Journal of the Korean
OR/MS Society
Vol. 13 No. 2 December 1988

—Research Paper—

AN APPLICATION OF LAGRANGIAN RELAXATION ALD SUBGRADIENT METHOD FOR A DYNAMIC UNCAPACITATED FACILITY COCATION PROBLEM

Jae-wook, Song*
Sheung-Kown, Kim**

ABSTRACT

The dynamic uncapacitated facility location model is formulated by a mixed integer programming. It has the objective of minimizing total discounted costs for meeting demands specified in different time periods at various demand centers. Costs include those for operation of facilities to demand centers and a fixed cost associated with the capital investment. The problem is decomposed into two simple Lagrangian relaxed subproblems which are coordinated by Lagrangian multipliers. We explored the effect of using the subgradient optimization procedure and a viable solution approach is proposed. Computational results are presented and further research directions are discussed.

1. INTRODUCTION

Dynamic Uncapacitated Facility Location(DUFL) problems are a type of the problem that we would encounter when we try to setup an optimal long range planning for a firm's operation. It can be considered as the generalization of a static plant location problem in the dynamic context[1, 11, 12]. It involves the determination of the time-staged establishment of facilities at different locations so as to minimize the total discounted costs for meeting demands specified over time at various demand centers. This problem is closely related to the capacity expansion problems. Capacity expansion is the addition of similar facilities over time to meet increasing demand. Discussions of these problems with several applications are given by many researchers[2, 5, 6, 9, 10].

Ever since solution methods for the DUFL problem have been developed by Roodman and Schwartz[14, 15], different aspects of this problem have been investigated under various assumptions[3, 6, 9, 17].

Van Roy & Erlenkotter(1982) proposed powerful dual-based procedures for the DUFL problems. In this paper, we investigated the effectiveness of using Lagrangian relaxation with the subgradient method. We assume that the facility opened in period t, will remain opened through a planning horizon as in the capacity expansion context.

We consider the case where the fixed capital investment cost is proportional to the amount shipped, in other words, the cost associated with the capital investment consists of a fixed cost and a linear portion of it.

*) Department of Industrial Engineering, R. O. K. Air Force Academy.

**) Department of Industrial Engineering, Korea University.

2. MODEL

2.1 MATHEMATICAL FORMULATION

The assumptions for our study are as follows ;

1) Each facility can be expanded without any restriction on capacity, that is, it can be built to large enough specification if this is suggested by the analysis. (Uncapacitated Facility Location Problem)
2) The expansion cost occurred is ignored. This will be discussed at later section.
3) Time is treated as a discrete variable.
4) The planning horizon and the number of facility and demand center is finite.
5) Variable operation and distribution cost is proportional to the amount supplied from facility to demand region.
6) Initial capacity of each facility is zero.
7) The investment cost F_{it} is incurred at the start of time period that facility is operated and the construction time is taken as being small.
8) The demand D_{jt} is deterministic and non-decreasing.

Under these assumptions, the mathematical model of the dynamic uncapacitated facility location problem may be formulated as follows ;

[P] Min. $\sum_i \sum_j \sum_t C_{ijt} X_{ijt} + \sum_i \sum_t F_{it}(Y_{it} - Y_{it-1})$ (1)

S.T. $\sum_i X_{ijt} = 1 \quad \forall j \in J,\ t \in TT$ (2)

$X_{ijt} \leqq Y_{it} \quad \forall i \in I,\ j \in J,\ t \in TT$ (3)

$Y_{it} - Y_{it-1} \leqq 0 \quad \forall i \in I,\ t \in TT$ (4)

$X_{ijt} \geqq 0 \quad \forall i \in I,\ j \in J,\ t \in TT$ (5)

$Y_{it} = 0 \text{ or } 1 \quad \forall i \in I,\ t \in TT$ (6)

where, $Y_{i0} = 0 \quad \forall i \in I$ (7)

where, I = (1,...,m) : the set of facilities to be open

J = (1,...,n) : the set of demand centers

TT = (1,...,T) : the set of time periods

And for $i \in I$, $j \in J$, $t \in TT$, we define the following notations.

X_{ijt} = the fraction of demand center j 's demand, D_{ij} from facility i, in period t ;

Y_{it} = $\begin{cases} 1, \text{ if facility i is open in period t ;} \\ 0, \text{ otherwise ;} \end{cases}$

UC_{ij} = the unit cost of producing and shipping center j from facility i ;

D_{jt} = demand at each demand center j in time period t ;

C_{ijt} = the total discounted cost of producing and shipping demand center j 's demand in time period t from facility i.

$C_{ijt} = UC_{ij} * D_{jt} * exp[-r(t-1)]$ (r : continuous discount rate)*

F_{it} = the discounted fixed cost for opening facility i in time period t ;

Constraints (2) mean that each region's demand should be met through the planning horizon. Constraints (3) require that a facility cannot be operated before it is opened. Constraints (4) state that any facility once opened remain open in all successive periods.

2.2 LAGRANGIAN SUBPROBLEM

We know that the optimal value of the Lagrangian relaxed problem is the lower bound of the original minimization problem. Now, choosing the disaggregated (strong) constraint, (3) as the set of complicating constraints to be relaxed, let $U_{ijt}(i = 1,\ldots,m ; j = 1,\ldots,n ; t = 1,\ldots,T)$ be nonnegative Lagrangian multipliers associated with the *ijt*-th constraint in the equation. Then the primal problem [*P*] can be converted to the Lagrangian relaxation problem [*LR*] as follows ;

$$[LR]\ Z_D(U) = Min.\ \sum_i\sum_j\sum_t (C_{ijt}+U_{ijt})X_{ijt} + \sum_i\sum_t [F_{it}(Y_{it}-Y_{it-1}) - \sum_j U_{ijt}Y_{it}] \quad (8)$$

S.T. (2), (4), (5), (6), (7)

Taking advantage of the seperation of X_{ijt} and Y_{it}, [*LR*] can be decomposed into two independent subproblems, associated with X_{ijt} in [*LR1*] and Y_{it} in [*LR2*].

$$[LR1]\ Min.\ \sum_i\sum_j\sum_t (C_{ijt}+U_{ijt})X_{ijt} \quad (9)$$

S.T. (2), (5)

$$[LR2]\ Min. \sum_i\sum_t [F_{it}(Y_{it}-Y_{it-1}) - \sum_j U_{ijt}Y_{it}] \quad (10)$$

S.T. (4), (6), (7)

Hence, the Lagrangian dual problem [*D*] to the primal [*P*] is as follows ;

$$[D]\ Z_D = Max.\ Z_D(U) \quad (11)$$

$$= Max.\ [[LR1]+[LR2]] \quad (12)$$

S.T. U≧0

2.3 ADDITIONAL EXPANSION COST

If we consider the additional expansion cost, which is assumed to be proportional to the amount shipped from each facility, the objective function (1) would include the following additional term.

$$\sum_i\sum_{t=2}^{T} [\sum_j (D_{jt}X_{ijt} - D_{jt-1}X_{ijt-1}) \cdot \alpha \cdot exp(-r(t-1))] \quad (13)$$

*) It is assumed that the time value of money is continuously discounted for the convenience of abstraction and exp(-rt) is used as a discount factor.

where, α : expansion cost incurred for a unit capacity is increased.

Equation (13) is a function of X_{ijt} only and the modified objective function can be rearranged as follows ;

$$\sum_i \sum_j [C_{ij1} - \alpha D_{j1} exp(-r)] X_{ij1} + \sum_i \sum_j \sum_{t=2}^{T-1} [C_{ijt} + \alpha \cdot D_{jt} exp(-rt) \ (exp(r)-1)] X_{ijt} + \sum_i \sum_j [C_{ijT} + \alpha D_{jT} exp(-r(T-1))] \ X_{ijT} \quad (14)$$

As long as the additional expansion cost is proportional to the amount shipped from each facility, the additional expansion cost considered changes only the coefficients of X_{ijt}'s in the objective function. In this study, we ignore the additional expansion cost.

3. COMPUTATION OF LOWER BOUNDS

3.1 SOLUTION PROCEDURE OF [LR1]

The problem [$LR1$] can be decomposed into T independent subproblems according to the time period t. Let [$LR1$] be the subproblem of time period $t = k$, then it can be represented as follows ;

$$[LR1]_k \qquad L1 = \text{Min.} \sum_i \sum_j (C_{ijk} + U_{ijk}) \ X_{ijk} \quad (15)$$

$$\text{S.T.} \quad \sum_i X_{ijk} = 1 \qquad \forall j \in J \quad (16)$$

$$X_{ijk} \geqq 0 \qquad \forall i \in I, \ j \in J$$

This $[LR1]_k$ has the similar structure of an assignment problem and its optimal solution can be easily found by simple additions and comparisons. The optimal solution of $[LR1]_k$, L_k is as follows ;

$$L_k = \sum_j \ (C_{i(j)^*jk} + U_{i(j)^*jk}) \quad (17)$$

$$\text{where, } i(j)^* = \text{Min}_i. \ (C_{ijk} + U_{ijk}) \quad (18)$$

Therefore, computing L_k for all $k (k \in TT)$ and summing them, we can obtain the optimal solution value of [$LR1$], $L1^*$ as follows ;

$$L1^* = \sum_t [\sum_j (C_{i(j)^*jt} + U_{i(j)^*jt})] \quad (19)$$

3.2 SOLUTION PROCEDURE OF [LR2]

We can decompose the problem [$LR2$] into m subproblems for each facility i. For some $i = q$, the subproblem $[LR2]_q$ can be represented as follows ;

$$[LR2]_q \qquad L2 = \text{Min.} \sum_t [F_{qt}(Y_{qt} - Y_{qt-1}) - \sum_j U_{qjt} Y_{qt}] \quad (20)$$

$$\text{S.T.} \quad Y_{qt} - Y_{qt-1} \geqq 0 \ \forall t \in TT \quad (21)$$

$$Y_{qt} = 0 \text{ or } 1 \ \forall t \in TT$$

$$\text{where, } Y_{q0} = 0$$

Considering all possible cases that the decision variables "Y_{qt}", "Y_{qt-1}", can take, we can evaluate the value of the objective function as in Table 1. Thus if the facility q is set up in time period t, the optimal value of $[LR2]_q$ is

$$F_{qt} - \sum_j U_{qjt} + \sum_{k}^{T} \sum_{t+1} (-\sum_j U_{qjt})$$

$$= F_{qt} - \sum_{k}^{T} \sum_{t} (\sum_j U_{qjk}) \quad (22)$$

Table 1. The Value of The Objective Function Coefficients of L2.

	Decision	Variable	Coefficients of
	Y_{qt}	Y_{qt-1}	Objective Function
CASE 1	0	0	0
CASE 2	1	0	$F_{qt} - \sum U_{qjt}$?
CASE 3	1	1	$-\sum_j U_{qjt} < 0$

and L_q, the optimal function value of $[LR2]_q$ is

$$L_q = \text{Min.}[0, \ \underset{t \in q^*}{\text{Min}} \ F_{qt} - \sum_{k=t}^{t} \sum_j U_{qjk}] \quad (23)$$

So the optimal solution of $[LR2]$, $L2^*$ is

$$L2^* = \sum_{i \in I_M} [F_{it(i)^*} - \sum_{k=t(i)^*}^{T} \sum_j U_{ijk}] \quad (24)$$

where, I_M = the set of facilities whose optimal solution value of $[LR2]_q$ is negative.

$t(i)^*$ = the optimal set up time of facility i having negative L_q.

Therefore, a lower bound of primal problem $[P]$ is ;

$$Z_D(U) = L1^* + L2^* \quad (25)$$

4. CONSTRUCTING UPPER BOUNDS FROM PRIMAL FEASIBLE SOLUTIONS

4.1 FINDING AN INITIAL UPPER BOUND

At the start, we must find an initial primal feasible solution of $[P]$ for an initial upper bound. It can be determined by the following simple steps.

STEP 1° Select i^* satisfying the following condition.

$$i^* = \underset{i}{Min.} \{F_{il}\} \quad \forall i \in I$$

STEP 2° Get an initial upper bound by computing

$$\sum_j \sum_t C_{i*jt} + F_{i*1} \quad (26)$$

where, $\sum_j \sum_t C_{i*jt}$ = the total producing and shipping cost

F_{i*1} = the total fixed-charge cost

4.2 COMPUTATION OF UPPER BOUNDS

Unless the values of the decision variables obtained from the Lagrangian dual problem satisfy the relaxed constraints (3) of the primal problem, a feasible solution of the primal cannot be obtained readily. However, in this study, even though the values of X_{ijt} and Y_{it} obtained by the Lagrangian relaxation do not satisfy the relaxed constraint, we may heuristically find a primal feasible solution so long as more than one facility is established(i. e. more than one Y_{it} is 1). The heuristic feasible solution finding procedure is as follows ;

First of all, let I_t be the set of facilities whose Y_{it} values are 1 as was determined in [*LR2*], then the primal problem [*P*] can simply be expressed as [*P'*] where 'A' contains the rest of the terms of [*P*].

$$[P'] \ \text{Min.} \ \sum_t \sum_{i \in It} \sum_j C_{ijt} X_{ijt} + A \tag{27}$$

$$\text{S.T.} \ \sum_{i \in It} X_{ijt} = 1 \ \forall j \in J, \ t \in TT \tag{28}$$

$$X_{ijt} \geq 0 \qquad \forall i \in I, \ j \in J, \ t \in TT$$

In 'A', the Y_{it}'s have the fixed value as was determined in [*LR2*], and the constraints (3), (4) of the primal problem [*P*] are also satisfied accordingly. Since this simplified [*P'*] is in the same form as the Lagrangian subproblem [*LR1*], the procedure of finding an optimal solution for [*P'*] is exactly same as that of [*LR1*]. So, the optimal solution value of [*P'*] is as follows ;

$$\sum_t [\sum_j C_{i(j)^*jt}] + A \tag{29}$$

where, $i(j)^* = \underset{i \in It}{\text{Min}}(C_{ijt})$

If the optimal value of equation (29) is less than the existing upper bound, it replaces the existing upper bound.

4.3 IMPROVEMENT OF UPPER BOUNDS

In this section, an efficient heuristic procedure that improves the primal feasible solution (upper bounds) is suggested. Here, two possible cases that can decrease the objective function value of the primal problem is considered.

First is the case when the values of Y_{it}'s from [*LR*2] indicate a facility is set up and yet, is not utilized immediately, the fixed charge cost addition is postponed until the moment that X_{ijt} has the value of 1.

Second is the case that the facility is screened out during a certain time period when the cost reduction due to delaying the set up time is greater than the shipping cost increase without it. The screening process is carried out for each facilities set up by [*LR*2] in the order of the largest cost savings. Thus, unnecessary facilities can be found if more facilities were open than required.

5. THE SUBGRADIENT OPTIMIZATION PROCEDURE

The objective of the subgradient optimization procedure is to find a better Lagrangian multiplier vector U_{ijt} which improves the lower bound obtained from Lagrangian relaxation problem. The subgradient optimization method is a brazen adaptation of the gradient method treated in non-linear programming, in which the gradient is replaced by the subgradient [4].

In general, when an inital Lagrangian multiplier vector U is given, a series of a vector U^k generated is as follows ;

$$U^{k+1} = Max.\ [0,\ U^k + t_k(AX^k - b)] \tag{30}$$

where $AX = b$ is the relaxed constraint of the Lagrangian relaxation.

In equation(30), t_k is the positive stepsize and $U \geqq 0$. The stepsize t_k is computed by the following formula as was suggested by Fisher [4]

$$t_k = [\pi_k \cdot (UB - Z_D(U^k))] / \| S \|^2 \tag{31}$$

In the above formula, π_k is a constant($0 \leq \pi_k \leq 2$) and halved whenever the Lagrangian dual problem [LR] has failed to increase in some prespecified number of iterations[8]. UB is an upper bound of the primal problem, and $\| S \|$ is any norm of subgradient vector S (in this study, we used the Euclidean norm). So

$$\| S \|^2 = \sum_i \sum_j \sum_t (S^k_{ijt})^2 \tag{32}$$

$$= \sum_i \sum_j \sum_t (X^k_{ijt} - Y^k_{it}) \tag{33}$$

Therefore, the improved Lagrangian multiplier vector U^{k+1} is described as follows ;

$$U^{k+1}_{ijt} = Max.\ [0,\ U^k_{ijt} + t_k(X^k_{ijt} - Y^k_{it})] \tag{34}$$

In this study, we applied some stopping rules to enhance the computational efficiency of the algorithm. Practically, however, there is no way of proving the optimality by this Lagrangian relaxation method as long as the positive duality gap $(UB - Z_D(U^k))$ exists. In fact, when the size of the problem becomes larger, equating UB to $Z_D(U^k)$ is hardly expected. Therefore, it is hopeful to find a feasible solution close to an optimal value more rapidly by making stopping rules appropriate to each problem. Stopping rules considered in this study is as follows ;

(1) If all subgradient values $X_{ijt} - Y_{it}$ are zero, it will stop.

(2) If $(UB - LB)/LB \leq$ k, it will stop (generally, $0.005 \leq k \leq 0.05$).

(3) If the best feasible solution is obtained repeatedly several times (in general, 5 times), it will stop.

(4) When it meets the prespecified iteration limit (in general, 100 times), it will stop.

(5) In case of $\pi_k \leq 0.0001$, it will stop.

A flow chart for the proposed solution procedure is depicted in Figure 1.

6. COMPUTATIONAL RESULTS

The computational efficiency of our solution procedure proposed in this stuty is evaluated by solving example problems. Eleven example problems(7 phase-in and 4 phase-out problems) of Roodman & Schwartz are solved to see the effectiveness of the algorithm. The sizes of these examples considered are from $8 \times 15 \times 5$ to $15 \times 30 \times 8$ (640－3720 variables). Throughout the solution procedure the maximum number of iterations were set to 30 times and the iteration stops if the best uppper bound repeats 5 times or the tolerance limit between UB and LB (duality gap) is less than 0.05. And as initial input data, π_1 is 2 and the discount rate used is 0.07. In setting initial Lagrangian multiplier value, U_{ijt}, two cases are considered.

CASE 1) $U_{ijt} = 0 \quad \forall i \in I,\ j \in J,\ t \in TT$

CASE 2) $U_{ijt} = \underset{t}{Min.}\ C_{ijt} \quad \forall i \in I,\ j \in J,\ t \in TT$

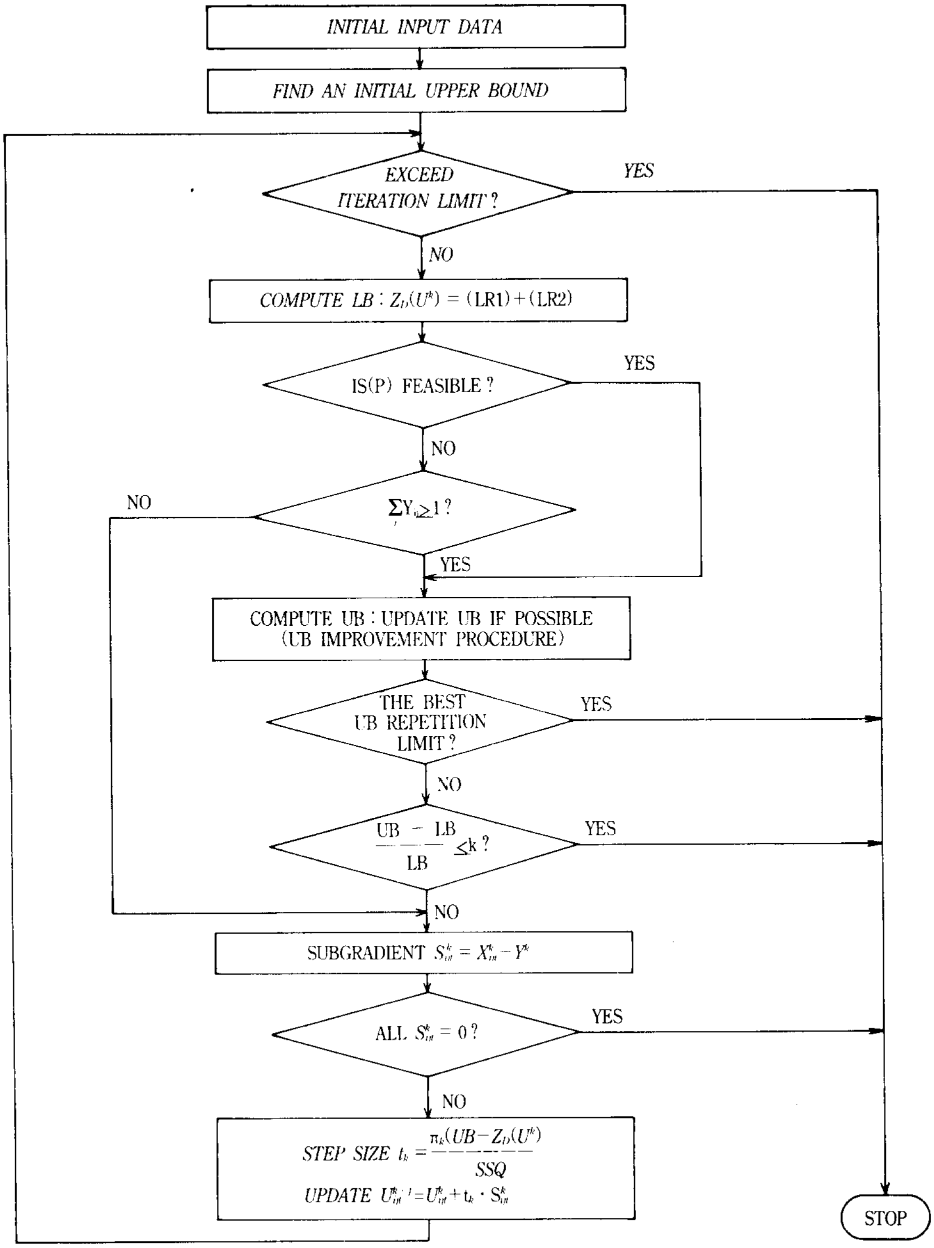

Figure 1. A Flow Chart of the Solution Procedure.

The proposed Lagrangian relaxation procedure is programmed in FORTRAN IV and run on 16-bit personal computer (SPC-3200). The computational results were compared with dual-based procedure of Van Roy & Erlenkotter[16] and are summarized in Table 2. Figures in parentheses have been obtained by taking initial U_{ijt} as in *CASE 2*. The proposed solution procedure was not so efficient as DUALOC of Van Roy & Erlenkotter. DUALOC is a solution algorithm devised by Erlenkotter[3] for a special class of probelm such as the simple plant location problem and has been extended to solve the dynamic plant location problem by Van Roy and Erlenkotter. The solution scheme is centered around meeting the complementary slackness condition adjusting dual variables efficiently at a time by dual ascent procedure. And a primal feasible solution is readily obtained upon termination of dual ascent. For the completeness of the algorithm they added dual adjustment and branch and bound procedure. Interested reader should read their paper.

For a time being it seems to be the most powerful and efficient solution algorithm for the class of problem. It is because the DUALOC adjusts each component of dual variables with exact admissible step size exploiting some special structure of the uncapacitated plant location problem, unlike the subgradient algorithm does on the Lagrangian multipliers. If we take a close look of the proposed solution procedure, it very often finds optimal solution within the first few iterations. It reveals that the updating rule of the subgradient method has poor convergence rate.

Table 2. Computational Results

*Ex. No.	**Size (ixjxt)	CPU TIME		Total # of Iter.	Iter.# finding opt. sol.	SOLUTION		
		Lag.Rel. Method	Dual-Based			UB	LB	(LB−LB)/LB
1	8×15×5	44 (57)	11	21 (25)	20 (22)	24372	23839 (23783)	0.0223 (0.0247)
2	8×30×5	63 (69)	11	14 (15)	1 (2)	39819	37931 (38044)	0.0497 (0.0466)
3	12×30×5	85 (66)	48	14 (7)	1 (2)	38030	36342 (22501)	0.0464 (0.6901)
4	12×30×8	204 (107)	72	17 (7)	1 (2)	49362	47299 (35273)	0.0436 (0.3994)
5	15×30×8	95 (136)	31	6 (7)	1 (2)	50054	42476 (39024)	0.1784 (0.2826)
6	12×20×5	53 (86)	39	13 (20)	1 (3)	27015	25767 (25821)	0.0484 (0.0462)
7	8×20×5	41 (42)	29	15 (15)	1 (5)	25437	24301 (24421)	0.0467 (0.0416)
8	12×30×5	213 (88)	123	24 (7)	22 (2)	79893	76491 (57357)	0.0444 (0.3382)

9	8×30×5	95 (255)	26	16 (30)	1 (3)	73473	70053 (54902)	0.4882 (0.3382)
10	8×30×5	100 (183)	41	21 (30)	10 (3)	62970	60275 (47962)	0.0447 (0.3129)
11	8×15×5	67 (88)	2	30 (30)	6 (8)	28733	0 (22606)	– (0.2710)

*) 1– 7 : Phase-in problems
8–11 : Phase-out problems

**) i = The number of facility sites
j = The number of demand centers
t = The number of planning periods

7. CONCLUDING REMARKS

We have developed a solution procedure for a dynamic uncapacitated facility location problem that deals with size, location, and time-phasing decisions for establishment of service facilities. The solution procedure can readily be used to solve a problem that we would face when we try to establish a long-range investment planning strategy for a public sector project where increasing service demands are dispersed in several locations.

The mathematical model for this problem was formulated by a mixed integer programming. Generally the large scale mixed integer programming problem cannot be solved easily. However, Lagrangian relaxation method overcomes this obstacle and offers an optimal solution with great ease in computation. We tried to solve the Lagrangian dual problem. With careful use of its special problem structure, the Lagrangian dual problem is decomposed into two subproblems which can be solved simply by enumeration. Primal feasible solution is obtained from dual solutions by heuristics. And the subgradient optimization procedure is employed for the maximization of the dual function.

The main purpose of the subgradient algorithm is to construct dual feasible solutions that come close to maximizing $Z_D(U^*)$. The largest found value Z_D^* is the lower bound of the objective function of the integer program.

Immediately after the dual procedure, the generation of primal feasible solution is followed in order to reduce the ever present "duality gap", which is ultimately the amount of discrepency from the complementary slackness condition for the relaxed equation (3).

However, the computational results show that our method finishes in no better CPU time than DUALOC (dual based procedure for dynamic facility location) which has been known as the most powerful tool for this kind of problems. It is because the subgradient method does not adjust each component of Lagrangian dual vector with exact admissible step size as DUALOC does. Therefore, the subgradient method oscillates around dual optimal solution resulting in slow convergence. Nontheless, the proposed solution algorithm found all the optimal solutions. Besides, it is simple to understand and easy to execute.

In closing, we feel it is necessary to caution the reader not to infer from the computational results that Lagrangian relaxation is an inappropriate way to solve the problem. It is rather the subgradient algorithm that makes slow convergence. As long as a solution scheme that would find dual optimal step sizes faster, is utilized, the simple decomposition structure we implemented in this study can be useful.

ACKNOWLEDGEMENT

We would like to thank Professor HyungBong Ro at Hong-ik University for letting us use the DYNALOC computer source code which was programmed by himself following the solution procedure of Erlenkotter[3]. This research was supported by Korea Science and Engineering Foundation.

REFERENCES

[1] Efroymson, M. A. and T. L. Ray, "A Branch-bound Algorithm for Plant Location," Operations Research, Vol. 14, pp. 361-368(1966).

[2] Erlenkotter, D., "Capacity Planning for Large Multilocation Systems : Approximate and Incomplete Dynamic Programming Approaches," Management Science, Vol. 22, pp. 247-258(1975).

[3] Erlenkotter, D., "A Dual-based Procedure for Uncapacitated Facility Location," Operations Research, Vol.26 No. 6, pp. 992-1009(1978).

[4] Fisher, M. L., "The Lagrangian Relaxation Method for Solving Integer Programming Problems," Management Science, Vol. 27, pp. 1-18(1981)

[5] Fong, C. O. and Rao, M. R., "Capacity Expansion with Two Producing Regions and Concave Costs," Management Science, Vol. 22, pp. 331-339(1975).

[6] Fong, C. O. and V. Srinivasan, "The Multiregion Dynamic Capacity Expansion Problem : An Improved Heuristic," Management Science, Vol. 32, No, 9. pp. 1140-1152(1986).

[7] Geoffrion, A. M., "Lagrangean Relaxation for Interger Programming," Mathematical Programming Study, Vol. 2, pp. 82-114, North-Holland and Publishing Co.(1974).

[8] Held, M., Wolfe, P. and Crowder, H. P., "Validation of Subgradient Optimization", Mathematical Programming, Vol. 6, pp. 52-88(1974).

[9] Jacobsen, S., "Heuristic Procedures for Dynamic Plant Location", The Institute of Mathematical Statistics and Operations Research, The Technical University of Denmark DK-2800 Lyngby, Denmark, Working Paper (1977).

[10] Kim, S. K. and Yeh, W. W., "A Heuristic Solution Procedure for Expansion Sequencing Problems", Water Resources Research, Vol. 22, No. 8, pp. 1197-1206(1986).

[11] Khumawala, B. M, "An Efficient Branch and Bound Algorithm for the Warehouse Location Problem," Management Science, Vol. 18, pp. 718-731(1972).

[12] Krarup, J., and P. M. Pruzan, "The simple plant location problem ; Survey and Synthesis," European Journal of Operations Research, Vol. 12, pp. 36-81, 1983.

[13] Manne, A. S, Investments for Capacity Expansion : Site, Location and Time-Phasing, MIT Press, Cambridge, Mass(1967).

[14] Roodman, G. M. and Schwartz, L. B., "Optimal and Heuristic Facility Phase-out Strategies," AIIE Transaction, Vol. 7, No. 2, pp. 177-184(1975).

[15] Roodman, G. M. and Schwartz, L. B., "Extension of the Multiperiod Facility Phase-out Model : New Procedure and Applications to a Phase-in/Phase-out Problem," AIIE Transaction, Vol. 9, No. 1, pp. 103-107(1977).

[16] Van Roy, T. J. and Erlenkotter, D., "A Dual-Based Procedure for Dynamic Facility Location," Management Science, Vol. 28, pp. 1091-1105(1982).

[17] Wesolowsky, G. O. and Truscott, W. G., "The Multiperiod Location-Allocation Problem with Relocation of Facilities", Management Science, Vol. 28, No. 1, pp. 57-65(1975).