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Approximate Solutions of a Nonlinear Population Dynamics by Finite Difference Methods

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1. Introduction

We connsider a nonlinear population dynamics problem

$$
(1.1)\quad \begin{cases} \rho t+\rho a+\lambda(a,\ p(t)\rho=0 & a>0, \quad t>0 \\ \rho(a,\ 0)=\rho(a) & a\geqq 0 \\ \rho(0,\ t)=\int_0^\infty \beta(a,\ p(t))\ \rho(a,t)\,da\ , & t>0, \end{cases}
$$

,where $\rho(a,t)$ is the pupulation at age a, time t, $p(t)$ the total population at t, $\lambda(a,\ p(t))$ the death modulus, and $\beta(a,\ p(t))$ the birth modulus.

Problem (1.1) has been first studied by Gurtin-MacCamy's [1] in 1974.
And Marcus [3] obtained an equivalent solution of (1.1) as Gurtin-MacCamy's.

In this paper, we will construct an approximate solution of (1.1) to the solution of Marcus sense in the spirit of Kannan and Ortega [2].

2. Definitions of Solutions

We construct approximate solutions of (1.1) using finite difference methods. We assume the following conditions as Gurtin-MacCamy did.

(2.1) $\lambda,\ \beta$: $[0,\infty]\times[0,\infty]\rightarrow(0,\infty)$ are continuous functions.

(2.2) $\sup\beta=\bar{\beta}<\infty$, $\sup\ \lambda=\bar{\lambda}<\infty$

(2.3) λ and β satisfy the Lipschitz condition with respect to p, that is, there exists a positive number L such that

$$|\beta(a,\ p_1)-\beta(a,\ p_2)|\ \leqq L|\ p_1-p_2|$$
$$|\lambda(a,\ p_1)-\ \lambda(a,\ p_2)|\ \leqq L|\ p_1-\ p_2|$$

(2.4) The initial population $\psi(a)$ is nonnegative and measurable.

We now introduce concepts of solutions of the problem (1.1)

Definition 2.1. we define the solution of (1.1) in the sense of Marcus.

For $\rho\in C([0,\ T]),\ L(0,\ \infty)$, we define

$$
(2.5)\quad T_\rho(a,\ t)=\begin{cases}\psi(a-t)-\int_0^t \lambda(a-\tau,\ p(t-\tau)\ \rho(a-\tau,\ t-a)\ d\tau, & a>t \\ B(a-t)-\int_0^a \lambda(a-\tau,\ p(t-\tau)\ \rho(a-\tau,\ t-a)\ d\tau, & a\leqq t\end{cases}
$$

, where $B(t)=\int_0^\infty \beta(a,\ p(t)\ \rho(a,\ t)\ da.$

Then $T_\rho(a,\ t)$ is the solution of (1.1)

Definition 2.2. We call $\rho(a, t)$ is an ε-approximate solution of $T\rho(a, t)$ if for any $\varepsilon(\delta)$,
$\int_0^\infty |(a, t)-T\rho(a, t)|\, da \leqq \varepsilon(\delta)$

Then From Definitions 2.1 and 2.2, we obtain the following lemma using Gronwall-Bellman's integral inequality.

Lemma 2.1. *Let $\rho_1(a, t)$ and $\rho_2(a, t)$ be $\varepsilon_1(\delta)$and $\varepsilon_2(\delta)$ -approximate solution, respectively. Assume that $\lambda(a, p(t)$ and $\beta(a, p(t)$ satisfy the Lipschitz condition with the constant L for $p(t) \leqq \bar{p}(0)e^{\bar{\beta}T}$, $t \in [0, T]$
Then $(\int_0^\infty |\rho_1(a, t)-\rho_2(a, t)|\, da) \leqq (\varepsilon_1+\varepsilon_2)e^{Kt}$, a. e. $t \in [0, T]$, where $K=\bar{\lambda}+\bar{\beta}+2L\bar{p}$*

3. Construction of Approximate Solutions.

From the character of population, we may approximate

(3.1) $\rho_t+\rho_a=\lim_{\delta\to 0}\dfrac{\rho(a+\delta, t+\delta)-\rho(a, t)}{\delta}$

Hence we can construct an approximate solution of (1.1) as follows :

(3.2) $\rho(a+\delta, t+\delta)=\rho(a, t)\cdot(1-\delta\lambda(a, p(t)))$

(3.3) $p(t)=\sum\rho(a, t)\delta$

(3.4) $\rho(0, t)=\sum\beta(a, p(t))\,\rho(a, t)\cdot\delta$

But we know that the population is always nonnegative, we need restriction on δ.
We choose δ and n so that $\delta\bar{\lambda}\leqq 1$, $\delta \leqq \dfrac{T}{n}$.
We denote $\rho_{k,j}=\rho(k\delta, j\delta)$ $p_j=p(j\delta)$
Then we have following lemmas.

Lemma 3.1. *Under the above restriction on δ,*

$\rho_{k,j}>0$

Proof. By the mathematical induction, we can provethe reuired result.

Lemma 3.2. *If we let $\underline{\lambda}=\inf\lambda(a, p(t))$, then*

$$p(t) \leqq p(0)e^{(\bar{\beta}-\underline{\lambda})t} \text{ for } t \in (0, T]$$

From Lemma 3.2, We can weaken the restriction on δ, so we have to choose δ so that

$\delta p(0)\, e^{(\bar{\beta}-\underline{\lambda})T} \leqq 1$

And we now have the following lemma which shows that $p(t)$ is continuous.

Lemma 3.3. *For* t, t' *such that* $|t-t'| < h\delta$ $|p_\delta(t')-p_\delta(t)| < M_1 h\delta$,

where M_1 *is independent of* δ.

Lemma 3.4. *efine* $\tilde{B}_\delta(t)=\rho_{0j}$ *for* $t\in[j\delta, (j+1)\delta]$

Then $|\tilde{B}_\delta(t')-\tilde{B}_\delta(t)|\leqq M_2 h\delta+\int_0^\infty w\beta(a+h\delta, h\delta)\,\rho_\delta(a, t)\,da$

if $|t-t'|\leqq h\delta$, *where* $w_\beta(a:\delta)=sup\{|\beta(a_1, p_1)-\beta(a_2, p_2)|$ $a_1, a_2[\theta, a+\delta]$, $|a_1-a_2|\leqq\delta$, $p_1, p_2\in[0,\bar{p}]$, $|p_1-p_2|\leqq M_1\delta\}$. *Where* M_2 *is independent of* δ.

Lemma 3.5 *If* $|t-t'| < h\delta$, *then* $\int_0^\infty |\rho_\delta(a, t)-\rho_\delta(a, t')|\,da \leqq M_3 h\delta+|B_\delta(t)-B_\delta(t')|$

$$+\int_0^\infty|\varphi_\delta(a+h\delta)-\varphi_\delta(a)|\,da+\int_0^{h\delta}\varphi_\delta(\xi)d\xi,$$

where M_3 *is independent of* δ.

Lemma 3.6. *If we define* $\rho_\delta(a, t)=\rho_{k,j}$ *for* $(a, t)\in[k\delta, (k+1)\delta)\times[j\delta, (j+1)\delta)$,

Then $\rho_\delta(a, t)$ *is an* ε*-approximate solution of (1.1) In fact,*

$$\int_0^\infty |\rho_\delta(a, t)-T_\rho(a, t)|\,da\leqq\bar{\delta}(\bar{\lambda}\,\bar{p}+\bar{\lambda}\,\bar{\beta}\,\bar{p})$$

We are now ready to state the main theorem.

Theorem 3.1. *Let* $\{\rho_{\delta n}\}$ *be a sequence of* ε*-approximate solutions depending on* δ.

Then the limit of the sequence is a solution of (1.1)

Proof. From Lemma 1.1., we can show that $\{\rho_{\delta n}\}$ is a Cauchy sequence in $L_\infty([\theta, T], L^1(\theta, \infty))$. And by Lemma 3.5, the limit ρ is in $C([\theta, T], L^1(\theta, \infty))$. Furthermore, we can show that the approximate solution is continuously dependent on initial data. In fact, we have the following lemma.

Theorem 3.2. *Let* ρ_φ *and* ρ_ψ *be approximate solutions of (1.1) depending on initial data* $\varphi(a)$ *and* $\psi(a)$, *respectively. Then* $\|\rho_\varphi(a, t)-\rho_\psi(a, t)\|_1\leqq\|\varphi-\psi\|_1 e^{(2L\bar{P}+\bar{\lambda}+\bar{\beta})t}$

Proof. Let φ and ψ be in $L_1(\theta, \infty)$ and let ρ_φ and ρ_ψ · be · corresponding solutions, respectively. Assume that

$$|\lambda(a, p_1)-\lambda(a, p_2)| \leq L|p_1-p_2|, \quad |\beta(a, p_1)-\beta(a, p_2)| \leq L|p_1-p_2|, \quad p_1, p_2 \leq \bar{p}$$

$$\sup \lambda(a, p)=\bar{\lambda}<\infty \qquad \sup \beta(a, p)=\bar{\beta}<\infty$$

, where $\bar{p}=\max\{\bar{p}_\varphi, \bar{p}_\psi\}$, $\bar{p}=p$

(1) $a>t$

$$|\rho_\varphi(a-t)-\rho_\psi(a-t)|$$

$$\leq |\varphi(a-t)-\psi(a-t)| + \int_0^t |\lambda(a-\tau, p_\varphi(t-\tau))\, \rho_\varphi(a-\tau, t-\tau) - \lambda(a-\tau, p_\psi(t-\tau))\, \rho_\psi(a-\tau, t-\tau)|\, d\tau$$

$$\leq |\varphi(a-t)-\psi(a-t)| + L\int_0^t |p_\varphi(t-\tau)\, p_\psi(t-\tau)| - 1 \mid p_\varphi(a-\tau, t-\tau)d\tau + \bar{\beta}\int_0^t |\rho_\varphi(a-\tau, t-\tau) - |\rho_\psi(a-\tau, t-\tau)|\, d\tau$$

We get the following inequality by integrating both sides,

$$\int_t^\infty |\rho_\varphi(a, t)-\rho_\psi(a, t)|\, da$$

$$\leq \int_t^\infty |((a-t)-\psi(a-t)|\, da + L\int_t^\infty \int_0^t |p_\varphi(t-\tau)-p_\psi(t-\tau)|\, \rho_\varphi(a-\tau, t-\tau)d\tau\, da + \bar{\lambda}\int_t^\infty \int_0^t |\rho_\varphi(a-\tau, t-\tau)-\rho_\psi(a-\tau, t-\tau)|\, d\tau\, da$$

$$\leq \|\varphi-\psi\|_1 + L\int_0^t |p_\varphi(t-\tau) \mid p_\psi(t-\tau \mid \int_t^\infty p_\psi(a-\tau, t-\tau)da d\tau + \bar{\lambda}\int_0^t \int_t^\infty |\rho_\varphi(a-\tau, t-\tau)-\rho_\psi(a-\tau, t-\tau)|\, da d\tau$$

(2) $a\leq t$

$$|\rho_\varphi(a, t)-\rho_\psi(a, t)|$$

$$\leq |B_\varphi(t-a)-B_\psi(t-a)| + \int_0^a |\lambda(a-\tau, p_\varphi(t-a))\, \rho_\varphi(a-\tau, t-\tau) - \lambda(a-\tau, p_\psi(t-\tau))\, \rho_\psi((a-\tau, t-\tau)|\, d\tau$$

$$\leq |\int_0^\infty |\beta(a, p_\varphi(t-a))\, \rho_\varphi(a, t-a) - \beta(a, p_\psi(t-a))\rho_\psi(a, t-a)|\, da + \int_0^a |\lambda(a-\tau, p_\varphi(t-a))\, \rho_\varphi(a-\tau, t-\tau) - \lambda(a-\tau, p_\psi(t-\tau))\, \rho_\psi(a-\tau, t-\tau)|\, d\tau$$

$$\leq L\int_0^\infty |p_\psi(t-a) \mid p_\psi(a, t-a)da + \bar{\beta}\int_0^\infty |\rho_\varphi(a, t-a) - \rho_\psi(a, t-a)|\, da + L\int_0^a |p_\varphi(t-\tau, p_\varphi(t-\tau)|\, \rho_\varphi(a-\tau, t-\tau)d\tau + \bar{\lambda}\int_0^\infty |\rho_\varphi(a-\tau, t-\tau)-\rho_\psi(a-\tau, t-\tau)|\, d\tau$$

Hence

$$\int_0^t |p_\varphi(a, t)-p_\psi(a, t)|\, da$$

$$\leq L\int_0^t \int_0^\infty |p_\varphi(a-t)-p_\varphi(t-a) \mid \rho_\varphi(a, t-a)\, da\, da + \bar{\beta}\int_0^t \int_0^\infty |\rho_\varphi(a, t-a)-\rho_\varphi(a, t-a)|\, da\, da + L\int_0^t \int_0^a |p_\varphi(t-\tau)-p_\psi(t-\tau)|\, \rho_\varphi(a-\tau, t-\tau)d\tau\, da + \bar{\lambda}\int_0^t \int_0^a |\rho_\varphi(a-\tau, t-\tau)-\rho_\psi(a-\tau, t-\tau)|\, d\tau\, da$$

$$\leq L \int_0^t \int_0^\infty | p_\varphi(t-\tau) - p_\psi(t-\tau)| \rho_\psi(\tau, t-\tau) d\tau\, da$$
$$+\bar{\beta} \int_0^t \int_0^\infty | \rho_\varphi(\tau, t-\tau)| \rho_\varphi(\tau, t-z)| dz\, da\, d\tau\, da$$
$$+L \int_0^t | p_\varphi(t-a) - p_\psi(t-\tau)| \int_z^t \rho_\varphi(a-\tau, t-\tau) da\, d\tau$$
$$+\bar{\lambda} \int_z^t \int_t^\tau | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da\, d\tau$$

Adding (1) and (2)

$$\int_0^\infty | \rho_\varphi(a, t) - \rho_\psi(a, t)| da$$
$$\leq \|\varphi-\psi\|_1$$
$$+L \int_0^t | p_\varphi(t-\tau) - p_\psi(t-\tau)| \int_t^\infty \rho_\varphi(a-\tau, t-\tau) da\, d\tau$$
$$+L \int_0^t | p_\psi(t-\tau) - p_\varphi(t-\tau)| \int_\tau^t \rho_\varphi(a-z, t-\tau)\, da d\tau$$
$$+\bar{\lambda} \int_0^t \int_t^\infty | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da d\tau$$
$$+L \int_0^t \int_0^\infty | p_\varphi(t-\tau) - p_\psi(t-\tau) \rho_\varphi(\tau, t-\tau) da d\tau$$
$$+\bar{\beta} \int_0^t \int_t^\infty | \rho_\varphi(z, t-\tau) - p_\psi(z, t-\tau) \rho_\varphi(\tau, t-\tau) da d\tau$$
$$\leq \|\varphi-\psi\|_1$$
$$+L\bar{p} \int_0^t | p_\varphi(t-\tau)| p_\psi(t-z) \int_\tau^\infty \rho_\varphi(a-\tau, t-\tau)\, da\, d\tau$$
$$+\bar{\lambda} \int_0^t \int_\tau^\infty | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da\, d\tau$$
$$+\bar{\beta} \int_0^t \int_0^\infty | \rho_\varphi(a, t-a) - \rho_\psi(a, t-a)| da\, d\tau$$
$$+L \int_0^t \int \int_0^\infty | p_\varphi(t-\tau) - p_\psi(t-\tau)| \rho_\varphi(\tau, t-\tau)\, da\, d\tau$$
$$\leq \|\varphi-\psi\|_1$$
$$+L\bar{p} \int_0^t | p_\varphi(t-\tau) - p_\psi(t-z)| d\tau$$
$$+\bar{\lambda} \int_0^t \int_0^\infty | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da d\tau$$
$$+\bar{\beta} \int_0^t \int_0^\infty | \rho_\varphi(a, t-a) - \rho_\psi(a, t-a)| da\, d\tau$$
$$+L\, \bar{p} \int_0^t | p_\varphi(t-\tau) - p_\psi(t-\tau)| d\tau$$
$$\leq | \varphi-\psi|_1$$
$$+L\bar{p} \int_0^t \int_0^\infty | \rho_\varphi(a, t-\tau) - \rho_\psi(a, t-\tau)| d\tau$$
$$+\bar{\lambda} \int_0^t \int_0^\infty | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da d\tau$$
$$+\bar{\beta} \int_0^t \int_0^\infty | \rho_\varphi(a, t-a) - \rho_\psi(a, t-a)| da\, d\tau$$
$$+L\bar{p} \int_0^t \int_t^\infty | \rho_\varphi(t-\tau) - \rho_\psi(t-\tau)| da\, d\tau$$
$$+L\bar{p} \int_0^t \int_0^\infty |\rho_\varphi(a, t-\tau) - \rho_\psi(a, t-\tau)| da d\tau$$
$$\leq | \varphi-\psi|_1$$
$$+2L\bar{p} \int_0^t \int_0^\infty |\rho_\varphi(a, t-\tau) - \rho_\psi(a, t-\tau)| d\tau$$
$$+\bar{\lambda} \int_0^t \int_0^\infty | \rho_\varphi(a-\tau, t-\tau) - \rho_\psi(a-\tau, t-\tau)| da d\tau$$
$$+\bar{\beta} \int_0^t \int_0^\infty | \rho_\varphi(a, t-a) - \rho_\psi(a, t-a)| da\, d\tau$$

$$=|\varphi-\psi|_1$$
$$+L\bar{p}\int_0^t\int_0^\infty|\rho_\varphi(a,t-\tau)-\rho_\psi(a,t-\tau)|\,da\,d\tau$$
$$+\bar{\lambda}\int_0^t\int_0^\infty|\rho_\varphi(a-\tau,t-\tau)-\rho_\psi(a-\tau,t-\tau)|\,d\tau\,da$$
$$+\bar{\beta}\int_0^t\int_0^\infty|\rho_\varphi(a,t-a)-\rho_\psi(a,t-a)|\,da\,da$$
$$\leq|\varphi-\psi|_1$$
$$+(2L\bar{p}+\bar{\lambda}+\bar{\beta})\int_0^t\int_0^\infty|\rho_\varphi(a,\tau)-\rho_\psi(a,\tau)|\,da\,d\tau$$

Now applying Gronwall Bellman's inequality, we get

$$\int_0^\infty|\rho_\varphi(a-t)-\rho_\psi(a,t)|\,da$$
$$\leq|\varphi-\psi|_1+\int_0^t(2L\bar{p}+\bar{\lambda}+\bar{\beta})\cdot|\varphi-\psi|\,e^{\int_s^t(2L\bar{p}+\bar{\lambda}+\bar{\beta})du}ds$$
$$=|\varphi-\psi|_1+|\varphi-\psi|_1(2L\bar{p}+\bar{\lambda}+\bar{\beta})\left(-\frac{1}{2L\bar{p}+\bar{\lambda}+\bar{\beta}}\right)(1-e^{(2L\bar{p}+\bar{\lambda}+\bar{\beta})t}$$
$$=|\varphi-\psi|_1\cdot e^{(2L\bar{p}+\bar{\lambda}+\bar{\beta})t}$$

i. e.

$$|\rho_\varphi(a,t)-\rho_\psi(a,t)|_1<|\varphi-\psi|_1e^{(2L\bar{p}+\bar{\lambda}+\bar{\beta})t}$$

We prove the continuous dependence of solution on the initial condition.

Finally, from Theorem 3.2. we obtain the uniquencess of approximate solutions of (1.1).

Corollary 3.1. *The approximate solution obtained by (3.2)−(3.4) is unique.*

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