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A Study On The Hypersurface Of *g-SEX$_n$(*)

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I. Generalized n-dimensional $^*g$-manifold x_n

Let x_n be an n-dimensional generalized Riemannian space referred to be a real coordinate system x^ν, which obeys coordinate transformation $x^\nu \leftrightarrow x^{\nu'}$ for which

(1—1) $\quad \mathrm{Det}(\frac{\partial \chi'}{\partial \chi}) = 0.$

The space x_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$;

(1-2) $\quad g_{\lambda\mu}\, h_{\lambda\mu} + k_{\lambda\mu,}$ (***)

where

(1-3) $\quad \mathrm{Det}(g_{\lambda\mu}) = 0, \ \mathrm{Det}(h_{\lambda\mu}) = 0.$

The algebraic structure is imprsed on x_n by the basic real tensor $^*g^{\lambda\mu}$ defined by

(1-4) $\quad g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\lambda\mu}{}^*g^{\nu\lambda} = \delta^\nu{}_\mu$

in virtue of (1-3). It may also split into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$;

(1-5) $\quad ^*g^{\lambda\nu} = {}^*h^{\lambda\mu} + {}^*k^{\lambda\nu}.$

Since Det($^*h^{\lambda\nu} = 0$, we may define a unique tensor $^*h_{\lambda\mu,}$ by

(1-6) $\quad ^*h_{\lambda\mu,}\, {}^*h^{\lambda\nu} = \delta^\nu{}_{\mu,}$

which together with $^*h_{\lambda\mu,}$ will serve for raising and / or lowering indices of all tensors defined in x_n in the usual manner.

The space x_n is assumed to be connected by a general real connection $\Gamma^\nu{}_{\lambda\mu}$ with the following transformation rule;

(1-7) $\quad \Gamma^{\nu'}{}_{\lambda'\mu'} = \frac{\alpha\chi^{\nu'}}{\alpha\chi^\alpha} \left(\frac{\alpha\chi^\beta}{\alpha\chi^{\lambda'}} \frac{\alpha\chi^\gamma}{\alpha\chi^{\mu'}} \Gamma^{\alpha\beta\gamma} + \frac{\alpha^2\chi^\alpha}{\alpha\chi^{\lambda'}\alpha\chi^{\mu'}} \right).$

(*) 이 논문은 1987년도 문교부 학술연구조성비 지원에 의하여 연구되었음

(***). Throuhout the present paper, Geek indices are used for the tensors in x_n and take the values 1, 2,⋯, n, while Roman indices are used for the components of tensors in a hypersurface x_{n-1} of x_n and take the values 1, 2, ⋯, $_{n-1}$. They both follow the summation convention.

II. n-dimensional $^*g$-SEmanifold x_n

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be Einstein if it satisfies the Einstein equations;

(2-1) $D_w {}^*g^{\lambda\mu} = -2\, S_{w\alpha}{}^{\mu}\, {}^*g^{\lambda\alpha}$

which is equivalent to the system of equations

(2-2) $D_w g_{\lambda\mu} = 2\, S_{w\mu}{}^{x} g_{\lambda\alpha},$

where $S_{\lambda\mu}{}^{\nu}$ is the torsion tensor of $\Gamma^{\nu}_{\lambda\mu}$, and D_w is symbolic vector of the covarient derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$.

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be semi-symmetric if its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form

(2-3) $S_{\lambda\mu}{}^{\nu} = 2\, \delta^{\nu}_{[\lambda} x_{\mu]}$ for an arbitrary vector x_{λ}.

The connection which is both semi-symmetric and Einstein is called a SE-connection. An n-dimensional *g-SE manifold (*g—SEX$_n$) is a space X_n on which the differential geometric structure is imposed by $^*g_{\lambda\mu}$ through a SE-connection.

III. The SE-connection $\Gamma^{\nu}_{\lambda\mu}$ *g—SEX$_n$

We introduce the following abbreviation for any real vector y_{λ};

(3-1) $y_{\lambda}^{(p)} = {}^{(p)*}k^{\alpha}{}_{\lambda} y_{\alpha}, (p=0,1,\cdots).$

We need the symmetric real tensor

(3-2) $c_{\lambda\mu}{}^{\nu} = {}^{(1-n)*}h_{\lambda\mu} + {}^{(2)*}k_{\lambda\mu}$

which is of rank n [see 3.], so that there exists a unique inverse tensor $D^{\lambda\nu} = D^{\nu\lambda}$ satisfying

(3-3) $c_{\lambda\mu} D^{\lambda\nu} = \delta^{\nu}_{\mu}$

In the next we state two theorems, proofs of which are given in [3].

Theorem(3.1) *If there exists a SE-connection* $\Gamma^{\nu}_{\lambda\mu}$, *it must be of the form*

(3-4) $\Gamma^{\nu}_{\lambda\mu}\ {}^*\{^{\nu}_{\lambda\mu}\} + 2\delta^{\nu}{}_{[\lambda} x_{\mu]} - {}^*h_{\lambda\mu} x^{\nu(1)}$ *For a vector* x, *where* ${}^*\{^{\nu}_{\lambda\mu}\}$ *are the Christoffel symbols defined by* ${}^*h_{\lambda\mu}$.

Theorem 3.2. *There exists a unique SE-connection* ${}^*\{^{\nu}_{\lambda\mu}\}\ \Gamma^{\nu}_{\lambda\mu}$ *if and only if there is a vector* x_{λ} *such that*

(3-5) $\nabla_w {}^*k_{\lambda\mu} = 2\, {}^*h_{w[\lambda} x_{\mu]} + 2\, {}^*K_{w[\mu} {}^*k_{\lambda]}{}^{\alpha} x_{\alpha}.$

The vector x_{λ} satisfying (3-5) is unique and may be given by

(3-6) $x_{\lambda} = D^{\alpha}{}_{\lambda} \nabla_{\beta} {}^*k^{\beta}{}_{\alpha},$

Where ∇_w is the symbolic vector of the covarient derivative with respect to ${}^*\{^{\nu}_{\lambda\mu}\}$.

Ⅳ. Geometry on a hypersurface in *g−SEX$_n$

Let a hypersurface X_{n-1} be embedded in a general X_n. Then X_{n-1} may be given by real parametric equations

(4-1) $\quad x^\alpha = x^\alpha (y^1, y^2, \cdots\cdots, y^{n-1})$

provided the matrix (B^α_i) where $B^\alpha_i = \frac{\partial x^\alpha}{\partial y^i}$, is assumed to be of rank n—1 in order to exclude singular points. Let the metric for X_n and X_{n-1} be given by $*h_{\alpha\beta} dx^\alpha dx^\beta$ and $*h_{ij} dy^i dy^j$ respectively. Then we have

(4-2) $\quad *h_{ij} = *h_{\alpha\beta} \frac{\partial X^\alpha}{\partial y^i}, \frac{\partial X^\beta}{\partial y^j}, = *h_{\alpha\beta} \beta^\alpha_i \beta^\beta_j.$

Since the rank of β^α_i is n—1, so is the matrix (h_{ij}). Hence we may define a unique tensor $*h^{ik}$ by

(4-3) $\quad *h_{ij} *h^{ik} = \delta^k_i$ which together with $*h^{ij}$ will serve for raising and / or lowering indices of the tensors in X_{n-1} in the usual manner.

Now we need the quantities B^i_α and B^α_β defined by

(4-4) $\quad B^i_\alpha = *h^{ij} *h_{\alpha\beta} B^\alpha_\beta, B^\alpha_i = B^\alpha_i B^i_\beta.$

Definition 4.1 If $T^{i\cdots}_{j\cdots}$ are the components of a tensor in X_n, the components of induced tensor on X_{n-1} derived form it are defined by

(4-5) $\quad T^{i\cdots}_{j\cdots} = T^{\alpha\cdots}_{\beta\cdots} B^i_\alpha \cdots B^\beta_j \cdots.$

Definition 4.2 If $\Gamma^\nu_{\lambda\mu}$ is a connection on X_n the connection defined by

(4-6) $\quad \Gamma^k_{ij} = B^k_r (B^r_{ij} + \Gamma^r_{\alpha\beta} B^\alpha_i B^\beta_j),$

where $B^\alpha_{ij} = \frac{\partial^2 X^\alpha}{\partial y^i \partial y^j}$, is called the induced connection on X_{n-1} derived from $\Gamma^r_{\alpha\beta}$ on X_n.

Theorem 4.3 *The induced connection* Γ^k_{ij} *on* X_{n-1} *derived form a SE-connection* $\Gamma^r_{\alpha\beta}$ *on* X_n *is of the from*

(4-7) $\quad \Gamma^k_{ij} = *\{ {}^{k}_{ij} \} + 2\, \delta^k_{[i} X_{j]} - 2 *h_{ij} \overset{(1)}{X}{}^k,$

where $*\{ {}^{k}_{ij} \}$ *are the induced Christoffel symbols on* X_{n-1} *and the* X_j *is the induced vector of that determining the SE-connection on* X_n.

Proof. Substituting from (3-3) in (4-6), we obtain

$$\Gamma^k_{ij} = B^k_r (B^r_{ij} + *\{ {}^{r}_{\alpha\beta} \} B^\alpha_i B^\beta_j) + 2\delta^r_{[\alpha} X_{\beta]} B^\alpha_i B^\beta_j B^k_r$$

$$+ 2 *h_{\alpha\beta} \overset{(1)}{X}{}^r B^\alpha_i B^\beta_j B^k_\alpha.$$

Making use of (4-5), (4-6), we have(4-7).

Theorem 4.4 *The induced connection* Γ^k_{ij} *on* X_{n-1} *derived from a SE-connection on* X_n *is a semi-symmetric. That is,* (4-8) $S^k_{ij} = 2\delta^k_{[i}X_{j]}$.

Proof. It is easily verified from (2-3)and (4-5).

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