain, *Martingales in conjugate Banach spaces* (unpublished).
2. J. Diestel and J.J. Uhl, Jr., Vector Measures, Math surveys 15, *Amer. Math. Soc.* Providence (1977).
3. N. Dunford and J.T. Schwartz, *Linear Operators, Part I,* Interscience, New York, 1958.
4. L.H. Riddle, Weak Radon-Nikodym sets in dual Banach spaces, PH.D. *Thesis, University of Illinois,* 1982.
5. J.J. Uhl, Jr., *Pettis Representability* (unpublished).
6. Walter Schachermayer, *Operators from L_1 to Banach spaces and Subsets of* L_∞ (reprint).