韓國數學教育學會誌(數學教育)
1988. 12. Vol. XXVII, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1988. Vol. XXVII, No. 1

Linear Operator represented by an infinite matrix in the Hilbert Space l_2*

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I. Introduction

Let $A(a_{jk})$ be an arbitrary infinite matrix with complex entries $a_{jk}(j, k=1, 2, 3, \cdots\cdots)$. If the matrix A represents a bounded linear operator A (For the corresponding operator we shall coveniently use the same symbol A since there will be no ambiguity) in the complete Hilbert space l_2 with respective to the standard basis $\{e_i\}$, then $\|A_j\| \leqq \|A\|$ and $\|A^k\| \leqq \|A\|$ hold for all i, j. Here A_i and A^j are the i-th row and the k-th column vectors of the matrix A respectively. $\|A_j\| = \sqrt{\sum_k |a_{jk}|^2}, \|A^k\| = \sqrt{\sum_j |a_{jk}|^2}$, $\|A\|$ is operator-norm.

So we will concentrate our attention to the case that both $\|A_i\|$ and $\|A^j\|$ are bounded above, and throughout this paper we will assume that there is a positive constant M such that

(1) $\|A_j\| \leqq M$ and $\|A^k\| \leqq M$

for all j, k

unless otherwise mentioned.

Under this assumption we can prove that the vector $y=Ax$ with components $y_j = \sum_k a_{jk} x_k$, $x=(x_k) \in l_2$, belongs to c_0, the vector space of all null sequences (components are complex numbers of course).

Theorem 1. *Suppose the matrix* $A=(a_{jk})$ *satisfies the condition (1), and* $x=(x_k) \in l_2$. *Then the component* $y_j=\sum_k a_{jk}x$ *of* $y=Ax$ *converges to* 0 *as* $j \to \infty$, *i. e.* $y \in c_0$.

Proof.

$$|y_j| \leqq |\sum_{k=1}^{n} a_{jk}x_k| + |\sum_{n+1}^{\infty} a_{jk}x_k| \quad \text{And} \quad |\sum_{n+1}^{\infty} a_{jk}x_k| \leqq \sum_{n+1}^{\infty} a_{jk}x_k|$$

$$\leqq \sqrt{(|a_{j,n+1}|^2 + |a_{j,n+2}|^2 + \cdots)} \sqrt{(|x_{n+1}|^2 + |x_{n+2}|^2 + \cdots)}$$

$$\leqq \sqrt{M} \sqrt{(|x_{n+1}|^2 + |x_{n+2}|^2 + \cdots)}$$

can be made arbitrarily small by taking sufficiently large n.

Next $|\sum_{k=1}^{n} a_{jk}x_k|$ can be made arbitrarily small by taking sufficiently large j.

* This research is supported by kying Hee Unirersity research grant in 1988.

According to Theorem 1, $x \in l_2$ implies $y = Ax \in c_0$. But $y = Ax$ may not belong to l_2. So let $D(A)$ be the sub-space consisting of all the vectors $x \in l_2$ for which $y = Ax \in l_2$. Then we have the following theorem.

Theorem 2. *Suppose that A satisfies the conditin(1), and D (A) = l_2. Then A is bounded on l_2.*

Proof. Appealing to the Closed Graph Theorem it is enough to show that

if $\|x^{(n)} - x\| \to 0$ and $\| Ax^{(n)} - y\| \to 0$, then $y = Ax$.

Now for each fixed j

$$|\sum_k a_{jk}x_k - y_j| \leq | \sum_k a_{jk}x_k - \sum a_{jk}x^{(n)}{}_k| + |\sum a_{jk}x^{(n)} - y_j|$$

$$\leq \|A_j\| \cdot \|x - x^{(n)}\| + \|Ax^{(n)} - y\| \to 0 (n \to \infty)$$

Hence $y_j = \sum_k a_{jk}x_k$.

That is $y = Ax$.

II. Componentwise convergence in l_2

Let $y = Ax$, where A satisfies the condition (1) and $x = (x_k)$ is a vector of l_2. This means that the components y_j of the vector y are given by the formulae

(2) $y_j = \sum_k a_{jk}x^k$ $(j = 1, 2, 3 \cdots\cdots)$.

Now let

$$y^{(n)} = x_1A^1 + x_2A^2 + \cdots + x_nA^n$$

Then (2) means that for each j, the j-th component of y^n converges to the j-th component of y. That is, (2) means that

$$y^{(n)} = x_1A^1 + \cdots\cdots + x_ny^n \to y \text{ (component-wise)}$$

It is clear that $x^{(n)} \to x$ (weakly) implies $x^{(n)} \to x$(componentwise). But as we can see in the following example, the converse need not to hold.

Example. Let $x^{(n)} \in l_2$ be such that its k-th component is $x^{(n)}{}_k = \frac{k}{n^2}\left(\frac{n^2}{n^2+1}\right)^k$

Then $x^{(n)} \to x = 0$ componentwise as $n \to \infty$.

Let $\psi \in l_2$ be such that its k-th component is $\psi_k = \frac{1}{k}$,

then

$$<x^{(n)}, \psi> = \sum_{k=1}^{\infty} \frac{1}{n^2}\left(\frac{n^2}{n^2+1}\right)^k = 1$$

However $< x, \psi> = 0$

Therefoe $<x^{(n)}, \psi>$ does not tend to $<x, \psi>$,

That is, $x^{(n)}$ does not converge to $x = 0$ weakly as $n \to \infty$.

We know that every vector of l_2 is mapped into c_0 by A if A satisfies the condition (1). Furthermore A becomes bounded on l_2 if every vector of l_2 is mapped into l_2 by A.

Let $C = A^{abso}$ be the matrix formed by the entries $c_{jk} = |a_{jk}|$, absolute values of the entries of $A = (a_{jk})$.

Then $D(A)=l_2$ when $D(C)=l_2$. For suppose that $D(C)=l_2$, $x=(x_k)\in l_2$. Then since $z=(|x_k|)\in l_2=D(C)$, we have $u=Cz\in l_2$.
That is, the vector u with components $u_j=|a_{jk}|.|x_k|$ belongs to l_2.
Therefore the vector $y=Ax$ with components $y_j=\sum_k a_{jk}x_k$ also belongs to l_2. Hence $x\in D(A)$.

Next let $l_2^+=\{x\in l_2|$ all the components $x_j\geqq 0\}$

It is evident that $D(A)=l_2$ if every vector of l_2^+ is mapped into l_2(in this case every vector of l_2^+ is actually mapped into l_2^+) by $C=A^{abso}$.

Suppose that a vector $x=(x_k)$of l_2^+ is mapped to $y\in c_0$ by A with non-negative entries $a_{jk}\geqq 0$. Then

$$x_1A^1\cdots+x_n\rightarrow y(\text{component-wise})$$

That is

$$x_1a_{k1}+\cdots+x_na_{kn}\rightarrow y_k(k=1,2,3,\cdots\cdots\cdots\cdots)$$

This convergence is uniform with respective to k as one can see from the following theorem.

Theorem 3. *Assume that a sequence of vectors of l_2, $y^n(n=1,2,3,\cdots\cdots)$ converges componentwise to a vector y of c_j. i.e.*

(3) $\quad y^n{}_k\rightarrow y_k(n\rightarrow\infty)$ *for all* $k\in N=\{1,2,3,\cdots\cdots\}$.

Assume furthermore that the convergence in (3) is increasing one, i.e. $y^{n+1}{}_k\geqq y^n{}_k(n=1,2,3,\cdots\cdots\}$. *Then the convergence in (3) is uniform with respective to k.*

Proof Let us assume by contradiction that there are a positive number α as well as two sequences of positive integers $n(1)<n(2)<\cdots\cdots$ and $k(1),k(2),\cdots\cdots$ such that

(4) $\quad y_{k(i)}-y^{n(i)}{}_{k(i)}\geqq\alpha(i=1,2,3,\cdots\cdots)$.

There are two cases with respective to the sequence $k(1),k(2),\cdots\cdots$ (A) there are only finitely many distinct values of $k(i)$, in this case we may assume without loss of generality that all the $k(i)=k$; (B) there are infinitely many distinct values of $k(i)$, in this case we may assume without loss of generality that $k(i)\rightarrow\infty(i\rightarrow\infty)$.

Now we will derive a contradiction from (4).
First in the case (A) $y^{n(i)}{}_k\leqq y_k-\alpha$ from(4). Letting $i\rightarrow\infty$ we arrive at the conclusion $y_k\leqq y_k-\alpha$, which is absurd. Next in the case(B) $y^1{}_{k(i)}\leqq y^{n(i)}{}_{k(i)}\leqq y_{k(i)}-\alpha$ from(4).
Since $y^1\in l_2$, by letting $i\rightarrow\infty$ we arrive at the conclusion $0\leqq 0-\alpha$, which again is absurd.

Theorem 4. *Suppose that a sequence of vectors of l_2, $y^{(n)}=(y^{(n)}{}_1, y^{(n)}{}_2,\cdots\cdots)$ converges componentwise to a vector of l_2, $y=(y_1,y_2,y_3,\cdots\cdots)$. If the convergence*

$$|y^{(n)}{}_q|^2+|y^{(n)}{}_{q+1}|^2+\cdots\cdots\rightarrow 0\ (q\rightarrow\infty)$$

is uniform with respective to n, then $y^{(n)}$ converges to y strongly.

Proof. Let ε be an arbitrarily small positive number. Then we can find a number p such that

$$|y_p|^2+|y_{p+1}|^2+\cdots\cdots<\varepsilon \mid y^{(n)}{}_p|^2+|y^{(n)}{}_{p+1}|^2\cdots\cdots<\varepsilon(n=1,2,3,\cdots\cdots)$$

Also we can find a number N such that $n\geqq N$ implies

$$|y^{(n)}{}_1-y_1|^2\leqq\frac{\varepsilon}{p},\ |y^{(n)}{}_2-y_2|^2\leqq\frac{\varepsilon}{p},\cdots |y^{(n)}{}_p-y_p|^2\leqq\frac{\varepsilon}{p}.$$

Then when $n\geqq N$ we have

$$\|y^{(n)}-y\|^2=|y^{(n)}{}_1-y_1|^2+\cdots\cdots+|y^{(n)}{}_p-y_p|^2+\sum_{q=p+1}^{\infty}|y^{(n)}{}_q-y_q|^2\leqq p\cdot\frac{\varepsilon}{p}+4\varepsilon=5\varepsilon$$

Thoerem 5. *Suppose that* A *satisfies the condition (1), and suppose that all the entries* a_{jk} *of* A *are non-negative, If* $(x_1, x_2, x_3, \cdots\cdots)\in l_2^+\cap D($ *then* $y^{(n)}=x_1A^1+x_2A^2+\cdots\cdots+x_nA^n$ *converges strongly.*

Proof. There is a vector $y=(y_1, y_2, y_3, \cdots\cdots)\in l_2$ such that

$$y^{(n)}=x_1A^1+\cdots\cdots+x_nA^n\rightarrow y\,(\text{component-wise})$$

And

$$|y^{(n)}{}_q|^2+|y^{(n)}{}_{q+1}|^2+\cdots\cdots\leqq|y_q|^2+|y^{(n)}{}_{q+1}|^2\cdots\rightarrow 0\ (q\rightarrow\infty)$$

That is, the convergence

$$|y^{(n)}{}_q|^2+|y^{(n)}{}_{q+1}|^2+\cdots\cdots\rightarrow 0\ (q\rightarrow\infty)$$

is uniform with respect to n. By appealing to Theosem 4 we conclude that $y^{(n)}\rightarrow y$ strongly.

Corollary. *Suppose that* A *Satisfies the condition (1). If*

$x^{abso}=(|x_1|, |x_2|, \cdots\cdots)\in D(A^{abso})$, *then* $y^{(n)}=x_1A^1\cdots\cdots+x_nA^n$ *converges strongly.*

Proof. $\|y^{(n)}-Ax\|=\sum_j |x_{n+1}a_{j,n+1}+x_{n+2}a_{j,n+2}+\cdots|^2$

$\leqq \sum_j (|x_{n+1}|\cdot|a_{j,n+1}|+|x_{n+2}|\,a_{j,n+2}|+\cdots\cdots)^2\rightarrow 0\ (n\rightarrow\infty)$

since $|a_{jk}|\geqq 0$ and $x^{abso}\in l_2^+\cap D(A^{abso})$.

Theorem 6. *Suppose that* A *satisfies the condition (1). Then a necessary and sufficient condition that* $y^{(n)}=x_1A^1+\cdots\cdots+x_nA^n$ *storongly converges for all vector* $x=(x_1, x_2, x_3, \cdots\cdots)\in l_2$ *is that* A *should be bounded on* l_2

Proof. First suppose that $y^{(n)}$ converges strongly to some $y=y_1, y_2, y_3,\cdots\cdots)$ for every $x\in l_2$. Then $y\in l_2$, $y_j=\sum_{k=1}^{\infty}a_{jk}x_k$

&o from theorem (2) we conclude that A is bounded on l_2.

Conversely suppose that A is bounded on l_2, $x=(x_1, x_2, x_3\cdots\cdots)\in l_2$ and $x^{(n)}=(x_1, x_2,\cdots,x_n, 0, 0, \cdots)$.

Then $x^{(n)}$ converges strongly to x Hense $\|Ax^{(n)}-Ax\|\leqq\|A\|\cdot\|x^{(n)}-x\|\rightarrow 0$.

That is, $Ax^{(n)}=x_1A^1+\cdots\cdots+x_nA^n$ converges strongly.

III. Norm of an operator

Suppose that A is bounded on l_2. Let $b_{jk}=<A^j, A^k>$
That is, $A^t\bar{A}=(b_{jk})$ *or* $A^*A=(\bar{b}_{jk})$,
where A^t and $\bar{A}$ are the transposed and the complex conjugate matrices of A respectively, and $A^*=\bar{A}^t$. Then the following theorem holds.

Theorem 7. *Suppose that A is bounded on l_2. Then*

(i) $b_{jj}\geqq 0$

(ii) $b_{jk}=\overline{b_k}$

(iii) $|b_{jk}|\leqq\sqrt{b_{jj}\,b_{kk}}$

(iv) $\| A \|^2=sup \sum_{j,k} b_{jk}\, x_j\, \bar{x}_k$,

sup being taken for all $x=(x_1, x_2, x_3, \cdots\cdots)$ of l_2 with $\| x \|\leqq 1$.

Proof. Since $\sum_{j=1}^{n}A^jx_j$ Strongly converges by Theonem 6, we have

$$\| Ax \|^2=<Ax, Ax>=<\sum_j A^jx_j, \sum_k A^kx_k>=\sum_{j,k}<A^j, A^k>x_j\,\bar{x}_k=\sum b_{jk}x_j\,\bar{x}_k,$$

Hence

$$\| A \|^2=sup\ \| Ax \|^2=\ sup\ \sum b_{jk}\ x_j\ \bar{x}_k(\| x \|\leqq 1)$$

Proof of the remaining part is evident.

The value of $\| A \|$ lies between $sup\ \| A^j\|$ and $\sqrt{\sum_{j,k}|a_{jk}|^2}\sqrt{\sum_k\| A^k\|^2}$ when A is bounded on l_2.
That is,

$$sup\ b_{jj}\leqq sup\ \sum_{j,k} b_{jk}x_j\bar{x}_k\leqq\sum_j b_{jj}$$

For when $x=l_p$, i.e. when its component $x_k=\delta_{pk}$, we obtain

$$\sum_{j,k} b_{jk}\ x_j\ \bar{x}_k=b_{pp}$$

Hence

$$b_{pp}\leqq\ sup\ \sum b_{jk}\ x_j\bar{x}_k$$

Also

$$\sum b_{jk}x_j\bar{x}_k\leqq\sum|b_{jk}|\cdot|x_j|\cdot|x_k|$$
$$\leqq\sum\sqrt{b_{jj}\,b_{kk}}\ |x_j|\cdot|x_k|=(\sum_j\sqrt{b_{jj}}\,|x_j|)^2$$
$$\leqq(b_{11}+b_{22}+\cdots\cdots)\ (|x|^2+|x_2|^2+\cdots\cdots)=\sum b_{jj}$$

for an artitrary unit vector $x=(x_k)$ of l_2.

Theorem 8. *Let* $A\neq 0$ *be bounded on l_2. Then a necessary and sufficient condition that $\| A\|^2=\sum_{j,k}|a_{jk}|^2$ is that all the column vectors A^j should be parallel to each other.*

Proof. First suppose that $\| A \|^2\sum_{j,k}|a_{jk}|^2$. Then

(5) $$sup\ \sum_{j,k} b_{jk}x_j\bar{x}_k=\| A\|^2=\sum_k\ \sum_j|a_{jk}|^2=\sum_k<A^k, A^k>=\sum_k b_{jk}$$

Here as well as in the rest of this proof, sup is always taken over all unit vectors $A=(A_k)$ of l_2. On the other hand

(6) $\sup \sum_{j,k} b_{jk} x_j x_k \leq \sup \sum_{j,k} |b_{jk}| \, |x_j| \cdot |x_k| \leq \sup \sum_{j,k} \sqrt{b_{jj}} \sqrt{b_{kk}} \, |x_j| \cdot |x_k|$

$\leq \sup(\sum_j \sqrt{b_{jj}} \, |x_j|^2) \leq \sum b_{jj} \sum |x_j|^2 = \sum b_{kk}$

Comparing (5) with (6) we conclude that

(7) $\sup \sum_{j,k} |b_{jk}| \cdot |x_j| \cdot |x_k| = \sup \sum_{j,k} \sqrt{b_{jj}} \sqrt{b_{kk}} \, |x_j| \cdot |x_k| = \sup (\sum_j \sqrt{b_{jj}} \, |x_j|)^2$

But the sup of the R.H.S. is $\sum b_{jj}$, and occurs when and only when

$|x_j| = x^0_j = \sqrt{(b_{jj} / \sum b_{jj})}$, $\sum b_{jj} > 0$, since $A \neq 0$.

Now A is a compact operator since $\sum |a_{jk}|^2 < \infty$. [Ref. 1, p. 86].

Therefore $B = A^t \bar{A}$ is a compact self-adjoint operator. Hence $e\| B \| = e\| A \|^2$ (*e=either +1 or −1*) becomes an eigen-value of B. That is, there exists an unit vector $\xi \in l_2$ such that

$A^t \bar{A} \xi = B\xi = e\| B \| \xi = e\| A \|^2 \xi$

Then

$<B\xi, \xi> = e\| A \|^2 \cdot \| \xi \|^2 = e\| A \|^2$

That is,

$\sum_{j,k} b_{jk} \xi_k \bar{\xi}_j = e\| A \|^2$

Therefore

(8) $\| A \|^2 \leq \sum_{j,k} |b_{jk}| \cdot |\xi_j| \cdot |\xi_k|$

By comparing (7) with (8) we obtain

$\| A \|^2 \leq \sum_{j,k} |b_{jk}| \cdot |\xi_j| \cdot |\xi_k| \leq \sum \sqrt{b_{jj}} \sqrt{b_{kk}} \, |\xi_j| \cdot |\xi_k| \sum \sqrt{b_{jj}} \sqrt{b_{kk}} \, x^0_j x^0_k = \| A \|^2$

Hence all the inequality symbol $\leq$ become equality symbol $=$ in the above formula, and $|\xi_j| = x^0_j (j=1, 2, 3, \cdots)$. Therefore

(9) $\sum_{j,k} |b_{jk}| x^0_j x^0_k = \sum \sqrt{b_{jj} b_{kk}} \, x^0_j \cdot x^0_k$

If some $|b_{jk}| < \sqrt{b_{jj}} \sqrt{b_{kk}}$, then from (9) $x^0_j x^0_k$ must be 0. However this implies $b_{jj} b_{kk} = 0$ which is absurd, since $|b_{jk}| < \sqrt{b_{jj}} \sqrt{b_{kk}}$. Therefore we conclude that

$|b_{jk}| = \sqrt{b_{jj}} \sqrt{b_{kk}}$ *for all* j, k.

And this means that all the column vectors A^j are parallel to each other.

Conversely when A^j are parallel to each other, then the entries a_{jk} of A can be written $a_{jk} = s_j t_k$, with $\sum |t_k|^2 = 1$, Hence

$b_{jk} = <A^j, A^k> = \| s \|^2 t_j \bar{t}_k$ where $\| s \|^2 = |s_1|^2 + |s_2|^2 + \cdots\cdots$.

Therefore $\| A \|^2 = \sup \sum b_{jk} x_j \bar{x}_k = \| s \|^2 \sup \sum t_j \bar{t}_k x_j \bar{x}_k = \| s \|^2 \sup |\sum t_j x_j|^2 = \| s \|^2 \cdot \| t \|^2 \cdot \| s \|^2$ where $\| t \|^2 = |t_1|^2 + |t_2|^2 + \cdots = 1$ That is,

$\| A \|^2 = \| s \|^2 = \sum_{j,k} |a_{jk}|^2$

Theorem 9. *Let* A *be an arbitrary matrix not necessarily satisfying the condition (1), but satisfying* $\sum_k |a_{jk}|^2 < \infty (j=1, 2, \cdots)$, *And suppose that the row vectors* A_j *are orthogonal to each other,*

i.e. $<A_j, A_k> = 0$ *whenever* $j \neq k$.

(I) If $\sup \| A_j \| < \infty$, *then* A *is bounded on* l_2.

(II) If A *is bounded on* l_2*, then sup* $\| A_j \| < \infty$, *and furthermore* $\sup \| A_j \| = \| A \|$

Proof.(I) Suppose that $sup \| A_j \| < \infty$.

When a row vector A_j is different from the null vector, let us represent it by

(10) $A_j = s_j c_j, \quad s_j = \| A_j \|$

Those c_j form a system of ortho-normal vectors, and for an arbitary $x \in l_2$ we have (summation being taken over all j with $A_j \neq 0$)

$$\| Ax \|^2 = |< s_1 c_1, x >|^2 + |< s_2 c_2, x >|^2 + \cdots\cdots$$
$$\leq (sup\ s_j^2)(|< c_1, x >|^2 + |< c_2, x >|^2 + \cdots\cdots$$
$$\leq (sup \| A_j \|^2) \| x \|^2$$

Therefore A is bounded on l_2 and

(11−a) $\| A \| \leq sup \| A_j \|$

(II) Suppose that A is bounded on l_2. Using the representation (10) whenever $A_j \neq 0$,

$$\| A \|^2 \geq \| Ac_j \|^2 = \| \frac{AA_j}{s_j} \|^2$$

$$\frac{1}{s_j^2} (|< A_j, A_j >|^2 + |< A_j, A_j >|^2 + \cdots) = \frac{1}{s_j^2} |< A_j, A_j >|^2 = s_j^2 = \| Aj \|^2$$

Hence

(11−b) $\| A \| \geq sup \| A_j \|$

That is, $sup \| A \| < \infty$

Furthermore form (11−a) and (11−b) we obtain

$sup \| A_j \| = \| A \|$

Theorem 10. *Let* A *be an arbitrary matrix not necessarily satisfying the condition (1), but satisfying* $\sum_{j,k} |a_{jk}|^2 < \infty$ *(*$k = 1, 2, 3, \cdots\cdots$*).*

And suppose that the column vectors A^k *are orthogonal to each other, i.e.*

$< A^j, A^k > = 0$ *whenever* $j \neq k$.

(I) If sup $\| A^k \| < \infty$, *then* A *is bounded on* l_2.

(II) If A *is bounded on* l_2*, then sup* $\| A^k \| < \infty$, *and furthermore* $sup \| A^k \| = \| A \|$

Proof. *Under the assumptions of the theorem the transpose* A^t *of* A *satisfies the corresponding assumptions of Theorem 9.*

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