Journal of the Korean
OR/MS Society
Vol. 13 No. 2 December 1988

— Research Paper —

A Modified Gradient Procedure for Multifacility Euclidean Distance Location Problems

Chae Y. Lee *

Abstract

An efficient heuristic solution procedure is developed for the minisum location problems. The gradient direction method and modified gradient approach are developed due to the differentiability of the objective functions. Suboptimal step size is obtained analytically. A Modified Gradient Procedure (MGP) is presented and compared with the hyperboloid approximation procedure (HAP) which is one of the best known methods.

1. Introduction

Location problems are classified as either a minisum location problem or a minimax problem. The minisum location problems are concerned with locating plants, warehouses, or service centers such that they minimize the total cost of servicing customers. The minimax location problem however, minimizes the maximum weighted distance to the fixed facilities. Typical examples include locaing public schools or emergency facilities such as fire stations or ambulance centers. Francis and White [4], Hensen and Thisse [5] and many others [1, 7, 8] developed efficient algorithms for these problems.

In this paper we consider an analytical solution procedure to the minisum location problems. We obtain an improvig direction and suboptimal step size at each point in the search process. Computational results are presented for various size of problems. The approach is compared to the well-known ε-approximation procedure.

2. Multifacility Euclidean Distance Location Problems

The problem we are concerned with is a general capacitated munisum location problems. It involves interactions between sources and destinations as well as interactions between sources.

Suppose that we are to locate m new facilites (sources) which will interact with n existing facilities (destinations). Let the new facilities be numbered 1 through m and the existing facilities be numbered $m+1$ through $m+n$. Let u_{ij} be the allocation from new facility i to new or exiting facility j. Also suppose that the transportation

*Korea Institute of Technology

cost for one unit of the product per unit distance is constant. The problem is to determine the location of m new facilities so as to minimize the total transportaion cost.

For a mathematical statement of the problem, let $X_i=(x_i, y_i)$ denotes the location of the i-th new facility and $P_j=(a_j, b_j)$ be the location of the j-th new or existing facility. Also let $d(X_i, P_j)$ be the distance measure between the two points X_i and P_j. Note that P_j is a decision variable for $j=1, \ldots, m$ and a parameter for $j=m+1, \ldots, m+n$.

We may then formulate the multifacility Euclidean distance location problem with $X=(X_1, \ldots, X_m)$ as follows :

$$\text{minimize } f(X)=\sum_{i=0}^{m}\sum_{j=1}^{m+n} u_{ij}\, d(X_i, P_j) \tag{1}$$

$$\text{where} \quad d(X_i, P_j)=[(x_i-a_j)^2+(y_i-b_j)^2]^{1/2}$$

It is well known that f is strictly convex in E^2 if the points P_j are not collinear [4]. Hence the minimum of f is achieved at a unique point X.

In the objective function given above, the allocation $u_{ij}=0$, for $j=i$. Hence, if we define a set J_i as

$$J_i=\{j : j=1, \ldots, m+n, j\neq i\} \tag{2}$$

(1) can be given by

$$\text{minimize } f(X)=\sum_{i=1}^{m}\sum_{j\in J_1} u_{ij}\, d(X_i, P_j). \tag{3}$$

2.1 A Modified Gradient at the Point X

Kuhn [6] used a modified gradient to find a single point that solves the general Fermat problem. We will apply the Kuhn's modified gradient to this multifacility location problem. The partial dervatives of $f(X)$ given in (3) respect to x_i *and* y_i is given by

$$\begin{aligned} \nabla_{i1} f(X) &= \sum_{j\in Ji} \frac{u_{ij}(x_i-a_j)}{d(X_i P_j)} \qquad i=1,\ldots,m \\ \nabla_{i2} f(X) &= \sum_{j\in Ji} \frac{u_{ij}(y_i-b_j)}{d(X_i P_j)} \qquad i=1,\ldots,m \end{aligned} \tag{4}$$

Clearly, the gradient at X_i is expressed as

$$R(X_i)^{\bullet}=(\nabla_{i1} f(X),\ \nabla_{i2} f(X)) \tag{5}$$

Note that $R(X_i)$ is not defined if $X_i=P_j$ for some i and j. In other words, if either any two new facilites or a new facility and an existing facility have the same location, then $d(X_i, P_j)$ is equal to zero, and the partial derivatives given in (4) are undefined.

Here, let us define $c(i)$ as the existing facility j which coincides with the i-th new facility. Also, we define a set J_2 as

$$J_2=\{j : j=1, \ldots, m+n,\ j\neq i\ j\neq c(i)\}. \tag{6}$$

Then, when $X_i=P_{c(i)}$ the following modified gradient is considered :

$$R(X_i)=R(P_{C(i)})=\begin{cases}\dfrac{\|R_i\|-u_{ic(i)}}{\|R_i\|}R_i, & \text{if } \|R_i\|>u_{ic(i)}\\ 0, & \text{if } \|R_i\|\geq u_{ic(i)}\end{cases} \tag{7}$$

where

$$R_i=\sum_{j\in J_2}\frac{u_{ij}(X_i-P_j)}{d(X_i,\ P_j)} \tag{8}$$

In Equation (7) the length of R_i is compared with the interaction $u_{ic(i)}$ and the resultant weight is defined in the direction of R_i. Francis and Cabot [3] prove that a necessary condition for X_i, $i=1, \ldots, m$ to be an optimal new facility location is $R(X_i)$, $i=1, \ldots, m$, given in (5) and (7) are equal to zero. The question of whether the condition is also sufficient remains an open one for the multifacility case.

2.2 An Approximate Optimal Step Size

In previous section we have seen that the gradient at a new facility X_i is defined in a different fashion due to the coincidence between the location of new facility X_i and the existing facility P_j. Thus, to classify the m new facilities we introduce a set I as follows :

$$I=\{i : X_i=P_{c(i)}\} \tag{9}$$

For X_i $i\notin I$, the gradient $R(X_i)$ is defined as in (4). Also, for X_i, $i\in I$, the modified gradient as in (7).

Here, we will define

$$R(X)=(R(X_1), \ldots, R(X_m)) \tag{10}$$

Note that $-R(X)$ is an improving direction at the point $X=(X_1, \ldots, X_m)$.

Thus, an iterative solution procedure to search the optimal point is given by

$$X^{h+1}=X^h-\lambda R(X^h) \quad h=1,\ 2,\ldots \tag{11}$$

Where λ is the step size taken along the direction $-R(X^h)$ and h denotes the iteration number. We will now show below that an estimate of the optimal step size is given by

$$\hat{\lambda}=\frac{\sum_{i=1}^{m}\|R(X_i)\|^2}{\sum_{i=1}^{m}\sum u_{ij}\|R(X_i)\|^2/d(X_i P_j)} \tag{12}$$

In the denominator of Equation (12) the index i and j have the following relationship

a) if $i \notin I$, then $j \notin J_1$

b) if $i \notin I$, then $j \notin J_2$

let $d_{i1}=-\nabla_{i1}f(X)$ and $d_{i2}=-\nabla_{i2}f(X)$ such that $d_i=-R(X_i)$. By setting $d=(d_1,\ldots,d_m)$ we will solve the following minimization problem :

$$\text{minimize } f(X+\lambda d)$$
$$\text{subject to } \lambda\varepsilon E^1$$

Note that $f(X+\lambda d)$ is a differentiable convex function, and can be written as

$$f(X+\lambda d)=\sum_{i=1}^{m}\sum_{j} u_{ij}[(x_i+\lambda d_{i1}-a_j)^2+(y_i+\lambda d_{i2}-b_j)^2]^{1/2}.$$

By taking the derivative of f with respect to λ, we obtain the following expression :

$$\frac{df(X+\lambda d)}{d\lambda}=\sum_{i=1}^{m}\sum_{j}u_{ij}\frac{d_{i1}(x_i+\lambda d_{i1}-a_j)+d_{i2}(y_i+\lambda d_{i2}-b_j)}{[(x_i+\lambda d_{i1}-a_j)^2+(y_i+\lambda d_{i2}-b_j)^2]^{1/2}} \tag{13}$$

By letting the derivative equal to zero, we get

$$\lambda\sum_{i=1}^{m}\sum_{j}\frac{u_{ij}(d_{i1}+d_{i2})}{[(x_i+\lambda d_{i1}-a_j)^2+(y_i+\lambda d_{i2}-b_j)^2]^{1/2}}=-\sum_{i=1}^{m}\sum_{j}\frac{u_{ij}[d_{i1}x_i-a_j)+d_{i2}(y_i-b_j)]}{[(x_i+\lambda d_{i1}-a_j)^2+(y_i+\lambda d_{i2}-b_j)^2]^{1/2}}$$

If we approximate $[(x_i+\lambda d_{i1}-a_j)^2+(y_i+\lambda d_{i2}-b_j)^2]^{1/2}$ with $[(x_i-a_j)^2+(y_i+b_j)^2]^{1/2}$, then solving for λ gives

$$\hat{\lambda}=-\sum_{i=1}^{m}\sum_{j}\frac{u_{ij}(X_i-P_j)d_i}{d(X_i P_j)}\bigg/\sum_{i=1}^{m}\sum_{j}\frac{u_{ij}\|d_i\|^2}{d(X_i P_j)} \tag{15}$$

By substituting (4) for the numerator of (15) we obtain the approximate optimal step size $\hat{\lambda}$ given in (12).

2.3 Hyperboloid Approximation Procedure (HAP)

As discussed previously, the partial derivatives of the objective function given in (4) are not defined of $X_i = P_j$ for some i and j. Hence, an alternative minimization problem is employed in HAP as follows :

$$\text{minimize } \hat{f}(X_1, \ldots, X_m) = \sum_{i=1}^{m} \sum_{j=1}^{m+n} u_{ij}[(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon]^{1/2}$$

$$\text{where } \lim_{\varepsilon \to 0} \hat{f}(X_1, \ldots, X_m) = f(X_1, \ldots, X_m).$$

The modified objective function $\hat{f}(X_1, \ldots, X_m)$ is differentiable at any point in the plane, and its gradient can be used in developing an iterative scheme. The procedure which is due to Eyster et. al [2] also uses an arbitrary small positive perturbation constant. It is known that the iterative procedure converged to the optimal point for all problems it was used on. However, no convergence proof has been given by the authors.

3. Modified Gradient Procedure (MGP) for Multifacility Location Problems

We propose an iterative scheme to solve the multifacility location problems based on the direction and the step size we have developed in Section 2.

3.1 Direction of Movement

The direction $-R(X)$ at the point $X = (X_1, \ldots, X_m)$ is expressed as

$$R(X) = (R(X_1), \ldots, R(X_m))$$

where each component $R(X_i)$ is defined as follows :

(a) If $X_i \neq P_j$ then

$$R(X_i) = \sum_{j \in J_i} \frac{u_{ij}(X_i - P_j)}{d(X_i, P_j)}$$

(b) If $X_i = P_{c(i)}$, then

$$R(X_i) = \begin{cases} \dfrac{\| R_i \| - u_{ic(i)}}{\| R_i \|} R_i, & \text{if } \| R_i \| > u_{ic(i)} \\ 0, & \text{if } \| R_i \| \leq u_{ic(i)} \end{cases}$$

where

$$R_i = \sum_{j \in J_2} \frac{u_{ij}(X_i - P_j)}{d(X_i, P_j)}$$

3.2 Step Sizes Along the direction $-R(X)$

In Section 2.2 we have obtained the suboptimal step size as follows :

$$\hat{\lambda} = \frac{\sum_{i=1}^{m} \| R(X_i) \|^2}{\sum_{i=1}^{m} \sum_{j} u_{ij} \| R(X_i) \|^2 / d(X_i, P_j)}$$

3.3 Modified Gradient Procedure (MGP)

By using the direction and step size, we here develop an algorithm to solve the multifacility Euclidean distance location problems.

Initialization : Choose a starting point $X_0 = (X_1, \ldots, X_m)$ and terminating scalar Δ. Compute $f(X^0)$ with given u_{ij} and P_j. Set $h=0$ and go to Step 1.

Step 1. Set $I = \{i : X_i^h = P_{c(i)}\}$ and let $i=1$ and go to Step 2.

Step 2. If $i \in I$, go to Step 4. Otherwise, if $i \notin I$, go to Step 3.

Step 3. Compute

$$R(X_i^h) = \sum_{j \in J_1} \frac{u_{ij}(X_i^h - P_j)}{d(X_i^h, P_j)}$$

where $J_1 = \{j : j=1, \ldots, m+n, j \neq j\}$

Replace i by $i+1$. If $i > m$, go to Step 5. Otherwise, if $i \leq m$, go to step 2.

Step 4. Compute

$$R_i = \sum_{j \in J_2} \frac{u_{ij}(X_i^h - P_j)}{d(X_i^h, P_j)}$$

where $J_2 = \{j : j=1, \ldots, m+n, j \neq i, j \neq c(i)\}$

Let

$$R(X_i^h)=\begin{cases}\frac{\| R_i \| - u_{ir(i)}}{\| R_i \|}R_i, & if \ \| R_i \| > u_{ir(i)} \\ 0, & if \ \| R_i \| \leq u_{ir(i)}\end{cases}$$

Replace i by $i+1$. If $i>m$ then go to Step 5. Otherwise, if $i\leq m$, go to Step 2.

Step 5. Compute the step size λ as

$$\hat{\lambda}=\frac{\sum_{i=1}^{m} \| R(X_i) \|^2}{\sum_{i=1}^{m}\sum_{j} u_{ij} \| R(X_i) \|^2 \| / d(X_i, P_j)}$$

and go to Step 6.

Step 6. Let $X_i^{h+1}=X^h-\hat{\lambda}R(X_i^h)$, for $i=1, \ldots, m$ and let $X^{h+1}=(X^{h+1}, \ldots, X^{h+1})$. Obtain the objective function value $f(X^{h+1})$. If $f(X^h)-f(X^{h+1})<\Delta f(X^h)$, stop. Otherwise, replace h by $h+1$, and go to Step 1.

4. Computational Results

the computational experience of the MGP is presented and compared with the HAP. Four different problem types $(m\times n)$ are considered according to the number of new and existing facilities ; 2×8, 5×20, 10×50 and 25×100. For each type five independent random problems are generated. The weight u_{ij} is generated from a uniform distribution over [5, 15]. Each existing facility is located uniformly over [0.00, 10.00] in x and y coordinates. center of gravity solution is employed for the starting location of each new facility.

Table I Computational Results of MGP and HAP

Problem Type and Number		MGP		HAP	
		Objective function value	CPU time (second)	Objective function value	CPU time (second)
2×8	1	154.49	0.0105	153.05	0.0852
	2	175.24	0.0136	174.68	0.0401
	3	212.22	0.0162	211.86	0.1324
	4	236.65	0.0108	235.48	0.0896
	5	140.66	0.0108	140.29	0.0442
	1	240.17	0.0518	238.06	0.8487

	2	320.45	0.0517	316.03	1.2043
5×20	3	247.77	0.0518	244.17	0.5175
	4	294.32	0.0519	289.72	0.5007
	5	253.98	0.0518	251.07	0.3563
	1	636.15	0.2115	629.17	4.5462
	2	600.39	0.2116	593.23	2.9569
10×50	3	435.23	0.2115	431.18	2.7557
	4	638.53	0.2115	634.22	3.1626
	5	641.95	0.2111	633.53	2.9968
	1	656.87	1.0426	646.20	20.3739
	2	598.23	1.0462	586.92	16.1488
25×100	3	661.65	2.0285	655.34	24.3931
	4	607.80	1.0428	595.33	17.4116
	5	612.61	1.0418	600.18	24.1729

By implementing the procedure into a FORTRAN code and running on the CONVEX at Korea Institute of Technology we illustrate the computational results of the MGP. The operating system was UNIX and the code was compiled using the fc compiler. Table I shows the objective function value and the CPU time in second. From this table we see that the MGP is undoubtedly fast and compares well with the exact algorithm in view of the solution quality. For the problem of 25×100, MGP reduces the CPU time by a factor of 10 or 20 compared with the exact approach, with a margin of error in optimality of 1~2%.

5. Conclusion

An analytical approach to solve the multifacility Euclidean distance location problems is investigated. The improving direction to search the optimal point as well as the step size to the movement are derived both for the differentiable and nondifferentiable cases of the objective function. The algorithm based on the modified gradient procedure is developed using the suboptimal step size. It is illustrated that for big multifacility location problems the use modified gradient procedure is considerably superior to the epsilon-perturbation in terms of computation time. A comparable solution quality is also guaranteed by the procedure.

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