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Derivatives of Inner Functions on the Extension of H^p Spaces

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Let u denote the open unit disc in the complex plane. A function bounded and holomorphic in u is said to be an inner function $\psi(z)$ if its boundary values have modulus I almost everywhere. A Blaschke sequence is a (finite or infinite) sequence $\{a_n\}$ of complex numbers satisfying the conditions; $0<|a_n|<1$ and $\sum(1-|a_n|)<\infty$. An important class of inner functions is the Blaschke product. A Blaschke product $B(Z)$ with zeros $\{a_n\}$ is a function defined by a formula,

$$B(Z)=\prod_n \frac{|a_n|}{a_n}\ \frac{a_n-Z}{1-\ a_nZ}$$

for a Blaschke sequence $\{a_n\}$. The set of Blaschke products is uniformly dense in the set of inner function by the Frostman's theorem [6].

Now we introduce the definition of $A^p{}_q$ class and develop some of its properties. If $f(Z)$ is holomorphic in u and $0<P<1$ and $q>0$ we can define the weighted L^q norm by

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^q(1-r)^{1/P-2}\, d\theta dr.$$

If this is finite we say $f(Z)$ belongs to $A^p{}_q$. Especially, when $q=1$ we denote $A^p{}_q$ by B^p class. At that time $(q=1)$, there are many interesting results in B^p. For example, the hardy class $H^p(0<P<1)$ is dense subset of B^p, moreover, B^p is a Banach space with the norm of $A^p{}_q$ though H^p $(0<P<1)$ is not normable.

It is true the derivative of each finite Blaschke product is obviously in every H^p class. In the case of an infinite Blaschke product (abbreviate Blaschke product) D. Protas [12] showed that if $\sum(1-|a_n|)^P<\infty$ for some $P(0<P<1/2)$, then $B'\in B^{1-P}$ and $B'\in B^{1/P+1}$ for $P(0<P<1)$. See [9] and [8]. on the other hand the derivatives of inner functions have been established in [1], [2], [4]. P. Ahern [3] first considered the problems of determining the derivative of inner functions in $A^p{}_q$ classes.

In this paper we consider the problems of determining the values of P of ψ' and B' in $A^p{}_q$ for inner function and Blaschke product.

Theorem I gives sufficient conditions for the relation $\psi'\in A^p{}_q$ of inner function in some other case, and we show in Theorem 3 that codition $\sum(1-|a_n|)^q<\infty$ $(1/2<q<1)$ is necessary for $B'\in A^p{}_q$ for $0<P<1/2q$.

If $\psi(Z)$ is an inner function then the following fact is satisfied by the Sterling's formula

[3].

Lemma 1.

If $\psi(Z) = \sum a_n Z^n$, *then* $\int_0^1 \int_0^{2\pi} |\psi'(re^{i\theta})|^2 (1-r)^{1/P-1} d\theta dr = \sum |a_n|^2 n^{2-1/P}, \quad 0<P<1.$

Now to try to find the derivative value of inner function in A^p_q *classes, we introduce the next several statements.*

Lemma 2. *For any* $q>0, 0<r<1, r^q < \frac{1}{(1-r)^q}$

Lemma 3. $(a+b)^p \leqq a^p + b^p$ *for* $0<P<1$, $a\geqq 0$, $b\geqq 0$.

Since if $\psi(Z)=\sum a_n Z^n$ *is a member of* B^p, *then we truely know that*

$$a_n \leqq c_P \int_0^1 \int_0^{2\pi} |\psi(re^{i\theta})| (1-r)^{1/P-2} d\theta dr$$

and $a_n = o(n^{1/P-1})$.

Conversely, if $0<P<1$ *and* $a_n = o(n^K)$, $k\langle 1/P, -3/2$, *then* $\psi \in B^p$. *The exponent* $(\frac{1}{P} - \frac{3}{2})$ *is best possible because there exists a function* $\psi(Z) = \sum a_n Z^n$ *such that* $a_n = o(n^K)$ *with* $k = \frac{1}{P} - \frac{3}{2}$, *yet* $\psi \notin B^p$.

The condition $\psi' \in B^p$ for any inner function are known in [12], [1] but now we apply this to A^p_q class from obtaining ideas in methods of B^p.

Theorem 1. *Let* $\psi(Z) = \sum_{n>k} a_n Z^n$ *be an inner function then the necessary condition of* $\psi' \in A^p_q (q = \frac{1}{2})$ *is* $a_n = o(\frac{1}{n})$ *and* $0<P<\frac{2}{3}$.

Proof By Lemma 1, Lemma 2, and Lemma 3, we have that

$$\int_0^1 \int_0^{2\pi} |\psi'(re^{i\theta})|^{1/2} (1-r)^{1/P-2} d\theta dr$$
$$\leqq \sum_{n>k} n^{1/2} |a_n|^{1/2} \int_0^1 r^{\frac{n-1}{2}} (1-r)^{\frac{1}{P}-2} dr$$
$$\leqq \sum_{n>k} n^{1/2} |a_n|^{1/2} \int_0^1 r^{1/2} (1-r)^{\frac{1}{P}-2} dr$$
$$\leqq \sum_{n>k} n^{1/2} |a_n|^{1/2} \int_0^1 (1-r)^{\frac{1}{2}+\frac{1}{P}-2} dr, \quad (k= 1, 2, 3\cdots).$$

On the other hand, $\int_0^1 (1-r)^t dr$ is finite for any numbwes $t>-1$.

Therefore the theorem is complete.

· · Continuing this process, we claim the following fact.

Theorem 2. *If* $\psi(Z) = \sum_{n>k} a_n Z^n$ *is an inner function,* $q = \frac{1}{n}$ $(n = 2, 3\cdots)$ *and* $a_n = o(\frac{1}{n})$, *then* $\psi' \in A^p_q$ *when* $0<P<\frac{n}{n+1}$.

since $\int\int |\psi'(re^{i\theta})|^q (1-r)^{1/P-2} dr d\theta \leqq \int\int (1-r)^{-q+\frac{1}{p}-1} dr d\theta$

with $\frac{1}{g+1} \langle P \langle \frac{1}{g}$ *for* $1\leqq q \leqq 2$, *so Theorem 2 has the following restatement.*

Corollary 1. *If* $\frac{1}{q+1} < P < \frac{1}{q}$, *then* $\psi' \in A^P_q$ *if and only if* $\psi' \in B^t$ *with* $t = P/(1-P(q-1))$.

The equality $P = \frac{1}{q+1}$ is not satisfied, for example, if $q=2$ then $P = \frac{1}{3}$ and $\int\int |\psi'(re^{i\theta})|^1 (1-r) d\theta dr = \sum |a_n|^2 < \infty$ but if $q = 1$ then $\int_0^1 \int_0^{2\pi} |\psi'(re^{i\theta})| dr d\theta$ does not always converge.

Next we consider the derivative of Blaschke products. What conditions are necessary to do $B' \in A^P_q$?. Since $\int\int |B'(re^{i\theta})|^2 dr d\theta$ is finite if and only if B is a finite Blaschke product.

Suppose $\psi(Z)$ is an inner function and $P > \frac{1}{q}$ ($1 \leqq q \leqq 2$) then $\psi' \in A^P_q$ unless ψ is a finite Blaschke. But every inner function has decomposition $\psi(Z) = e^{ic}B(Z)s(Z)$,

Where c is a real constant, $S(Z)$ is a singular inner function.

Let us restrict our attention to finite Blaschke product then we have following results.

Theorm 3. *Let* $B(Z)$ *be a finite Blaschke product with zeros* $\{a_n\}$ *such that* $\sum_n (1-|a_n|)^q < \infty$ *for some* q ($\frac{1}{2} < q < 1$), *than* $B' \in A^P_q$ *when* $0 < P < \frac{1}{2q}$.

The proof requires the following statements.

Lemma 4. *For each* $P < \frac{1}{2}$,

$$\int_0^{2\pi} \frac{d\theta}{(1-2r\cos\theta + r^2)^P} = 0\left(1(1-r)^{2P-1}\right) \text{ as } r \to 1 \qquad [5].$$

Lemma 5. *For fixed* P ($\frac{1}{2} < P < 1$), *there exists a constant* M *such that*

$$\int_0^{2\pi} \frac{d\theta}{|1-\bar{a}_n re^{i\theta}|^{2P}} \; M(1-r)^{1-2P}$$

for $n = 1, 2, 3\cdots$, *and all* r $(0<r<1)$.

Proof. By Lemma 4,

$$\int_0^{2\pi} \frac{d\theta}{|1-\bar{a}_n re^{i\theta}|^{2P}} = \int_0^{2\pi} \frac{d\theta}{(1-r^2|a_n|^2 - 2r|a_n|\cos\theta)^P} < M(1-r)^{1-2P}$$

Proof of Theorem 3. The derivative of $B(Z)$ is the following formula;

$$B'(Z) = \sum_n (A_n(Z)(1-|a_n|^2)/(1-\bar{a}_n Z)^2$$

where $A_n(Z) = B(Z)(1-a_n Z)/(Z-a_n)$. This implies that

$|B'(Z)| < 2 \sum_n (1-|a_n|)/|1-\bar{a}_n Z|^2$ for all $|Z|<1$. Since $\frac{1}{2} < q < 1$,

$|B'(Z)|^q < 2^q \sum_n (1-|a_n|)^q / |1-\bar{a}_n Z|^{2q}$

Let us integrate each side and use Lemma 5, then we obtain the inequality

$$\int_0^1 \int_0^{2\pi} |B'(re^{i\theta})|^q (1-r)^{\frac{1}{P}-2} d\theta dr < 2^q M \sum_n (1-|a_n|^q \int_0^1 (1-r)^{-1-2q+\frac{1}{P}} dr.$$

Since $0 < P < \frac{1}{2q}$ implies $-1-2q+\frac{1}{P} > -1$ Thus the proof is complete.

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