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—Research Paper—

Sensitivity Analysis of Project Sequencing Problems

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Abstract

We consider sensitivity analysis of project sequencing problems, in which the sequence of a finite set of expansion projects is sought to meet a deterministic demand projection at minimum discounted cost. In particular, by characterizing the underlying network structure, we find analytically the sensitivity range for a project cost such that the optimal sequencing policy remains unchanged for any value in the range. A numerical example is presented.

1. Introduction

In this paper, we consider sensitivity analysis of project sequencing problems, in which the sequence of a finite set of expansion projects is sought to meet a deterministic demand projection at minimum discounted cost. In particular, we wish to find the sensitivity range for a project cost such that the optimal sequencing policy remains unchanged for any value in the range.

The project sequencing problem can be modelled by a forward dynamic programming formulation, which is similar to the backward version of Erlenkotter [1973]. The forward formulation also can be viewed as that for a shortest path problem. Hence the optimal sequencing policy can be easily obtained via any shortest path algorithm.

There is some inherent difficulty in sensitivity analysis of the project sequencing problem. Since any variation in a demand projection, project capacitites, or the discount rate entails simultaneous and nonlinear changes of discounted costs, a system of nonlinear inequalities needs to be solved to find the sensitivity ranges for these parameters. However, it is relatively simple to calculate sensitivity ranges for project costs if we characterize the network structure underlying the project sequencing problem. A numerical example of this calculation is presented.

2. The Project Sequencing Problem

Each project, indexed $k=1, \dots, n$ is defined by the pair (z_k, c_k) where $z_k > 0$ is the capacity of project k and $c_k > 0$ is the corresponding project cost. A demand projection $D(t)$ is known and assumed to be nondecreasing.

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Capacity must be sufficient to meet demand at all times; namely, $D(t) \leq \sum_{k=1}^{n} z_k$ at any time t during the planning horizon. We wish to establish projects to provide capacity to meet the demand at minimum discounted cost. Initial capacity is assumed to equal exactly the initial demand $D(0)$.

Let the project status set X contain the indices of projects that have already been established at some point. The set X may be represented conveniently by a binary status vector $(x_1, \ldots, x_n)$. $X^0=(0, \ldots, 0)$ is the initial status vector indicating that all projects remain to be undertaken. $X^n=(1, \ldots, 1)$ is the completion status vector indicating that all projects have been undertaken.

Investment costs are discounted to the present time 0 at the rate $r>0$, leading to a continuous discount factor for the time interval t of $exp(-rt)$. When we consider the time value of money, it is optimal to establish new projects only when the existing capacity is exhausted.[1] Let $\tau(X)$ be the earliest time at which demand exceeds the available capacity $\sum_{k \in X} z_k$. Mathematically,

$$\tau(X)=sup\ \{t \mid D(t) \leq \sum_{k \in X} z_k\}. \qquad (1)$$

Therefor, $\tau(X)$ is the optimal timing for the construction of any project that remains to be undertaken. The definition of $\tau(X)$ by equation (1) is restricted to the case where $D(t)$ is nondecreasing, as discussed in Erlenkotter [1974].

When project k is included in the set X, we define X_k to be the set X with project k excluded, or the status vector X with $x_k=0$. Let $c(X_k)$ be the discounted cost of implementing project k at time $\tau(X_k)$ such that

$$c(X_k)=c_k\ exp\ \{-r\ \tau(X_k)\}. \qquad (2)$$

Let $f(X)$ be the minimum discounted cost of sequencing those projects in the set X. The project sequencing problem then can be solved by the following forward dynamic programming formulation :

$$f(X^0) \equiv 0$$
$$f(X) = \min_{k \in X} \{f(X_k)+c(X_k)\} \text{ for all } X \subseteq X^n, \qquad (3)$$

which is a slight modification of the backward formulation proposed by Erlenkotter [1973]. Since $n2^{n-1}$ comparisons are needed to find the value of $f(X^n)$, the formulation by (3) runs in $O(n2^{n-1})$ time.[2] Still, this approach is very useful since a complete enumeration of all possible sequences would require $O(n!)$ time. In fact, the class of project sequencing problems was shown to be NP-complete by Akileswaran *et al.* [1983]. It is unlikely that a polynomial-time algorithm exists for solving such problems.

If we think of the status vector X as a node, the activity of implementing project k at time $\tau(X_k)$ as the arc joining X_k and X, then the project sequencing problem can be represented by an acyclic network that has the discounted cost $c(X_k)$ as the attribute of the arc joining X_k to X(see Luss 1982). The resulting network,

1. It is assumed that variable operating and distribution costs are proportional to the amount actually produced and identical for all projects.

2. When the set contains i projects, there are $n-i$ projects that remain to be undertaken. Since there are $\binom{n}{i}$ ways of having i projects in X, the number of comparisons to be made is $\sum_{i=0}^{n} (n-i)\binom{n}{i}=n2^{n-1}$.

denoted by G, consists of 2^n nodes and $n2^{n-1}$ arcs. The project sequencing problem now becomes the problem of finding the shortest path from X^0 to X^n on G.

3. Sensitivity Analysis

In practice, the project sequencing problem needs estimates of such model parameters as c_k, z_k, $D(t)$, and r. Since any of these parameters is unlikely to be known precisely, it is useful to examine whether the optimal sequencing policy remains stable for some variation in the value of each parameter.

In general, there are two ways of examining the sensitivity of the optimal policy with respect to a model parameter : (1) finding the closed-form sensitivity range for a parameter and (2) reoptimizing the original problem with different input data. The former method is clearly better, if available, but obtaining the sensitivity range analytically may be difficult. The latter method is heuristic in nature and less attractive, but still useful when no other approach is available. However, the repetitive use of reoptimization techniques seldom guarantees theoretic convergence to even a predetermined range, and is computationally prohibitive quite often.

While it is possible to find the closed-form sensitivity range for project cost c_k as we shall see in the next section, the sensitivity ranges for such parameters as the project capacity z_k, the demand projection $D(t)$, and the discount rate r are practically difficult to obtain since we need to solve systems of nonlinear inequalities resulting from parametric variations.

Any variation in z_k affects the optimal timing for the construction of a project that remains to be undertaken; thereby, all appropriate discounted costs associated with arcs emanating from nodes(status vectors) with $x_k=1$ should be updated accordingly. Any variation in r or $D(t)$ requires the recalculation of all discounted costs. All these parametric variations result in simultaneous changes of discounted costs in the form of exponential functions. Specifically, the number of nonlinear inequalities to be solved is $(n-1)2^{n-2}$ for z_k,[3] and $n2^{n-1}$ for r or $D(t)$. Because of these complexities involved, reoptimizing the original problem with different input data could be the only viable postoptimality tool for these parameters.

4. Sensitivity Ranges for Project Costs

Let $\triangle c$ denote the amount of variation in project cost c_k, We wish to find the sensitivity range for c_k, that is, the range of $\triangle c+c_k$ such that for any value in this range, the optimal sequencing policy remains unchanged. First, we characterize the network structure underlying the project sequencing problem.

We denote the reduced cost associated with arc (X_k, X) by $\bar{c}(X_k)$ so that

$$\bar{c}(X_k)=f(X_k)+c(X_k)-f(X). \tag{4}$$

Physically, the reduced cost $\bar{c}(X_k)$ measures the minimum possible opportunity cost of implementing the project

3. When $Z_k \rightarrow Z_k+\triangle Z$, the discounted costs associated with arcs starting from status vectors with $x_k=1$ are subject to change. The number of these arcs is $\sum_{n=0}^{n-2}(n-1-i)\binom{n-1}{i}=(n-1)2^{n-2}$.

k at time $\tau(X_k)$ instead of sequencing optimally all the projects in the set $X \subseteq X^n$. It is easy to see that $\bar{c}(X_k)=0$ if arc (X_k, X) is included in the shortest path from X^0 to X' where $X \subseteq X'$.

It is a well-known fact that any shortest path in a network is the same whether *original* or *reduced* costs are used. Algebraically, this is a consequence of the fact that the *standard form* LP's objective gradient c can be replaced by $c-\pi A$ for any π without changing the primal optimal solution x^*. This equivalence will be a key observation for deriving the sensitivity range for each project cost.

Given $\triangle c$ for c_k, the cost of arc (X_k, X) is subject to change. We denote the amount of such variation in $c(X_k)$ by $\triangle(X_k)$. Then

$$\triangle(X_k)=\triangle c \; exp\{-r \; \tau(X_k)\}. \tag{5}$$

The quantities $c(X_k)+\triangle(X_k)$ and $\bar{c}(X_k)+\triangle(X_k)$ are called the perturbed original cost and the perturbed reduced cost associated with the arc (X_k, X), respectively.

Since the aforementioned equivalence that any shortest path in a network is the same whether original or reduced costs are used is immediately extended to the perturbed cases, we can state:

> Observation : *The sum of perturbed reduced costs along the shortest path from* X^0 *to* X^n *is* $\triangle(X_k^*)$ *if and only if the optimal sequencing policy remains unchanged*

where X_k^* is the status vector with $x_k=0$ that is included in the shortest path from X^0 to X^n.

Since any project must be established at most once in a feasible sequencing policy, the total discounted cost of any feasible sequencing policy is increased by $\triangle(X_k)$ if the policy implements the project k at time $\tau(X_k)$. If we let $v^*(X)$ be the sum of reduced costs along the shortest path from X to X^n, the Observation can be specialized as follows :

Proposition 1 $\triangle c$ *for project cost* c_k *satisfies* $\triangle c \geq -c_k$ *and*

$$\triangle c \; exp\{-r \; \tau(X_k^*)\} - \triangle c \; exp\{-r \; \tau(X_k)\} \leq \bar{c}(X_k)+v^*(X) \tag{6}$$

for all the status vectors X *with* $x_k=1$ *if and only if the optimal sequencing policy remains unchanged.*

Proof : Given $\triangle c$ for c_k the sum of perturbed reduced costs along any path from X^0 to X^n that includes arc (X_k, X) is no less than

$$\bar{c}(X_k)+\triangle(X_k)+v^*(X)$$

since the minimum sum of reduced costs among paths from X^0 to X_k is always zero. It follows from the Observation that

$$\triangle(X_k^*) \leq \bar{c}(X_k)+\triangle(X_k)+v^*(X)$$

for every X with $x_k=1$ if and only if the optimal policy remains unchanged. Substituting $\triangle(X_k^*)$ and $\triangle(X_k)$ by equation (5), the proof is complete.

The value of $v^*(X)$ can be obtained by solving the shortest path problem on the nework G with reduced costs. Since there are 2^{n-1} status vectors with $x_k = 1$ and one of these vectors is X_k^*, we must solve $2^{n-1} - 1$ inequalities to find the sensitivity range for c_k. Hence the overall sensitivity algorithm takes $O(n2^{n-1})$ time, which is as diffcult as solving the original problem. However, this is superior to an *ad hoc* exploration of the sensitivity to c_k, which might require resolving the original problem indefinitely many times.

5. A Numerical Example

To demonstrate calculating the sensitivity ranges for project costs, we rewrite the four-project example of Erlenkotter [1973] as shown in Table 1.

project k	1	2	3	4
z_k	1	2	3	4
c_k	150	280	350	400

$$D(t) = \begin{cases} 0, & t=0 \\ 2, & 0<t\leq 10 \\ 5, & 10<t\leq 20 \\ 10, & 20<t\leq +\infty \end{cases} \quad \text{and } exp(-rt) = \begin{cases} 1.0000, & t=0 \\ 0.6065, & t=10 \\ 0.3678, & t=20 \end{cases}$$

Table 1 : A Four-project Sequencing Example

The problem of finding the sequence of the four projects that minimizes the total discounted cost can be represented by the acyclic network G as shown in Figure 1.

Each node X represents a project status vector X. The cost associated with arc (X_k, X) is the discounted cost of implementing project k at time $\tau(X_k)$ where $\tau(X_k)$ is determined by equation (1) independently of the capacity and the cost of project k. For example, the cost associated with the arc joining (0110) and (0111) is $c_4 \, exp\{-r\tau(0110)\} = (400)\, exp(-20r) = 147$.

The minimum discounted cost for this four-project example is the length of the shortest path from node (0000) to node (1111), which is computed as 694. The optimal path is

$$(0000) \rightarrow (0100) \rightarrow (0110) \rightarrow (0111) \text{ or } (1110) \rightarrow (1111).$$

Hence the optimal sequence and timing of the four projects can be stated as follows : Implement project 2 at time 0, project 3 at time 10, and projects 1 and 4 together at time 20.

Next, we calculate the sensitivity ranges for project costs using Proposition 1. By equation (4), we compute the reduced cost associated with each arc as shown in Figure 2. Table 2 shows how to calculate the sensitivity range for c_3. Note that $X_3^* = (0100)$ and $\tau(X_3^*) = 10$. Sensitivity ranges for all the project costs are summarized in Table 3.

X_3	X	$\tau(X_3)$	Inequality (6) of Proposition 1
(0000)	(0010)	0	$\triangle c$ (0.6065-1.0000)$\leq$0 +28
(0001)	(0011)	20	$\triangle c$ (0.6065-0.6065)$\leq$19+54
(1000)	(1010)	0	$\triangle c$ (0.6065-1.0000)$\leq$59+64
(0101)	(0111)	20	$\triangle c$ (0.6065-0.3678)$\leq$13+ 0
(1001)	(1011)	20	$\triangle c$ (0.6065-0.3678)$\leq$ 0 +29
(1100)	(1110)	10	$\triangle c$ (0.6065-0.6065)$\leq$36+ 0
(1101)	(1111)	20	$\triangle c$ (0.6065-0.3678)$\leq$13+ 0
Sensitivity Range : $278.84 \leq \triangle c + c_3 \leq 404.46$			

Table 2 : Calculating the Sensitivity Range for c_3

	$k=1$	$k=2$	$k=3$	$k=4$
Upper bound	$+\infty$	325.87	404.46	$+\infty$
Lower bound	28.51	0	278.84	354.13

Table 3 : Sensitivity Ranges for Project Costs

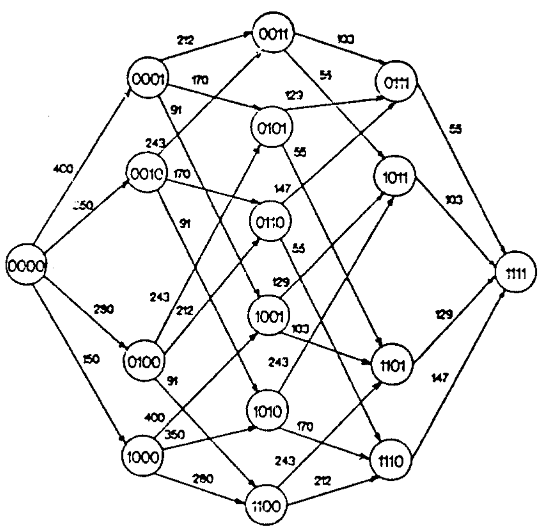

Figure 1 : The Acyclic Network G with Discounted Costs

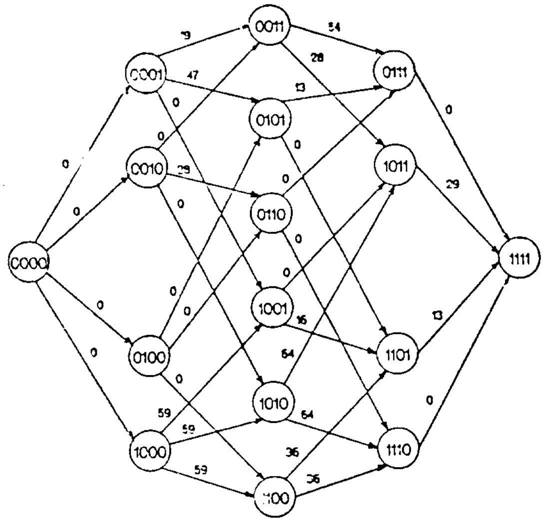

Figure 2 : The Acyclic Network G with Reduced Costs

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